

Research Article

GCR-Lightlike Product of Indefinite Sasakian Manifolds

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We study mixed geodesic *GCR*-lightlike submanifolds of indefinite Sasakian manifolds and obtain some necessary and sufficient conditions for a *GCR*-lightlike submanifold to be a *GCR*-lightlike product.

1. Introduction

The geometry of submanifolds is one of the most important topics of differential geometry. It is well known that the geometry of semi-Riemannian submanifolds have many similarities with their Riemannian case but the geometry of lightlike submanifolds is different since their normal vector bundle intersect with the tangent bundle making it more difficult and interesting to study. The lightlike submanifolds of semi-Riemannian manifolds were introduced and studied by Duggal and Bejancu [1]. Since contact geometry has vital role in the theory of differential equations, optics, and phase spaces of a dynamical system, therefore contact geometry with definite and indefinite metric becomes the topic of main discussion. In the process of establishment of theory of lightlike submanifolds, Duggal and Sahin [2] introduced the theory of contact *CR*-lightlike submanifold of indefinite Sasakian manifold. Later on, Duggal and Sahin [3] introduced generalized Cauchy-Riemann (*GCR*)-lightlike submanifold of indefinite Sasakian manifolds to find an umbrella of invariant, screen, real, contact *CR*-lightlike subcases, and real hypersurfaces. In the present paper we study *GCR*-lightlike submanifolds extensively and study mixed geodesic *GCR*-lightlike submanifolds of

indefinite Sasakian manifolds. We also obtain some necessary and sufficient conditions for a GCR-lightlike submanifold to be a GCR-lightlike product.

2. Lightlike Submanifolds

We recall notations and fundamental equations for lightlike submanifolds, which are due to the book [1] by Duggal and Bejancu.

Let $(\overline{M}, \overline{g})$ be a real $(m+n)$ -dimensional semi-Riemannian manifold of constant index q such that $m, n \geq 1$, $1 \leq q \leq m+n-1$ and (M, g) be an m -dimensional submanifold of \overline{M} and g be the induced metric of \overline{g} on M . If \overline{g} is degenerate on the tangent bundle TM of M then M is called a lightlike submanifold of \overline{M} . For a degenerate metric g on M

$$TM^\perp = \cup \left\{ u \in T_x \overline{M} : \overline{g}(u, v) = 0, \forall v \in T_x M, x \in M \right\}, \quad (2.1)$$

is a degenerate n -dimensional subspace of $T_x \overline{M}$. Thus both $T_x M$ and $T_x M^\perp$ are degenerate orthogonal subspaces but no longer complementary. In this case, there exists a subspace $\text{Rad}T_x M = T_x M \cap T_x M^\perp$ which is known as radical (null) subspace. If the mapping

$$\text{Rad}TM : x \in M \longrightarrow \text{Rad}T_x M, \quad (2.2)$$

defines a smooth distribution on M of rank $r > 0$ then the submanifold M of \overline{M} is called an r -lightlike submanifold and $\text{Rad}TM$ is called the radical distribution on M .

Let $S(TM)$ be a screen distribution which is a semi-Riemannian complementary distribution of $\text{Rad}(TM)$ in TM , that is

$$TM = \text{Rad}TM \perp S(TM), \quad (2.3)$$

and $S(TM^\perp)$ is a complementary vector subbundle to $\text{Rad}TM$ in TM^\perp . Let $\text{tr}(TM)$ and $\text{ltr}(TM)$ be complementary (but not orthogonal) vector bundles to TM in $\overline{TM}|_M$ and to $\text{Rad}TM$ in $S(TM^\perp)^\perp$, respectively. Then, we have

$$\text{tr}(TM) = \text{ltr}(TM) \perp S(TM^\perp), \quad (2.4)$$

$$\overline{TM}|_M = TM \oplus \text{tr}(TM) = (\text{Rad}TM \oplus \text{ltr}(TM)) \perp S(TM) \perp S(TM^\perp). \quad (2.5)$$

Let u be a local coordinate neighborhood of M and consider the local quasiorthonormal fields of frames of \overline{M} along M , on u as $\{\xi_1, \dots, \xi_r, W_{r+1}, \dots, W_n, N_1, \dots, N_r, X_{r+1}, \dots, X_m\}$, where $\{\xi_1, \dots, \xi_r\}, \{N_1, \dots, N_r\}$ are local lightlike bases of $\Gamma(\text{Rad}TM|_u)$, $\Gamma(\text{ltr}(TM)|_u)$ and $\{W_{r+1}, \dots, W_n\}, \{X_{r+1}, \dots, X_m\}$ are local orthonormal bases of $\Gamma(S(TM^\perp)|_u)$ and $\Gamma(S(TM)|_u)$, respectively. For this quasi-orthonormal fields of frames, we have the following.

Theorem 2.1 (see [1]). *Let $(M, g, S(TM), S(TM^\perp))$ be an r -lightlike submanifold of a semi-Riemannian manifold $(\overline{M}, \overline{g})$. Then there exists a complementary vector bundle $\text{ltr}(TM)$ of $\text{Rad}TM$ in $S(TM^\perp)^\perp$ and a basis of $\Gamma(\text{ltr}(TM)|_u)$ consisting of smooth section $\{N_i\}$ of $S(TM^\perp)^\perp|_u$, where u is a coordinate neighborhood of M such that*

$$\overline{g}(N_i, \xi_j) = \delta_{ij}, \quad \overline{g}(N_i, N_j) = 0, \quad \text{for any } i, j \in \{1, 2, \dots, r\}, \quad (2.6)$$

where $\{\xi_1, \dots, \xi_r\}$ is a lightlike basis of $\Gamma(\text{Rad}(TM))$.

Let $\overline{\nabla}$ be the Levi-Civita connection on \overline{M} . Then according to the decomposition (2.5), the Gauss and Weingarten formulas are given by

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y), \quad \forall X, Y \in \Gamma(TM), \quad (2.7)$$

$$\overline{\nabla}_X U = -A_U X + \nabla_X^\perp U, \quad \forall X \in \Gamma(TM), U \in \Gamma(\text{tr}(TM)), \quad (2.8)$$

where $\{\nabla_X Y, A_U X\}$ and $\{h(X, Y), \nabla_X^\perp U\}$ belong to $\Gamma(TM)$ and $\Gamma(\text{tr}(TM))$, respectively. Here ∇ is a torsion-free linear connection on M , h is a symmetric bilinear form on $\Gamma(TM)$ which is called the second fundamental form, and A_U is linear a operator on M and called a shape operator.

According to (2.4), considering the projection morphisms L and S of $\text{tr}(TM)$ on $\text{ltr}(TM)$ and $S(TM^\perp)$, respectively, (2.11) and (2.12) give

$$\overline{\nabla}_X Y = \nabla_X Y + h^l(X, Y) + h^s(X, Y), \quad (2.9)$$

$$\overline{\nabla}_X U = -A_U X + D_X^l U + D_X^s U, \quad (2.10)$$

where one puts $h^l(X, Y) = L(h(X, Y))$, $h^s(X, Y) = S(h(X, Y))$, $D_X^l U = L(\nabla_X^\perp U)$, $D_X^s U = S(\nabla_X^\perp U)$.

As h^l and h^s are $\Gamma(\text{ltr}(TM))$ -valued and $\Gamma(S(TM^\perp))$ -valued, respectively, therefore, these are called as the lightlike second fundamental form and the screen second fundamental form on M . In particular

$$\overline{\nabla}_X N = -A_N X + \nabla_X^l N + D^s(X, N), \quad (2.11)$$

$$\overline{\nabla}_X W = -A_W X + \nabla_X^s W + D^l(X, W), \quad (2.12)$$

where $X \in \Gamma(TM)$, $N \in \Gamma(\text{ltr}(TM))$ and $W \in \Gamma(S(TM^\perp))$. Using (2.4)-(2.5) and (2.8)–(2.12), one obtains

$$\bar{g}(h^s(X, Y), W) + \bar{g}(Y, D^l(X, W)) = g(A_W X, Y), \quad (2.13)$$

$$\bar{g}(h^l(X, Y), \xi) + \bar{g}(Y, h^l(X, \xi)) + g(Y, \nabla_X \xi) = 0, \quad (2.14)$$

$$\bar{g}(A_N X, N') + \bar{g}(N, A_{N'} X) = 0, \quad (2.15)$$

for any $\xi \in \Gamma(\text{Rad}TM)$, $W \in \Gamma(S(TM^\perp))$ and $N, N' \in \Gamma(\text{ltr}(TM))$.

Let P be the projection morphism of TM on $S(TM)$. Then using (2.3), one can induce some new geometric objects on the screen distribution $S(TM)$ on M as

$$\nabla_X PY = \nabla_X^* PY + h^*(X, Y), \quad (2.16)$$

$$\nabla_X \xi = -A_\xi^* X + \nabla_X^{*f} \xi, \quad (2.17)$$

for any $X, Y \in \Gamma(TM)$ and $\xi \in \Gamma(\text{Rad}TM)$, where $\{\nabla_X^* PY, A_\xi^* X\}$ and $\{h^*(X, Y), \nabla_X^{*f} \xi\}$ belong to $\Gamma(S(TM))$ and $\Gamma(\text{Rad}TM)$, respectively. ∇^* and ∇^{*f} are linear connections on complementary distributions $S(TM)$ and $\text{Rad}TM$, respectively. h^* and A^* are $\Gamma(\text{Rad}TM)$ -valued and $\Gamma(S(TM))$ -valued bilinear forms and called as the second fundamental forms of distributions $S(TM)$ and $\text{Rad}TM$, respectively, and one has the following equations:

$$\bar{g}(h^l(X, PY), \xi) = g(A_\xi^* X, PY), \quad \bar{g}(h^*(X, PY), N) = g(A_N X, PY) \quad (2.18)$$

$$g(A_\xi^* PX, PY) = g(PX, A_\xi^* PY), \quad A_\xi^* \xi = 0. \quad (2.19)$$

Next, one recalls some basic definition and results of indefinite Sasakian manifolds [4]. An odd dimensional semi-Riemannian manifolds (\bar{M}, \bar{g}) is called an contact metric manifold, if there is a $(1, 1)$ tensor field ϕ , a vector field V called characteristic vector field, and a 1-form η such that

$$\bar{g}(\phi X, \phi Y) = \bar{g}(X, Y) - \epsilon \eta(X) \eta(Y), \quad \bar{g}(V, V) = \epsilon, \quad (2.20)$$

$$\phi^2(X) = -X + \eta(X)V, \quad \bar{g}(X, V) = \epsilon \eta(X), \quad (2.21)$$

$$d\eta(X, Y) = \bar{g}(X, \phi Y), \quad \forall X, Y \in \Gamma(TM), \quad (2.22)$$

where $\epsilon = \pm 1$. Therefore it follows that

$$\begin{aligned} \phi V &= 0, \\ \eta \circ \phi &= 0, \quad \eta(V) = 1. \end{aligned} \quad (2.23)$$

Then (ϕ, V, η, \bar{g}) is called contact metric structure of \bar{M} . one says that \bar{M} has a normal contact structure if $N_\phi + d\eta \otimes V = 0$, where N_ϕ is the Nijenhuis tensor field then \bar{M} is called an indefinite Sasakian manifold and for which one has

$$\bar{\nabla}_X V = -\phi X. \quad (2.24)$$

$$(\bar{\nabla}_X \phi)Y = -\bar{g}(X, Y)V + \epsilon\eta(Y)X. \quad (2.25)$$

3. Generalized Cauchy-Riemann- (GCR-) Lightlike Submanifold

Calin [5] proved that if the characteristic vector field V is tangent to $(M, g, S(TM))$ then it belongs to $S(TM)$. We assume characteristic vector V is tangent to M , throughout this paper.

Definition 3.1. Let $(M, g, S(TM))$ be a real lightlike submanifold of an indefinite Sasakian manifold (\bar{M}, \bar{g}) then M is called generalized Cauchy-Riemann (GCR)-lightlike submanifold if the following conditions are satisfied.

(A) There exist two subbundles D_1 and D_2 of $\text{Rad}(TM)$ such that

$$\text{Rad}(TM) = D_1 \oplus D_2, \quad \phi(D_1) = D_1, \quad \phi(D_2) \subset S(TM). \quad (3.1)$$

(B) There exist two subbundles D_0 and \bar{D} of $S(TM)$ such that

$$S(TM) = \{\phi D_2 \oplus \bar{D}\} \perp D_0 \perp V, \quad \phi(\bar{D}) = L \perp S, \quad (3.2)$$

where D_0 is invariant nondegenerate distribution on M , $\{V\}$ is one dimensional distribution spanned by V , and L, S are vector subbundles of $\text{ltr}(TM)$ and $S(TM)^\perp$, respectively.

Then tangent bundle TM of M is decomposed as

$$TM = \{D \oplus \bar{D} \oplus \{V\}\}, \quad D = \text{Rad}(TM) \oplus D_0 \oplus \phi(D_2). \quad (3.3)$$

Let Q, P_1, P_2 be the projection morphism on $D, \phi S, \phi L$, respectively, therefore

$$X = QX + V + P_1X + P_2X, \quad (3.4)$$

for $X \in \Gamma(TM)$. Applying ϕ to (3.4), we obtain

$$\phi X = fX + \omega P_1X + \omega P_2X, \quad (3.5)$$

where $fX \in \Gamma(D)$, $\omega P_1X \in \Gamma(S)$, and $\omega P_2X \in \Gamma(L)$ and we can write (3.5) as

$$\phi X = fX + \omega X, \quad (3.6)$$

where fX and ωX are the tangential and transversal components of ϕX , respectively. Similarly

$$\phi U = BU + CU, \quad U \in \Gamma(\text{tr}(TM)), \quad (3.7)$$

where BU and CU are the sections of TM and $\text{tr}(TM)$, respectively.

Differentiating (3.5) and using (2.11)–(2.13), (2.14), and (3.7), we have

$$\begin{aligned} D^l(X, \omega P_1 Y) &= -\nabla_X^l \omega P_2 Y + \omega P_2 \nabla_X Y - h^l(X, fY) + Ch^l(X, Y), \\ D^s(X, \omega P_2 Y) &= -\nabla_X^s \omega P_1 Y + \omega P_1 \nabla_X Y - h^s(X, fY) + Ch^s(X, Y), \end{aligned} \quad (3.8)$$

for all $X, Y \in \Gamma(TM)$. Using Sasakian property of $\bar{\nabla}$ with (2.13) and (2.14), we have the following lemmas.

Lemma 3.2. *Let M be a GCR-lightlike submanifold of an indefinite Sasakian manifold \bar{M} . Then one has*

$$(\nabla_X f)Y = A_{\omega Y}X + Bh(X, Y) - g(X, Y)V + \varepsilon\eta(Y)X, \quad (3.9)$$

$$(\nabla_X^t \omega)Y = Ch(X, Y) - h(X, fY), \quad (3.10)$$

where $X, Y \in \Gamma(TM)$ and

$$(\nabla_X f)Y = \nabla_X fY - f\nabla_X Y, \quad (3.11)$$

$$(\nabla_X^t \omega)Y = \nabla_X^t \omega Y - \omega \nabla_X Y. \quad (3.12)$$

Lemma 3.3. *Let M be a GCR-lightlike submanifold of an indefinite Sasakian manifold \bar{M} . Then one has*

$$(\nabla_X B)U = A_{CU}X - fA_U X, \quad (3.13)$$

$$(\nabla_X^t C)U = -\omega A_U X - h(X, BU),$$

where $X \in \Gamma(TM)$, $U \in \Gamma(\text{tr}(TM))$ and

$$(\nabla_X B)U = \nabla_X BU - B\nabla_X^t U, \quad (3.14)$$

$$(\nabla_X^t C)U = \nabla_X^t CU - C\nabla_X^t U.$$

4. Mixed Geodesic GCR-Lightlike Submanifolds

Definition 4.1. A GCR-lightlike submanifold of an indefinite Sasakian manifold is called mixed geodesic GCR-lightlike submanifold if its second fundamental form h satisfies $h(X, Y) = 0$ for any $X \in \Gamma(D)$ and $Y \in \Gamma(\bar{D})$.

Definition 4.2. A GCR-lightlike submanifold of an indefinite Sasakian manifold is called D' geodesic GCR-lightlike submanifold if its second fundamental form h satisfies $h(X, Y) = 0$ for any $X, Y \in \Gamma(\overline{D})$.

Theorem 4.3. Let M be a GCR-lightlike submanifold of an indefinite Sasakian manifold \overline{M} . Then M is mixed geodesic if and only if A_ξ^*X and $A_WX \notin \Gamma(M_2 \perp \phi D_2)$, for any $X \in \Gamma(D \oplus V)$, $W \in \Gamma(S(TM^\perp))$ and $\xi \in \Gamma(\text{Rad}(TM))$.

Proof. Using definition of GCR-lightlike submanifolds, M is mixed geodesic if and only if, $\overline{g}(h(X, Y), W) = \overline{g}(h(X, Y), \xi) = 0$ for $X \in \Gamma(D \oplus V)$, $Y \in \Gamma(\overline{D})$, $W \in \Gamma(S(TM^\perp))$, and $\xi \in \Gamma(\text{Rad}(TM))$. Using (2.12) and (2.17) we get

$$\begin{aligned}\overline{g}(h(X, Y), W) &= \overline{g}(\overline{\nabla}_X Y, W) = -\overline{g}(Y, \overline{\nabla}_X W) = g(Y, A_W X), \\ \overline{g}(h(X, Y), \xi) &= \overline{g}(\overline{\nabla}_X Y, \xi) = -g(Y, \nabla_X \xi) = g(Y, A_\xi^* X).\end{aligned}\tag{4.1}$$

Therefore from (4.1), the proof is complete. \square

Theorem 4.4. Let M be a GCR-lightlike submanifold of an indefinite Sasakian manifold \overline{M} . Then M is \overline{D} geodesic if and only if A_ξ^*X and $A_WX \notin \Gamma(M_2 \perp \phi D_2)$, for any $X \in \Gamma(\overline{D})$, $\xi \in \Gamma(\text{Rad}(TM))$ and $W \in \Gamma(S(TM^\perp))$.

Proof. Proof is similar to the proof of Theorem 4.3. \square

Lemma 4.5. Let M be a mixed geodesic GCR-lightlike submanifold of an indefinite Sasakian manifold \overline{M} . Then $A_\xi^*X \in \Gamma(\phi D_2)$, for any $X \in \Gamma(\overline{D})$, $\xi \in \Gamma(D_2)$.

Proof. For $X \in \Gamma(\overline{D})$ and $\xi \in \Gamma(D_2)$, using (2.9) we have

$$\begin{aligned}h(\phi\xi, X) &= \overline{\nabla}_X \phi\xi - \nabla_X \phi\xi \\ &= (\overline{\nabla}_X \phi)\xi + \phi\overline{\nabla}_X \xi - \nabla_X \phi\xi \\ &= -g(X, \xi)V + \epsilon\eta(\xi)X + \phi\overline{\nabla}_X \xi - \nabla_X \phi\xi \\ &= \phi\nabla_X \xi + \phi h(X, \xi) - \nabla_X \phi\xi.\end{aligned}\tag{4.2}$$

Since M is mixed geodesic therefore $\phi\nabla_X \xi = \nabla_X \phi\xi$. Using (2.16) and (2.17) we get $\phi(-A_\xi^*X + \nabla_X^* \xi) = \nabla_X^* \phi\xi + h^*(X, \phi\xi)$ then using (3.6) we obtain $-fA_\xi^*X - \omega A_\xi^*X + \phi(\nabla_X^* \xi) = \nabla_X^* \phi\xi + h^*(X, \phi\xi)$. Comparing the transversal components we get

$$\omega A_\xi^*X = 0,\tag{4.3}$$

or

$$A_\xi^*X \in \Gamma(D_0 \oplus \{V\} \perp \phi(D_2)).\tag{4.4}$$

If $A_{\xi}^*X \in D_0$ then the nondegeneracy of D_0 implies that there must exist a $Z_0 \in D_0$ such that

$$\bar{g}(A_{\xi}^*X, Z_0) \neq 0. \quad (4.5)$$

But from (2.9) and (2.17) we get

$$\begin{aligned} \bar{g}(A_{\xi}^*X, Z_0) &= -\bar{g}(\nabla_X \xi, Z_0) \\ &= -\bar{g}(\bar{\nabla}_X \xi, Z_0) \\ &= \bar{g}(\xi, \bar{\nabla}_X Z_0) \\ &= \bar{g}(\xi, \nabla_X Z_0 + h(X, Z_0)) \\ &= 0. \end{aligned} \quad (4.6)$$

Therefore

$$A_{\xi}^*X \notin \Gamma(D_0). \quad (4.7)$$

Also using (2.20), (2.21), and (2.24), we get

$$\begin{aligned} \bar{g}(A_{\xi}^*X, V) &= -\bar{g}(\nabla_X \xi, V) \\ &= \bar{g}(\xi, \bar{\nabla}_X V) \\ &= -\bar{g}(\xi, \phi V) \\ &= g(V, \phi \xi) \\ &= 0. \end{aligned} \quad (4.8)$$

Therefore

$$A_{\xi}^*X \notin \{V\}. \quad (4.9)$$

Hence from (4.4), (4.7), and (4.9) the result follows. \square

Corollary 4.6. *Let M be a mixed geodesic GCR-lightlike submanifold of an indefinite Sasakian manifold \bar{M} . Then $\bar{g}(h^l(X, Y), \xi) = 0$, for any $X \in \Gamma(\bar{D})$, $Y \in \Gamma(M_2)$ and $\xi \in \Gamma(D_2)$.*

Proof. From (2.18) and above lemma, the result follows. \square

Lemma 4.7. *Let M be a GCR-lightlike submanifold of an indefinite Sasakian manifold \bar{M} . Then $\bar{g}(A_W \phi X, Y) = \bar{g}(A_W Y, \phi X) - \bar{g}(\phi X, D^l(Y, W))$, for any $X \in \Gamma(D \oplus \{V\})$, $Y \in \Gamma(\bar{D})$ and $W \in \Gamma(S(TM^\perp))$.*

Proof. Using (2.12), we have

$$\begin{aligned}
\bar{g}(A_W\phi X, Y) &= -\bar{g}(\bar{\nabla}_{\phi X}W, Y) \\
&= \bar{g}(W, h(\phi X, Y)) \\
&= \bar{g}(\bar{\nabla}_Y\phi X, W) \\
&= -\bar{g}(\phi X, \bar{\nabla}_Y W) \\
&= \bar{g}(\phi X, A_W Y) - \bar{g}(\phi X, D^t(Y, W)),
\end{aligned} \tag{4.10}$$

for $X \in \Gamma(D \oplus \{V\})$, $Y \in \Gamma(\bar{D})$ and $W \in \Gamma(S(TM^\perp))$. \square

Theorem 4.8. *Let M be a mixed geodesic GCR-lightlike submanifold of an indefinite Sasakian manifold \bar{M} . Then $A_U X \in \Gamma(D \oplus \{V\})$ and $\nabla_X^t U \in \Gamma(L \perp S)$, for any $X \in \Gamma(D \oplus \{V\})$ and $U \in \Gamma(L \perp S)$.*

Proof. Since M is mixed geodesic, therefore $h(X, Y) = 0$ for any $X \in \Gamma(D \oplus \{V\})$, $Y \in \Gamma(\bar{D})$, therefore (2.7) gives

$$0 = \bar{\nabla}_X Y - \nabla_X Y. \tag{4.11}$$

Since \bar{D} is anti-invariant there exist $U \in \Gamma(L \perp S)$ such that $\phi U = Y$. Thus from (2.12), (3.6), and (3.7) we get

$$\begin{aligned}
0 &= \bar{\nabla}_X \phi U - \nabla_X Y \\
&= (\bar{\nabla}_X \phi)U + \phi \bar{\nabla}_X U - \nabla_X Y \\
&= -g(X, U)V + \epsilon \eta(U)X + \phi(-A_U X + \nabla_X^t U) - \nabla_X Y \\
&= -g(X, U)V - f A_U X - \omega A_U X + B \nabla_X^t U + C \nabla_X^t U - \nabla_X Y.
\end{aligned} \tag{4.12}$$

Comparing transversal components we get

$$\omega A_U X = C \nabla_X^t U, \tag{4.13}$$

since $\omega A_U X \in \Gamma(L \perp S)$ and $C \nabla_X^t U \in \Gamma(L \perp S)^\perp$, hence $A_U X \in \Gamma(D \oplus \{V\})$ and $\nabla_X^t U \in \Gamma(L \perp S)$. \square

5. GCR-Lightlike Product

Definition 5.1. A GCR-lightlike submanifold M of an indefinite Sasakian manifold \overline{M} is called GCR-lightlike product if both the distributions $D \oplus \{V\}$ and \overline{D} defines totally geodesic foliation in M .

Theorem 5.2. *Let M be a GCR-lightlike submanifold of an indefinite Sasakian manifold \overline{M} . Then the distribution $D \oplus \{V\}$ defines a totally geodesic foliation in M if and only if $Bh(X, \phi Y) = 0$, for any $X, Y \in D \oplus \{V\}$.*

Proof. Since $\overline{D} = \phi(L \perp S)$, therefore $D \oplus \{V\}$ defines a totally geodesic foliation in M if and only if

$$g(\nabla_X Y, \phi \xi) = g(\nabla_X Y, \phi W) = 0, \quad (5.1)$$

for any $X, Y \in \Gamma(D \oplus \{V\})$, $\xi \in \Gamma(D_2)$ and $W \in \Gamma(S)$. Using (2.9) and (2.25), we have

$$\begin{aligned} g(\nabla_X Y, \phi \xi) &= -\overline{g}(\overline{\nabla}_X \phi Y, \xi) = -\overline{g}(h^l(X, fY), \xi), \\ g(\nabla_X Y, \phi W) &= -\overline{g}(\overline{\nabla}_X \phi Y, W) = -\overline{g}(h^s(X, fY), W). \end{aligned} \quad (5.2)$$

Hence, from (5.2) the assertion follows. \square

Theorem 5.3. *Let M be a GCR-lightlike submanifold of an indefinite Sasakian manifold \overline{M} . Then the distribution \overline{D} defines a totally geodesic foliation in M if and only if $A_N X$ has no component in $\phi S \perp \phi D_2$ and $A_{\omega Y} X$ has no component in $D_2 \perp D_0$, for any $X, Y \in \Gamma(\overline{D})$ and $N \in \Gamma(\text{ltr}(TM))$.*

Proof. We know that \overline{D} defines a totally geodesic foliation in M if and only if

$$g(\nabla_X Y, N) = g(\nabla_X Y, \phi N_1) = g(\nabla_X Y, V) = g(\nabla_X Y, \phi Z) = 0, \quad (5.3)$$

for $X, Y \in \Gamma(\overline{D})$, $N \in \Gamma(\text{ltr}(TM))$, $Z \in \Gamma(D_0)$, and $N_1 \in \Gamma(L)$. Using (2.9) and (2.11) we have

$$g(\nabla_X Y, N) = \overline{g}(\overline{\nabla}_X Y, N) = -\overline{g}(Y, \overline{\nabla}_X N) = g(Y, A_N X). \quad (5.4)$$

Using (2.9), (2.10), and (2.25) we obtain

$$\begin{aligned} g(\nabla_X Y, \phi N_1) &= -g(\phi \overline{\nabla}_X Y, N_1) = -g(\overline{\nabla}_X \omega Y, N_1) = g(A_{\omega Y} X, N_1), \\ g(\nabla_X Y, \phi Z) &= -g(\phi \overline{\nabla}_X Y, Z) = -g(\overline{\nabla}_X \omega Y, Z) = g(A_{\omega Y} X, Z), \end{aligned} \quad (5.5)$$

also

$$g(\nabla_X Y, V) = g(\overline{\nabla}_X Y, V) = -g(Y, \overline{\nabla}_X V) = g(Y, \phi X) = 0. \quad (5.6)$$

Thus from (5.4)–(5.6), the result follows. \square

Theorem 5.4. *Let M be a GCR-lightlike submanifold of an indefinite Sasakian manifold \overline{M} . If $\nabla f = 0$ then M is a GCR lightlike product.*

Proof. Let $X, Y \in \Gamma(\overline{D})$ therefore $fY = 0$, then using (3.11) with the hypothesis, we get $f\nabla_X Y = 0$ therefore the distribution \overline{D} defines a totally geodesic foliation. Let $X, Y \in D \oplus \{V\}$ therefore $\omega Y = 0$ then using (3.9), we get

$$Bh(X, Y) + g(X, Y)V + \epsilon\eta(Y)X = 0. \quad (5.7)$$

Equating components along the distribution \overline{D} of above equation, we get $Bh(X, Y) = 0$, therefore $D \oplus \{V\}$ define a totally geodesic foliation in M . Hence M is a GCR lightlike product. \square

Definition 5.5. A lightlike submanifold M of a semi-Riemannian manifold is said to be an irrotational submanifold if $\overline{\nabla}_X \xi \in \Gamma(TM)$ for any $X \in \Gamma(TM)$ and $\xi \in \Gamma\text{Rad}(TM)$. Thus M is an irrotational lightlike submanifold if and only if $h^l(X, \xi) = 0, h^s(X, \xi) = 0$.

Theorem 5.6. *Let M be an irrotational GCR-lightlike submanifold of indefinite Sasakian manifold \overline{M} . Then M is a GCR lightlike product if the following conditions are satisfied.*

$$(A) \overline{\nabla}_X U \in \Gamma(S(TM^\perp)), \forall X \in \Gamma(TM), \text{ and } U \in \Gamma(\text{tr}(TM)).$$

$$(B) A_\xi^* Y \in \Gamma(\phi(S)), \forall Y \in \Gamma(D).$$

Proof. Let (A) holds, then using (2.11) and (2.12), we get $A_N X = 0, A_W X = 0, D^l(X, W) = 0$, and $\nabla_X^l N = 0$ for $X \in \Gamma(TM)$. These equations imply that the distribution \overline{D} defines a totally geodesic foliation in M and with (2.13), we get $\overline{g}(h^s(X, Y), W) = 0$. Hence non degeneracy of $S(TM^\perp)$ implies that $h^s(X, Y) = 0$. Therefore $h^s(X, Y)$ has no component in S . Finally, from (2.14) and M being irrotational, we have $\overline{g}(h^l(X, Y), \xi) = \overline{g}(Y, A_\xi^* X)$, for $X \in \Gamma(TM)$ and $Y \in \Gamma(D)$. Assume (B) holds, then $h^l(X, Y) = 0$. Therefore $h^l(X, Y)$ has no component in L . Thus the distribution $D \oplus \{V\}$ defines a totally geodesic foliation in M . Hence M is a GCR lightlike product. \square

Definition 5.7 (see [6]). If the second fundamental form h of a submanifold tangent to characteristic vector field V , of an indefinite Sasakian manifold \overline{M} is of the form

$$h(X, Y) = \{g(X, Y) - \eta(X)\eta(Y)\}\alpha + \eta(X)h(Y, V) + \eta(Y)h(X, V), \quad (5.8)$$

for any $X, Y \in \Gamma(TM)$, where α is a vector field transversal to M , then M is called a totally contact umbilical submanifold of an indefinite Sasakian manifold.

Theorem 5.8. *Let M be a totally contact umbilical GCR-lightlike submanifold of an indefinite Sasakian manifold \overline{M} . Then M is a GCR-lightlike product if $Bh(X, Y) = 0$, for any $X \in \Gamma(TM)$ and $Y \in \Gamma(D \oplus \{V\})$.*

Proof. Let $X, Y \in \Gamma(D \oplus \{V\})$, then the hypothesis $Bh(X, Y) = 0$, implies that the distribution $D \oplus \{V\}$ defines totally geodesic foliation in M . Let $X, Y \in \Gamma(\overline{D})$ then using (3.9), we have

$$-f\nabla_X Y = A_{\omega Y} X + Bh(X, Y) - g(X, Y)V, \quad (5.9)$$

let $Z \in \Gamma(D_0)$ and using (2.8) and (2.25), then above equation becomes

$$\begin{aligned} -g(f\nabla_X Y, Z) &= g(A_{\omega Y} X + Bh(X, Y) - g(X, Y)V, Z) \\ &= g(\overline{\nabla}_X \phi Y, Z) \\ &= -g(\overline{\nabla}_X Y, \phi Z) \\ &= g(Y, \nabla_X Z'), \end{aligned} \quad (5.10)$$

where $\phi Z = Z' \in \Gamma(D_0)$. For $X \in \Gamma(\overline{D})$ from (3.10), we get

$$\omega P\nabla_X Z = h(X, fZ) - Ch(X, Z). \quad (5.11)$$

Using the hypothesis with (5.8), we get $\omega P\nabla_X Z = 0$, this implies $\nabla_X Z \in \Gamma(D)$ therefore, (5.10) becomes $g(f\nabla_X Y, Z) = 0$. Then non degeneracy of the distribution D_0 implies that the distribution \overline{D} defines a totally geodesic foliation in M . Hence the assertion follows. \square

Theorem 5.9. *Let M be a totally geodesic GCR-lightlike submanifold of an indefinite Sasakian manifold \overline{M} . Suppose there exists a transversal vector bundle of M which is parallel along \overline{D} with respect to Levi-Civita connection on M , that is, $\overline{\nabla}_X U \in \Gamma(\text{tr}(TM))$, for any $U \in \Gamma(\text{tr}(TM))$, $X \in \Gamma(\overline{D})$. Then M is a GCR-lightlike product.*

Proof. Since M is a totally geodesic GCR-lightlike, therefore $Bh(X, Y) = 0$, for $X, Y \in \Gamma(D \oplus \{V\})$, this implies $D \oplus \{V\}$ defines a totally geodesic foliation in M . Next, $\overline{\nabla}_X U \in \Gamma(\text{tr}(TM))$ implies $A_U X = 0$. Hence by Theorem 5.3, the distribution \overline{D} defines a totally geodesic foliation in M . Thus M is a contact GCR-lightlike product. \square

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