On the Spread of Viruses on the Internet

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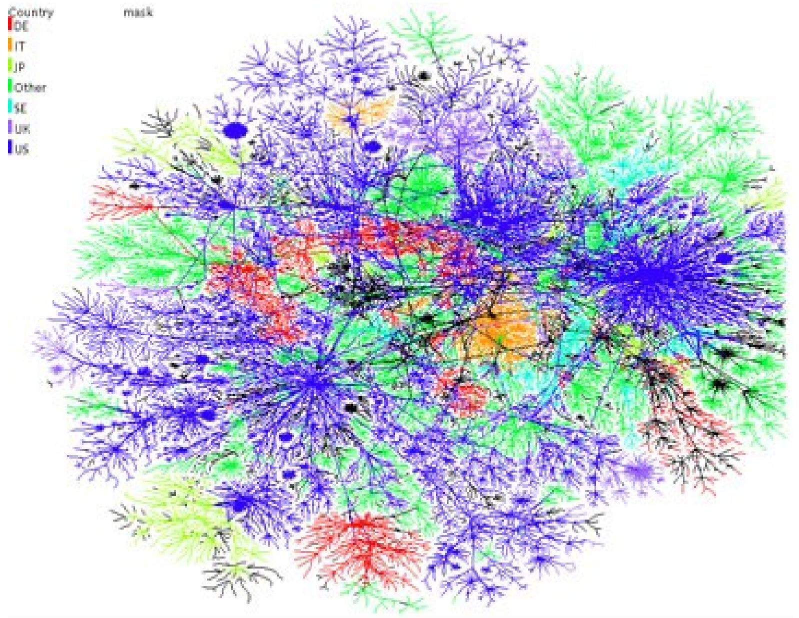
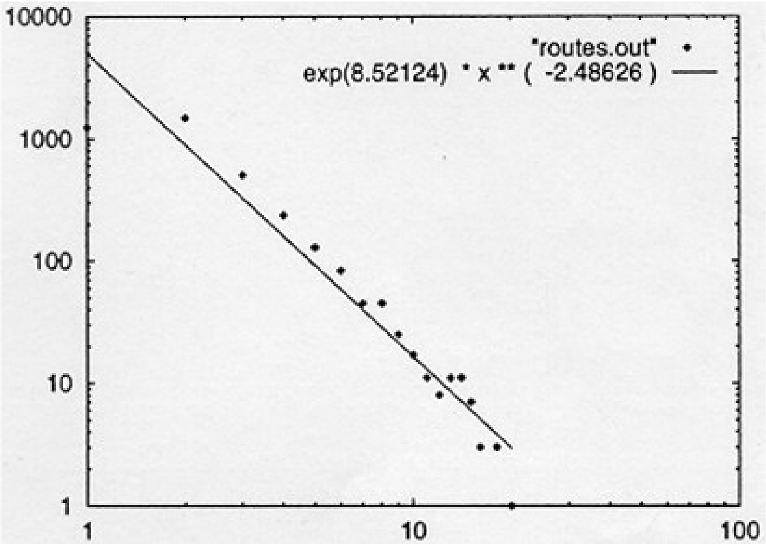
Joint work with C. Borgs, J.T. Chayes and A. Saberi

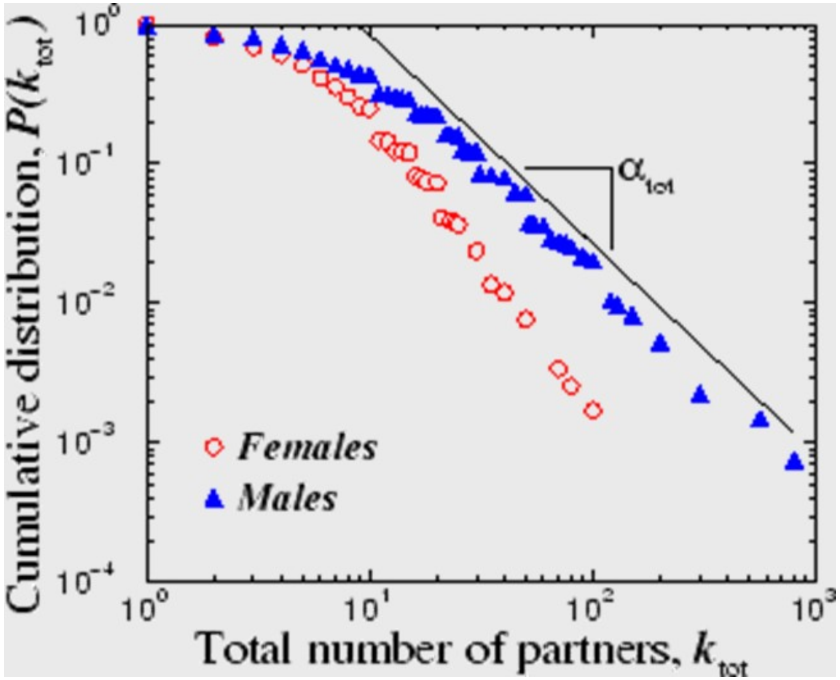
The Internet Graph





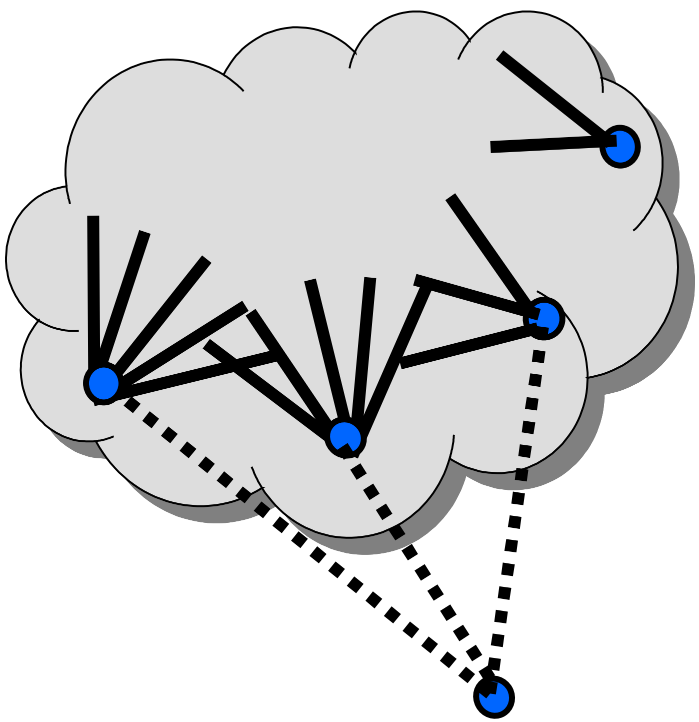
Faloutsos, Faloutsos and Faloutsos ‘99

The Sex Web



Lilijeros et. al '01

Model for Power-Law Graphs: Preferential Attachment



- Add one vertex at a time
- New vertex i attaches to m ¸ 1 existing vertices j chosen as follows: With probability α, choose j uniformly, and with probability 1-α, choose j according to

$$\text{Prob(i attaches to j)} / d_j$$

with d_j = degree(j)

non-rigorous: Simon ‘55, Barabasi-Albert ‘99,
measurements: Kumar et. al. ‘00,
rigorous: Bollobas-Riordan ‘00, Bollobas et. al. ‘03

Model for Spread of Viruses: Contact Process

- Definition of model:

infected ! healthy at rate 1

healthy ! infected at rate λ (# infected neighbors)

- Studied in probability theory, physics, epidemiology
- Kephart and White '93: modelling the spread of viruses in a computer network

Epidemic Threshold(s)

λ_1 λ_2

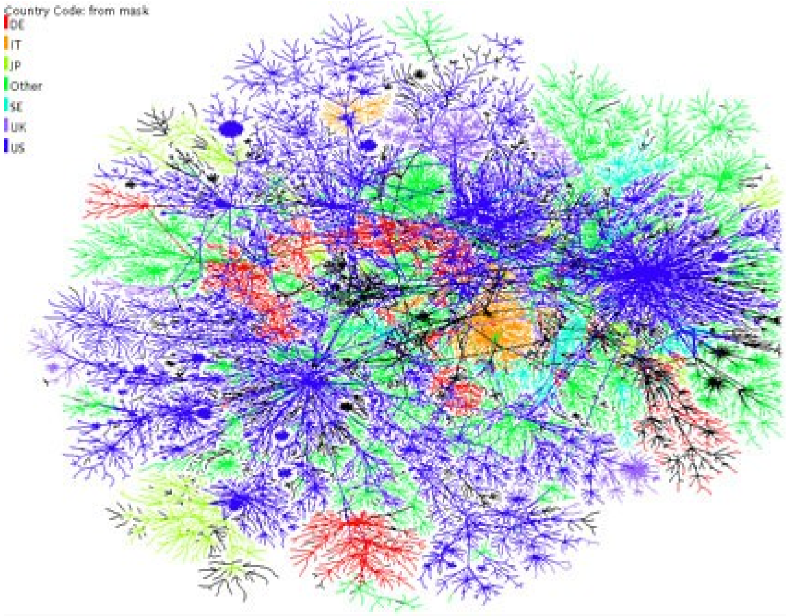
- Infinite graph: extinction | weak survival | strong survival

  Note: $\lambda_1 = \lambda_2$ on Z^d

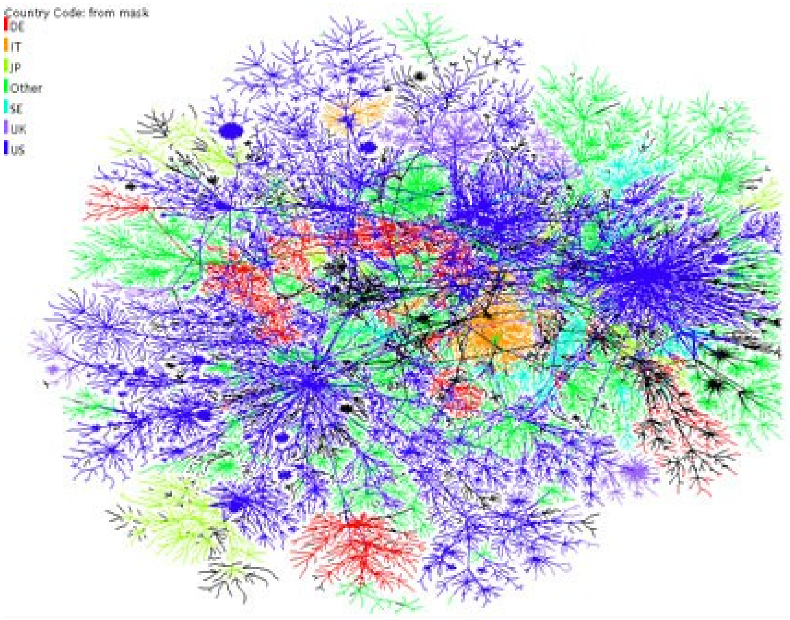
  $\lambda_1 < \lambda_2$ on a tree

- Finite subset of Z^d: logarthmic survival time | exponential (super-poly) survival time

The Internet Graph



The Internet Graph



What is the epidemic threshold of the Internet graph?

Epidemic Threshold in Scale-Free Network

In preferential attachment networks both thresholds are zero asymptotically almost surely, i.e.

$$\lambda_1 = \lambda_2 = 0 \quad \text{a.a.s.}$$

- Physics argument: Pastarros, Vespignani '01
- Rigorous proof: B., Borgs, Chayes, Saberi '04

Moreover, we get detailed estimates (matching upper and lower bounds) on the survival probability as a function of λ

Theorem 1. For every $\lambda > 0$, and for all n large enough, if the infection starts from a uniformly random vertex in a sample of the scale-free graph of size n, then with probability $1\text{-}O(\lambda^2)$, v is such that the infection survives longer than $e^{n^{0.1}}$ with probability at least

$$\lambda^{C_1 \frac{\log(1/\lambda)}{\log\log(1/\lambda)}}$$

and with probability at most

$$\lambda^{C_2 \frac{\log(1/\lambda)}{\log\log(1/\lambda)}}$$

where $0 < C_1 < C_2 < 1$ are independent of λ and n.

Typical versus average behavior

- Notice that we left out $O(\lambda^2 n)$ vertices in Theorem 1.
- **Question:** What is the effect of these vertices on the **average survival probability**?
- **Answer:** Dramatic.

Theorem 2. For every $\lambda > 0$, and for all n large enough, if the infection starts from a uniformly random vertex in a sample of the scale-free graph of size n, then the infection survives longer than $e^{n^{0.1}}$ with probability at least

$$\lambda^{C_3}$$

and with probability at most

$$\lambda^{C_4}$$

where $0 < C_3, C_4 < 1$ are independent of λ and n.

Typical versus average behavior

- The survival probability for an infection starting from a **typical** (i.e., $1 - O(\lambda^2)$) vertex is

$$\lambda^{\Theta\left(\frac{\log(1/\lambda)}{\log\log(1/\lambda)}\right)}$$

- The **average** survival probability is

$$\lambda^{\Theta(1)}$$

Key Elements of the Proof

1. Properties of the contact process
2. Properties of preferential attachment graphs

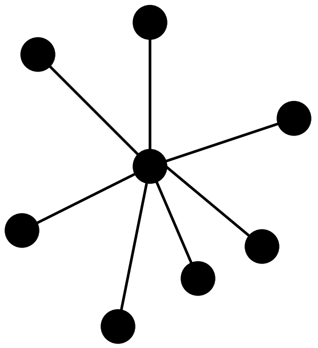
Key Elements of the Proof

1. Properties of the contact process

- If the maximum degree is much less than $1/\lambda$, then the infection dies out very quickly.
- On a vertex of degree much more than $1/\lambda^2$, the infection lives for a long time in the neighborhood of the vertex ("star lemma")

Star Lemma

If we start by infecting the center of a star of degree k , with high probability, the survival time is more than



$$\exp(Ck\lambda^2).$$

Key Idea: The center infects a constant fraction of vertices before becoming disinfected.

Consequence of star lemma

- If the virus is "lucky enough" to start at a vertex of degree higher than λ^{-2}, the process has a good chance of lasting for a long time. Since there are $\lambda^{\Theta(1)}$ such vertices, the average survival probability is $\lambda^{\Theta(1)}$.
- If the virus not as lucky, then if it can at least reach a vertex of degree $\lambda^{-\Theta(1)}$, it will survive for a long time in the neighborhood of that vertex.

Key Elements of the Proof:

2. Properties of preferential attachment graphs

Expanding Neighborhood Lemma (after Bollobas and Riordan):

With high probability, the largest degree in a ball of radius k about a vertex v is at most

$$(k!)^{10}$$

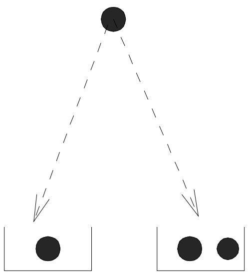
and at least

$$(k!)^{\gamma(m,\alpha)}$$

where $\gamma(m,\alpha) > 0$.

To prove this, we introduced a Polya Urn Representation of the preferential attachment graph.

Polya Urn Representation of Graph

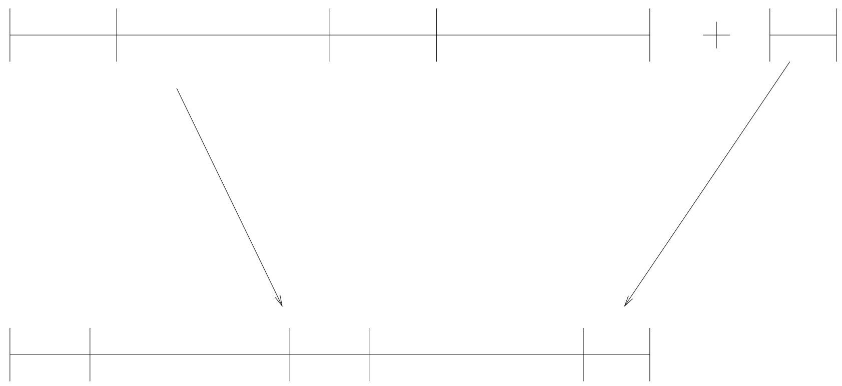


- Polya's Urn: At each time step, add a ball to one of the urns with probability proportional to the number of balls already in that urn.

- Polya's Theorem: This is equivalent to choosing a number p according to the β-distribution, and then sending the balls i.i.d. with probability p to the left urn and with probability 1– p to the right urn.

Polya Urn Representation of Graph

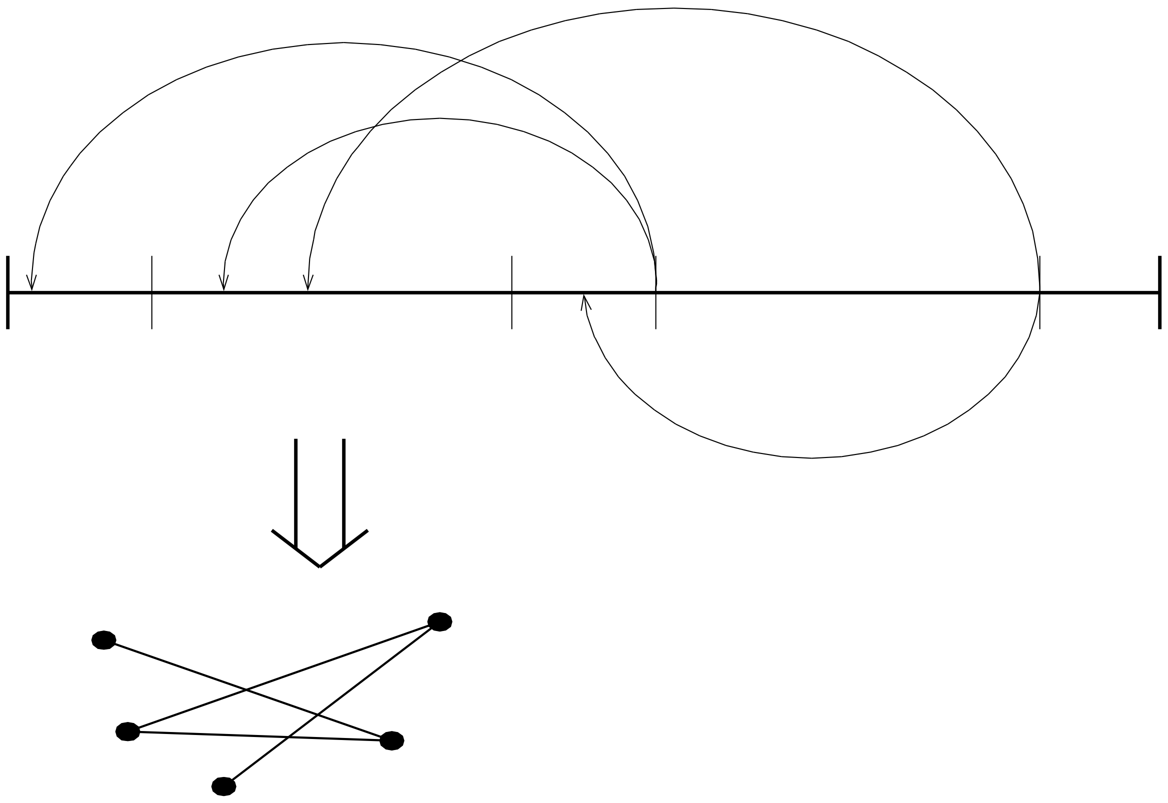
- For each new vertex we choose it strength according to the appropriate Beta distribution.
- Then we rescale all strengths so that they sum to 1.



Polya Urn Representation of Graph

- Then from every interval end we sample m i.i.d. uniform points left to it. We say that there is an edge between u and v if a variable from v fell in u.

Polya Urn Representation of Graph



When does it work

This works when the Barabasi-Albert graph
Is realized in an exchangeable way.

Exchangeable examples

- The m vertices are chosen one by one, where the distribution of the i-th is influenced by the previous ones.
- The m vertices are chosen simultaneously, only depending on the past, conditioned on being different from each other.

Non-exchangeable example

- The m vertices are chosen simultaneously and independently.

Open Problem:

Find a way to prove our estimates in the non-exchangeable regime.
(Or to prove they don't hold)

Consequence of the expanding neighborhood lemma

- The largest degree of a vertex within a ball of radius k around a typical vertex is $(k!)^{\Theta(1)}$

) the closest vertex of degree $\lambda^{-\Theta(1)}$ is at a distance Θ $(\log \lambda^{-1} / \log \log \lambda^{-1})$

) must take $k = \Theta$ $(\log \lambda^{-1} / \log \log \lambda^{-1})$ in the proof

"Proof" of Main Theorem:

Let $$k = \frac{\log(1/\lambda)}{\log\log(1/\lambda)}$$

By the preferential attachment lemma, the ball of radius C_1k around vertex v contains a vertex w of degree larger than

$$[(C_1k)!]^{\gamma} > \lambda^{-5}$$

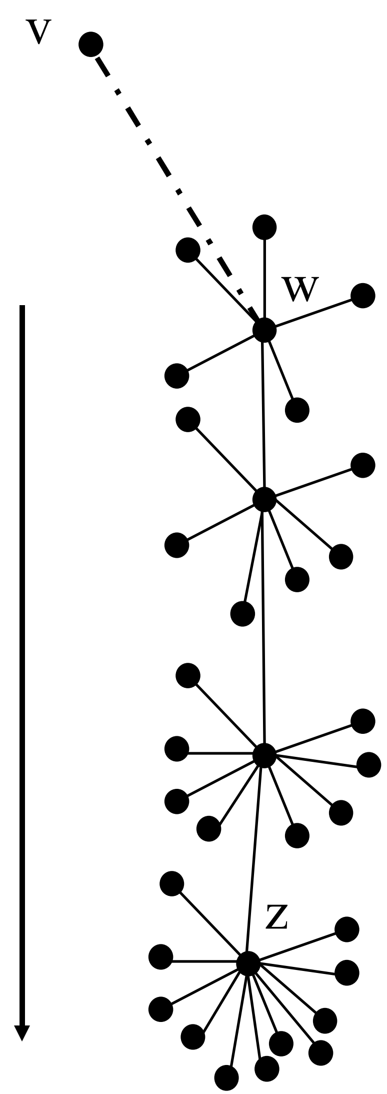
where the inequality follows by taking C_1 large.

The infection must travel at most C_1k to reach w, which happens with probability at least

$$\lambda^{C_1k},$$

at which point, by the star lemma, the survival time is more than $\exp(C\lambda^{-3})$.

Iterate until we reach a vertex z of



logn iterations to get
to high-degree vertex

Summary

- Developed a new representation of the preferential attachment model: Polya Urn Representation.

- Used the representation to:

  1. prove that any virus with a positive rate of spread has a positive probability of becoming epidemic

  2. calculate the survival probability for both typical and average vertices

Open Problems

- Show that there is exponential (rather than just super-polynomial) survival time
- Analyze other models for the spread of viruses (either with permanent immunity to one virus or with several distinct infected states)
- Design efficient algorithms for selective immunization

THE END

- Use this and some work to show that the addition of a new vertex can be represented by adding a new urn to the existing sequence of urns and adding edges between the new urn and m of the old ones.