Approximate Solutions to a Class of Nonlinear Stackelberg Differential Games

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Abstract—A two-player nonlinear Stackelberg differential game with player 1 and player 2 as leader and follower, respectively, is considered. The feedback Stackelberg solutions to such games rely on the solution of two coupled partial differential equations (PDEs) for which closed-form solutions cannot in general be found. A method for constructing strategies satisfying partial differential inequalities in place of the PDEs is presented. It is shown that these constitute approximate solutions to the differential game. The theory is illustrated by a numerical example.

I. INTRODUCTION

Game theory is the study of multi-player decision making [1]–[6]. Many problems involve decisions of some kind: these could be decisions between competing players or determining the best trade-offs for players with different goals. Thus, differential games have a large variety of applications in a wide range of areas. Typical areas are economics and management, defense, and robust control [2], [7]–[9]. Additionally, differential games play a role in many other areas, including problems involving multi-agent systems and biological systems [10], [11].

One of the most common solution concepts associated with game theoretic problems is that of Nash equilibrium solutions whereby it is assumed that all players are rational and announce their strategies simultaneously [1], [12], [13]. A different solution concept was introduced by Stackelberg in 1934 [14]. In Stackelberg differential games some decision makers are able to act prior to the other players which then react in a rational manner, i.e. the players act in a specific order [1]. Thus, unlike Nash equilibria, Stackelberg equilibria introduce a hierarchy between the players. Some simple examples of the different types of games are the game of “rock-paper-scissors”, where all players act simultaneously (Nash), and “tic-tac-toe” or chess, where the players act in a specific order (Stackelberg). In some cases the order in which the players act is irrelevant: the Nash and Stackelberg equilibria are equivalent. This is, however, not generally the case.

Different solution concepts for Stackelberg differential games exist, namely open-loop, closed-loop and feedback Stackelberg solutions [15]. Unlike closed-loop Stackelberg solutions, feedback Stackelberg solutions can be found via dynamic programming, and these are studied in this paper. Determining either Nash or Stackelberg feedback equilibrium solutions for a given differential game relies on the solution of coupled partial differential equations (PDEs), of the Hamilton-Jacobi-Isaacs type. Closed-form solutions of these cannot be found in general and it is often necessary to settle for approximate solutions. In [16]–[18] a method for obtaining solutions approximating the Nash equilibrium solution of a class of nonzero-sum differential games has been introduced. In what follows, the developed method is extended to the case in which feedback Stackelberg solutions are sought. In particular a two-player nonzero-sum differential game is considered.

The remainder of the paper is organised as follows. In Section II a formal definition of the problem and of its solution is given. A method for obtaining approximate solutions is presented in Section III, in which also the notion of the approximation is made precise. Simulations illustrating the method are then given in Section IV. Finally, directions for future work are given along with concluding remarks in Section V.

II. THE TWO-PLAYER STACKELBERG DIFFERENTIAL GAME

Consider the input-affine dynamical system

\[ \dot{x} = f(x) + g_1(x)u_1 + g_2(x)u_2, \]  

where \( x(t) \in \mathbb{R}^n \), with \( n > 0 \), is the state of the system, \( f(x) : \mathbb{R}^n \to \mathbb{R}^n \), \( g_1(x) \in \mathbb{R}^{n \times m} \) and \( g_2(x) \in \mathbb{R}^{n \times m} \), with \( m > 0 \), are smooth mappings and \( u_1(t) \in \mathbb{R}^m \) and \( u_2(t) \in \mathbb{R}^m \) are the control signals of player 1 and 2, respectively.

Assumption 1: The origin of \( \mathbb{R}^n \) is an equilibrium point of the vector field \( f(x) \), i.e. \( f(0) = 0 \).

A consequence of Assumption 1 is that we can write

\[ f(x) = F(x)x \]

for some (not unique) matrix-valued function \( F(x) : \mathbb{R}^n \to \mathbb{R}^{n \times n} \).

Suppose the game is such that the players announce their strategies in a predefined order. In particular suppose that player 1 acts before player 2: in this case player 1 is referred to as the leader, whereas player 2 is

1 All mappings and functions are assumed to be sufficiently smooth.
referred to as the follower. Similarly to [19], the leader seeks to minimise a cost functional of the form

\begin{equation}
J_1(x(0), u_1, u_2) \triangleq \frac{1}{2} \int_0^\infty \left( q_1(x(t)) + 2\theta u_1(t)^T u_2(t) + \|u_1(t)\|^2 \right) dt ,
\end{equation}

whereas the follower seeks to minimise a cost functional of the form

\begin{equation}
J_2(x(0), u_1, u_2) \triangleq \frac{1}{2} \int_0^\infty \left( q_2(x(t)) + 2\theta u_2(t)^T u_1(t) + \|u_2(t)\|^2 \right) dt ,
\end{equation}

where \( \theta \in \mathbb{R} \), and the running costs are such that \( q_1(x) = x^T Q_1(x)x \geq 0, q_2(x) = x^T Q_2(x)x \geq 0 \), with \( Q_1 \in \mathbb{R}^{n \times n} \) and \( Q_2 \in \mathbb{R}^{n \times n} \), and \( q_1(x) + q_2(x) > 0 \) for all \( x \neq 0 \). The cost functionals (2) and (3) are such that the agents seek to minimise their own running costs and control efforts. In addition, if \( \theta > 0 \), the players seek to minimise the cross-product between their control strategies (or maximise it if \( \theta < 0 \)).

The cost functionals remain bounded if \( J_1 + \lambda J_2 \geq 0 \) for some \( \lambda \in \mathbb{R}^+ \). This is guaranteed provided \( \theta \) and \( \lambda \) are such that \( \lambda - \theta^2 (1 + \lambda)^2 \geq 0 \). The range of allowable \( \theta \) is maximised by the selection \( \lambda = 1 \) and the resulting bound on \( \theta \) is given in the following statement, which is assumed to hold in the remainder of the paper.

Assumption 2: The parameter \( \theta \) is such that \( |\theta| < \frac{1}{2} \).

Remark 1: The range of \( \theta \) in Assumption 2 is more restrictive than that in [19], where \( |\theta| < \frac{1}{\sqrt{2}} \).

Remark 2: For the cost functionals to remain bounded it is necessary that \( J_1 + \lambda J_2 \geq 0 \) for some \( \lambda \in \mathbb{R} \). This is guaranteed provided \( \theta \) and \( \lambda \) are such that \( \lambda - \theta^2 (1 + \lambda)^2 \geq 0 \). The range of allowable \( \theta \) is maximised by the selection \( \lambda = 1 \) and the resulting bound is that given in Assumption 2.

In what follows feedback Stackelberg equilibria for the differential game are considered.

Definition 1: A pair of feedback strategies \((u_1^*, u_2^*)\) is said to be admissible if it renders the zero equilibrium of the closed-loop system (locally) asymptotically stable.

Problem 1: Consider the system (1) and the cost functionals of the leader and the follower, namely (2) and (3), respectively. The 2-player Stackelberg differential game consists in determining a pair of admissible feedback strategies, \((u_1^*, u_2^*)\) satisfying

\begin{equation}
J_1(x(0), u_1^*, u_2^*(u_1^*)) \leq J_1(x(0), u_1, u_2^*(u_1)) ,
\end{equation}

\begin{equation}
J_2(x(0), u_1^*, u_2^*(u_1^*)) \leq J_2(x(0), u_1^*(t), u_2^*(u_1^*)) ,
\end{equation}

for all \( u_1 \neq u_1^* \) such that \((u_1, u_2^*(u_1))\) is admissible, and \( u_2(u_1^*) \neq u_2^*(u_1^*) \), where \( u_1^* \) is any strategy such that \((u_1^*, u_2^*)\) and \((u_1^*, u_2^*)\) are admissible pairs.

The pair of strategies \((u_1^*, u_2^*)\) is said to be a Stackelberg equilibrium solution of the two-player differential game.

Remark 3: Since there is an ordering between the two players, the Stackelberg strategy of the follower, i.e. \( u_2(u_1^*) \), depends on the strategy of the leader, i.e. \( u_1^* \). This is not the case when solving for Nash equilibria, where both players act simultaneously [16]-[18].

The Hamiltonians associated with players 1 and 2 are

\begin{equation}
H_1(x, u_1, u_2, \lambda_1) = \frac{1}{2} \left( q_1(x) + 2\theta u_1^T u_2 + \|u_1\|^2 \right) + \lambda_1^T (f(x) + g_1(x) u_1 + g_2(x) u_2) ,
\end{equation}

\begin{equation}
H_2(x, u_1, u_2, \lambda_2) = \frac{1}{2} \left( q_2(x) + 2\theta u_2^T u_1 + \|u_2\|^2 \right) + \lambda_2^T (f(x) + g_1(x) u_1 + g_2(x) u_2) ,
\end{equation}

where \( \lambda_1 \) and \( \lambda_2 \) are the co-states. Feedback Stackelberg equilibria are such that the Hamiltonians are minimised [19]. Knowing the action of the leader, the follower’s response is

\begin{equation}
u_2^*(u_1) = \arg \min_{u_2} H_2(x, u_1, u_2, \lambda_2) \geq -g_2^T \lambda_2 - \theta u_1 .
\end{equation}

Anticipating this behaviour, the leader should select its strategy according to

\begin{equation}u_1^* = \arg \min_{u_1} H_1(x, u_1, u_2^*(u_1^*)) = -\frac{1}{2} \left( g_1(x) - \theta g_2(x) \right)^T \lambda_1 - \theta g_2(x)^T \lambda_2 .
\end{equation}

Let \( \Delta_{\theta}(x) = g_1(x) - \theta g_2(x) \) and consider the PDEs

\begin{equation}
\frac{\partial V_1}{\partial x} f(x) + \frac{1}{2} q_1(x) - \frac{\partial V_1}{\partial x} g_2(x) g_2(x)^T \frac{\partial V_2}{\partial x} \left\| \frac{\partial V_1}{\partial x} \Delta_{\theta}(x) - \theta \frac{\partial V_2}{\partial x} g_2(x) \right\|^2 = 0 ,
\end{equation}

\begin{equation}\frac{\partial V_2}{\partial x} f(x) + \frac{1}{2} q_2(x) + \frac{1}{1 - 2\theta^2} \theta \frac{\partial V_2}{\partial x} g_2(x) g_2(x)^T \frac{\partial V_1}{\partial x} \left\| \frac{\partial V_2}{\partial x} - \theta \frac{\partial V_1}{\partial x} \Delta_{\theta}(x) \right\|^2 = 0 ,
\end{equation}

with solutions \( V_i \) such that \( V_i(0) = 0 \) for \( i = 1, 2 \), and such that \( V_1(x) + V_2(x) > 0 \) for all \( x \neq 0 \). Then, the Stackelberg equilibrium solution of the game defined by the dynamics (1) and the cost functionals (2) and (3), with player 1 as leader and player 2 as follower, is given by

\begin{equation}u_1^* = -\frac{1}{1 - 2\theta^2} \left( \Delta_{\theta}(x)^T \frac{\partial V_1}{\partial x} - \theta g_2(x)^T \frac{\partial V_2}{\partial x} \right) ,
\end{equation}

\begin{equation}u_2^*(u_1^*) = -g_2(x)^T \frac{\partial V_2}{\partial x} - \theta u_1^* .
\end{equation}
Note that the conditions \( W = V_1(x) + V_2(x) > 0 \), for all \( x \neq 0 \), and \( \dot{W} < 0 \), for all \( x \neq 0 \), are implied by the PDEs (7). Thus, it follows from standard Lyapunov arguments that the feedback strategies (8) are admissible.

Consider now the linear-quadratic approximation of the problem. In a neighbourhood of the origin the system (1) can be approximated by a linear system, namely \( \dot{x} = Ax + Bu_1 + Bu_2 \), with

\[
A = \left. \frac{\partial f(x)}{\partial x} \right|_{x=0} = F(0), \quad B_1 = g_1(0), \quad B_2 = g_2(0).
\]

Furthermore, the running costs \( q_i(x) \) can be approximated as quadratic costs, namely \( q^{\text{lin}}_i(x) = x^\top Q_i x \), where \( Q_i = Q_i(0) \) for \( i = 1, 2 \). For the resulting linear-quadratic differential game, the PDEs (7) reduce to the coupled Algebraic Riccati Equations (AREs)

\[
P_1A + A^\top P_1 + \bar{Q}_1 - P_1B_2B_2^\top P_2 - P_2B_2B_2^\top P_1 - \frac{1}{1-2\theta^2}(P_1\Delta_B - \theta P_2B_2)(\Delta_B^\top P_1 - \theta B_2^\top P_2) = 0,
\]

\[
P_2A + A^\top P_2 + Q_2 + \frac{1}{1-2\theta^2}(2\theta P_2B_2B_2^\top P_2 - P_1\Delta_B B_1^\top P_2 - P_2B_1\Delta_B^\top P_1) - \frac{1}{1-2\theta^2}(1 - \theta^2) P_2B_2 - \theta P_1\Delta_B)((1 - \theta^2) B_2^\top P_2 - \theta \Delta_B^\top P_1) = 0,
\]

where \( \Delta_B = (B_1 - \theta B_2) \). Suppose \( P_1 = P_1^\top \) and \( P_2 = P_2^\top \), such that \( P_1 + P_2 > 0 \), solve (9). Then, the Stackelberg equilibrium solution is

\[
u_1^{\text{eq}}(x) = \frac{-1}{1 - 2\theta^2} \left( \Delta_B^\top P_1 - \theta B_2^\top P_2 \right) x,
\]

\[
u_2^{\text{eq}}(x) = -B_2^\top \tilde{P}_2 x - \theta \nu_1^{\text{eq}}.
\]

Note that the conditions \( W = \frac{1}{2} x^\top (P_1 + P_2) x > 0 \), for all \( x \neq 0 \), and \( \dot{W} < 0 \), for all \( x \neq 0 \), are implied by (9). It follows that the pair \((u_1^{\text{eq}}, u_2^{\text{eq}})\) is admissible.

Remark 4: In [19] a stochastic linear-quadratic differential game with \( B_1 = B_2 = I \) has been considered. The above results are consistent with [19].

Remark 5: When \( \theta = 0 \), the feedback Stackelberg equilibrium strategies and the feedback Nash equilibrium strategies coincide, i.e. in this case the order in which the players act is irrelevant to the corresponding outcomes [19], [20].

To conclude this section define the notions of \( \alpha \)-admissible strategies and \( \epsilon_\alpha \)-Stackelberg equilibrium solutions, similarly to what has been done in [18].

**Definition 2:** The pair of strategies \((u_1, u_2(u_1))\) is said to be \( \alpha \)-admissible for the non-cooperative Stackelberg differential game if the zero equilibrium of the system (1) in closed-loop with \((u_1, u_2(u_1))\) is (locally) asymptotically stable and\(^3\) \( \sigma(A_{\text{cl}} + \alpha I) \subseteq \mathbb{C}^- \), where \( A_{\text{cl}} \) is the matrix describing the linearisation of the closed-loop system around the origin.

**Definition 3:** A pair of admissible strategies \((u_1^*, u_2^*(u_1^*))\) is said to be an \( \epsilon_\alpha \)-Stackelberg equilibrium of the Stackelberg differential game with dynamics (1) and cost functionals (2) and (3), with player 1 as leader and player 2 as follower, if there exists a non-negative constant \( \epsilon_{x_0,\alpha} \), parametrised with respect to \( x(0) = x_0 \), and \( \alpha > 0 \) such that

\[
J_1(x_0, u_1^*, u_2^*(u_1^*)) \leq J_1(x_0, u_1, u_2^*(u_1)) + \epsilon_{x_0,\alpha},
\]

\[
J_2(x_0, u_1^*, u_2^*(u_1^*)) \leq J_2(x_0, u_1^*, u_2^*(u_1)) + \epsilon_{x_0,\alpha},
\]

for all \( \alpha \)-admissible strategies of the form \((u_1, u_2^*(u_1))\) and \((u_1^*, u_2(u_1^*))\), with \( u_1 \neq u_1^*, i = 1, 2 \).

### III. Constructing Approximate Solutions to the Stackelberg Differential Games

In general it is not possible to obtain closed-form solutions to the coupled PDEs (7) making it necessary to settle for approximate solutions. In this section a method for constructing a solution to a modified differential game that approximates the solution of the Stackelberg game defined by (1), (2) and (3) is introduced. The approximate solution relies on the notion of algebraic \( P \) Stackelberg solution which is defined in the following. Similar methods for obtaining \( \epsilon \)-Nash equilibrium strategies have been discussed in [16]-[18].

**Definition 4:** Consider the system (1), the cost functionals (2) and (3) and the resulting Stackelberg differential game with player 1 as leader and player 2 as follower. Let \( \Sigma_1(x) : \mathbb{R}^n \to \mathbb{R}^{n \times n} \) and \( \Sigma_2(x) : \mathbb{R}^n \to \mathbb{R}^{n \times n} \) be matrix-valued functions satisfying \( \Sigma_i(x) = \Sigma_i(x)^\top \geq 0 \) for all \( x \in \mathbb{R}^n \setminus \{0\} \) and \( \Sigma_i(0) = \Sigma_i \) for \( i = 1, 2 \). The continuous matrix-valued functions \( P_1(x) \in \mathbb{R}^{n \times n} \) and \( P_2(x) \in \mathbb{R}^{n \times n} \), such that \( P_1(x) = P_1(x)^\top \) and \( P_2(x) = P_2(x)^\top \), are said to constitute a \( \lambda \)-algebraic \( \mathcal{P} \) Stackelberg solution of the PDEs (7) if \( P_1(x) + \lambda P_2(x) > 0 \) for all \( x \in \mathcal{X} \), where \( \mathcal{X} \subseteq \mathbb{R}^n \) is a neighbourhood of the origin, and if the following conditions are satisfied.

i) For all \( x \in \mathcal{X} \)

\[
P_1(x)F(x) + F(x)^\top P_1(x) + Q_1(x)
\]

\[
- \left( \frac{1}{1 - 2\theta^2} \left\| P_1(x)\Delta_\theta(x) - \theta P_2(x)g_2(x) \right\|^2 \right)
\]

\[
- 2P_1(x)g_2(x)g_2(x)^\top P_2(x) + \Sigma_1(x) = 0,
\]

\[\text{(10)}\]

\(^{3}\)The spectrum of the matrix \( A \) is denoted by \( \sigma(A) \).
and
\[
P_2(x)F(x) + F(x)^\top P_2(x) + Q_2(x) - \frac{1}{(1 - 2\theta^2)^2} \left\| (1 - \theta^2)P_2(x)g_2(x) - \theta P_1(x)\Delta g(x) \right\|^2 + 2 \frac{1}{1 + 2\theta^2} \left( \theta P_2(x)g_2(x)g_2(x)^\top P_2(x) \right) 
\]
\[- \frac{1}{1 + 2\theta^2} \left( P_2(x)g_1(x)\Delta g(x)^\top P_1(x) \right) 
\]
\[- \frac{1}{1 + 2\theta^2} \left( P_1(x)\Delta g(x)g_1(x)^\top P_2(x) \right) + \Sigma_2(x) = 0. \tag{11}\]

ii) \( P_1(0) = \bar{P}_1 \) and \( P_2(0) = \bar{P}_2 \), where \( \bar{P}_1 \) and \( \bar{P}_2 \) are symmetric matrices, such that \( P_1 + P_2 > 0 \), and satisfy the coupled AREs
\[
P_1A + A^\top P_1 + Q_1 - P_1B_2B_2^\top - P_2B_2B_2^\top P_2 
\-
\frac{1}{1 - 2\theta^2} (\bar{P}_1\Delta B - \theta \bar{P}_2 B_2^\top (\Delta B^\top \bar{P}_1 - \theta B_2^\top \bar{P}_2) + \Sigma_1 = 0, \tag{12}\]
\[
P_2A + A^\top P_2 + Q_2 + \frac{1}{1 - 2\theta^2} \left( 2\theta P_2B_2B_2^\top P_2 - \bar{P}_2\Delta B B_2^\top \bar{P}_2 - \bar{P}_1 \Delta B B_2^\top P_2 - P_2\bar{B}_2\Delta B^\top \bar{P}_1 \right) 
\-
\frac{1}{1 - 2\theta^2} \left( (1 - \theta^2)P_2B_2 - \theta \bar{P}_1 \Delta B \right) \left( 1 - \theta^2 \right) B_2^\top - \Delta B \bar{P}_1 \right) + \Sigma_2 = 0. \]

If \( X = \mathbb{R}^n \), \( P_1(x) \) and \( P_2(x) \) are said to be an algebraic \( P \) Stackelberg solution.

In what follows it is assumed that an algebraic \( P \) Stackelberg solution exists.

In what follows we consider dynamic feedback strategies of the form
\[
\dot{x} = \alpha(x, \xi), \quad u_1 = \beta_1(x, \xi), \quad u_2 = \beta_2(x, \xi, \beta_1(x, \xi)), \quad \tag{13}\]
with \( \xi(t) \in \mathbb{R}^n \), for some \( \nu > 0 \), where \( \alpha, \beta_1 \) and \( \beta_2 \) are smooth mappings with \( \alpha(0,0) = 0, \beta_1(0,0) = 0 \) and \( \beta_2(0,0,0) = 0 \).

Definition 5: The dynamic feedback strategies \( (u_1, u_2(u_1), \xi) \) are said to be admissible if the zero equilibrium of the closed-loop system (1)-(13) is (locally) asymptotically stable.

Suppose \( P_1(x) \) and \( P_2(x) \) are an algebraic \( P \) Stackelberg solution and define the functions
\[
V_1(x, \xi) = \frac{1}{2}x^\top P_1(\xi)x + \frac{1}{2}\|x - \xi\|^2_{R_1}, 
\]
\[
V_2(x, \xi) = \frac{1}{2}x^\top P_2(\xi)x + \frac{1}{2}\|x - \xi\|^2_{R_2}, \tag{14}\]
where \( \xi \in \mathbb{R}^n \), and \( R_1 \in \mathbb{R}^{n \times n} \) and \( R_2 \in \mathbb{R}^{n \times n} \) are symmetric and positive-definite matrices, i.e. \( R_1 = R_1^\top > 0 \) and \( R_2 = R_2^\top > 0 \).

A modified problem in the extended state space is defined as follows.

Problem 2: Consider the system (1) and the cost functionals of the leader and the follower, i.e. (2) and (3). Solving the approximate dynamic Stackelberg differential game consists in determining dynamic feedback strategies \( (u_1, u_2, \xi) \) of the form (13) with \( \xi(t) \in \mathbb{R}^n \), and non-negative functions \( c_1 : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \) and \( c_2 : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \), such that for any \( x(t), (\xi(t)) \) and for any admissible \( (u_1, \beta_2, \xi) \), with \( u_1 \neq 0, \) and \( (\beta_1, u_2, \xi) \), with \( u_2 \neq 0, \)
\[
\dot{J}_1(x(t), \beta_1, \beta_2(\beta_1)) \leq \dot{J}_1(x(0), u_1, \beta_2(u_1)), 
\]
\[
\dot{J}_2(x(t), \beta_1, \beta_2(\beta_1)) \leq \dot{J}_2(x(0), \beta_1, u_2(\beta_1)), 
\]
with the modified cost functionals \( \dot{J}_1 \) and \( \dot{J}_2 \) given by
\[
\dot{J}_i \equiv \frac{1}{2} \int_0^\infty \left( q_i(x(t)) + 2\theta u_1(t)^\top u_2(t) + \|u_i(t)\|^2 
+ c_i(x(t), \xi(t)) \right) dt, \tag{15}\]
where \( c_i(x(t), \xi(t)) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \) are non-negative functions, for \( i = 1, 2 \).

Let \( \Phi_1(x) \in \mathbb{R}^{n \times n} \) be such that \( x^\top (P_1(x) - P_1(\xi)) = (x - \xi)^\top \Phi_1(x, \xi) \) and similarly let \( \Phi_2(x) \in \mathbb{R}^{n \times n} \) be such that \( x^\top (P_2(x) - P_2(\xi)) = (x - \xi)^\top \Phi_2(x, \xi) \). The next statement provides a method for constructing a solution for Problem 2.

Theorem 1: Consider the system (1) and the cost functionals (2) and (3) with \( \theta \in (-\frac{1}{2}, \frac{1}{2}) \). Let \( P_1(x) \) and \( P_2(x) \)
be algebraic \( P \) Stackelberg solutions and let \( R_1 \) and \( R_2 \) be such that
\[
R_i(R_1 + R_2) + (R_1 + R_2)R_i \geq 0, \tag{16}\]
for \( i = 1, 2 \). Then there exist \( \bar{k} \geq 0 \) and a neighbourhood of the origin \( \Omega \subset \mathbb{R}^n \times \mathbb{R}^n \) such that for all \( k \geq \bar{k} \) the functions (14) solve the system of partial differential inequalities
\[
\mathcal{H}J_1 = \frac{\partial V_1}{\partial x} f(x) + \frac{1}{2} q_1(x) - \frac{\partial V_1}{\partial x} g_2(x)g_2(x)^\top \frac{\partial V_2}{\partial x} 
\-
\frac{1}{2} \frac{1}{1 - 2\theta^2} \left\| \frac{\partial V_1}{\partial x} \Delta g(x) - \theta \frac{\partial V_2}{\partial x} g_2(x) \right\|^2 + \frac{\partial V_1}{\partial \xi} \dot{\xi} \leq 0, \tag{17}\]
\[
\mathcal{H}J_2 = \frac{\partial V_2}{\partial x} f(x) + \frac{1}{2} q_2(x) 
\+
\frac{1}{1 - 2\theta^2} \frac{\partial V_2}{\partial x} g_2(x)g_2(x)^\top \frac{\partial V_2}{\partial x} 
\-
\frac{1}{2} \frac{1}{(1 - 2\theta^2)^2} \left\| (1 - \theta^2)^2 \frac{\partial V_2}{\partial x} g_2(x) - \theta \frac{\partial V_1}{\partial x} \Delta g(x) \right\|^2 \tag{17}\]
\[
\-
\frac{1}{1 - 2\theta^2} \frac{\partial V_2}{\partial x} g_1(x)\Delta g(x)^\top \frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial \xi} \dot{\xi} \leq 0, \tag{17}\]
with \( \dot{\xi} = -k \left( \frac{\partial V_1}{\partial \xi} + \frac{\partial V_2}{\partial \xi} \right) ^\top \) and for all \( (x, \xi) \in \Omega. \)

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Furthermore, the dynamical system
\[
\dot{\xi} = -k \left( \frac{\partial V_1}{\partial \xi} + \frac{\partial V_2}{\partial \xi} \right)^\top,
\]
\[
u_1 = -\Delta g(x)^\top 1 - 2\theta_2 (P_1(x)x + (R_1 - \Phi_2(x)) - \xi)^\top,
\]
\[
u_2(u_1) = -g_2(x)^\top (P_2(x)x + (R_2 - \Phi_2(x))(x - \xi))^\top - \theta u_1,
\]
yields admissible dynamic feedback strategies that solve Problem 2. Finally, there exists a neighborhood of the origin in which the strategies (18) constitute an \(\epsilon_0\)-Stackelberg equilibrium solution of Problem 1 for all \(\alpha > 0\).

IV. SIMULATIONS

In this section a numerical example illustrating the theory is presented. Consider the dynamical system
\[
\dot{x} = \begin{bmatrix}
(1 + x_1^2) & 2x_1 x_2 \\
0 & 1 + x_2^2
\end{bmatrix} (u_1 + u_2),
\]
and suppose player 1 is the leader and player 2 is the follower, seeking to minimise the cost functionals (2) and (3), respectively, where the running costs are given by
\[
q_1(x) = a_1 \frac{x_1^2}{1 + x_2^2} + a_2 x_2^2,
q_2(x) = b_1 \frac{x_1^2}{1 + x_2^2} + b_2 x_2^2,
\]
with \(a_i \geq 0\) and \(b_i \geq 0\), for \(i = 1, 2\). Suppose \(a_1 = 18\), \(a_2 = 8\), \(b_1 = 9\), \(b_2 = 4\) and \(\theta = 0.5\).

The matrix-valued functions \(P_1(x) = \text{diag}(\{\alpha_1 x_1^2, \alpha_2 x_1^2 + d_1 x_2^2\})\) and \(P_2(x) = \text{diag}(\{\beta_1 x_1^2, \beta_2 (1 + d_3 x_1^2 + d_4 x_2^2)\})\), with \(d_i \geq 0\), \(i = 1, \ldots, 4\), \(\alpha_1 \beta_1 \geq \max\{\frac{1}{2} a_1, \beta_1^2 + \frac{1}{4} b_1\}\) and \(\alpha_2 \beta_2 \geq \max\{\frac{1}{2} a_2, \beta_2^2 + \frac{1}{2} b_2\}\), \(\beta_1 \leq \alpha_1\) and \(\beta_2 \leq \alpha_2\), constitute an algebraic \(P\) Stackelberg solution, with \(\Sigma_1(x) > 0\) and \(\Sigma_2(x) > 0\), for the differential game. Let \(\alpha_1 = 4.5\), \(\alpha_2 = 3\), \(\beta_1 = 2.5\), \(\beta_2 = 2\), \(d_i = 0\), \(i = 1, \ldots, 4\) and let \(u_1^q\) and \(u_2^q(u_1^q)\) denote the corresponding dynamic strategies given by (18).

Performing the change of coordinates \(\hat{x}_1 = x_1(1 + x_2)^2\) and \(\hat{x}_2 = x_2\), the problem can be transformed into a linear-quadratic Stackelberg differential game, with \(A = 0\), \(B_1 = B_2 = I\), \(Q_1 = \text{diag}(18, 8)\) and \(Q_2 = \text{diag}(9, 4)\), for which \(P_1 = \text{diag}(4, 2)\) and \(P_2 = \text{diag}(2, 2)\) are the solutions to the coupled AREs (9). It follows that the Stackelberg equilibrium strategies are
\[
u_1^* = -[\begin{array}{c}
6x_1 \\
2 x_2
\end{array}]^\top,
\]
\[
u_2(u_1^*) = -[\begin{array}{c}
2 x_1 \\
2 x_2
\end{array}]^\top - \theta u_1^*,
\]
whereas the linear-quadratic approximation of the problem yields the strategies
\[
u_1^{lq} = -[\begin{array}{c}
6x_1 \\
2 x_2
\end{array}]^\top,
\]
\[
u_2^{lq}(u_1^{lq}) = -[\begin{array}{c}
2 x_1 \\
2 x_2
\end{array}]^\top - \theta u_1^{lq}.
\]
Simulations have been run for the system with different combinations of the three control strategies for 25 initial conditions, \(x_0 = \begin{bmatrix}1, 0 \end{bmatrix}^\top\). These initial conditions are such that they form a uniform, square grid in the positive orthant, with \(x_{1,0}\) and \(x_{2,0}\) ranging from 0 to 4. Different values for \(k, R_1, R_2\) and \(\xi(0)\) have been selected for the different initial conditions.

To compare the dynamic strategies resulting from (18) with the linear strategies (21) consider the quantities \(C_1(x_0)\) and \(C_2(x_0)\) defined in (22). Since \(J_1(u_1^*, u_2^q(u_1^*)) \leq J_1(u_1^{lq}, u_2^q(u_1^{lq}))\) and \(J_2(u_1^*, u_2^q(u_1^*)) \leq J_2(u_1^{lq}, u_2^q(u_1^{lq}))\), \(C_1(x_0)\) quantifies the loss suffered by player 1 when it deviates from its Stackelberg strategy to the dynamic or linear strategy. More precisely, \(C_1(x_0) < 0\) indicates that player 1 loses less by deviating from \(u_1^*\) to \(u_1^{lq}\) than it does by deviating from \(u_1^*\) to \(u_1^{lq}\). Similarly, \(C_2(x_0) < 0\) indicates that player 2 loses less by deviating from \(u_2^q(u_1^*)\) to \(u_2^{lq}(u_1^{lq})\) than it does by deviating from \(u_2^q(u_1^*)\) to \(u_2^{lq}(u_1^{lq})\).

Figures 1 and 2 show the quantities \(C_1(x_0)\) and \(C_2(x_0)\), respectively, for the different initial conditions. Dark shades indicate small values of \(C_i < 0\), for \(i = 1, 2\). Close to the origin and along the line \(x_{2,0} = 0\), along which (21) and (20) are identical, the linear strategies perform better than the dynamic ones as expected. However, both \(C_1(x_0)\) and \(C_2(x_0)\) are small in this region. Further from the origin and the \(x_{2,0} = 0\) line, \(C_1(x_0) < 0\) and \(C_2(x_0) < 0\). This suggests that the dynamic strategies offer a relatively good approximation of the Stackelberg equilibrium solution compared to the linear strategies.
\[ C_1(x_0) = \frac{J_1(u_1^d, u_2^q) - J_1(u_1^q, u_2^q)}{|J_1(u_1^*, u_2^*(u_1^*))|}, \text{ if } J_1(u_1^*, u_2^*(u_1^*)) \neq 0 \]

\[ C_2(x_0) = \frac{J_2(u_1^d, u_2^q) - J_2(u_1^q, u_2^q)}{|J_2(u_1^*, u_2^*(u_1^*))|}, \text{ if } J_2(u_1^*, u_2^*(u_1^*)) \neq 0 \]  \hspace{1cm} (22)

![Graph](image)

**V. CONCLUSION**

Feedback Stackelberg equilibrium solutions for a class of nonlinear nonzero-sum differential games have been studied. A method for designing dynamic feedback strategies that satisfy partial differential inequalities in place of the partial differential equations associated with the differential game has been presented. It is shown that these strategies constitute c-Stackelberg equilibrium solutions for such games. The theory is illustrated by a numerical example for which the Stackelberg equilibrium strategies are known and it is demonstrated that the dynamic approximate solution yields a better approximation than the linear-quadratic approximation of the problem.

Directions for future work include considering Stackelberg solutions for games with more than two players or groups of players that may be playing either Nash or Stackelberg strategies among each other. It is also of interest to consider differential games with non-standard information structures. It is, for example, of interest to consider a differential game consisting of groups of players where the groups play a Stackelberg game between each other, whereas a Nash game is played between the members of each group, similar to what has been considered in [15].

**REFERENCES**


