Continuous Review Inventory Model with Dynamic Choice of Two Freight Modes with Fixed Costs

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We analyze a continuous review stochastic inventory model in which orders placed with a make-to-order manufacturer can be shipped via two alternative freight modes differing in lead-time and costs. The costs of placing an order and using each freight mode exhibit economies of scale. We derive an optimal policy for utilizing the two freight modes for shipping each order. This freight mode decision is delayed until the completion of manufacturing, and the optimal policy uses information about the demand incurred in the meantime. Further, given that the two freight modes are used optimally for shipping each order, we characterize the optimal reorder point and the optimal order quantity, and analyze the cost savings achieved due to postponement of the freight mode decision. We also provide analytical and numerical comparisons between the optimal solutions to our two freight model and the single freight models. Finally, we illustrate the properties of the optimal solution to our model using an extensive set of numerical examples.

Key words: Inventory/Production: Continuous review policy, lead-times. Transportation: Freight mode selection, Transportation economies of scale.

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1. Introduction
In the last few decades, the manufacturing industry has witnessed an ever growing trend of sourcing products from offshore locations. For a firm, one of the biggest challenges that accompany global sourcing is managing timely shipment of inventory over a far flung supply chain. The competitive edge attained by cost-effective sourcing can easily be eroded by the reduced ability to respond to surges and slumps in demand due to long lead-times of which transportation time is a major part.

In response, and aided by deregulation of the world-wide transportation industry, logistic service providers continue to develop innovative practices such as freight consolidation and multi-modal shipping. This has resulted in an increased number of logistic services, often offered by the same service provider, and typically differentiated over the trade-off between transportation time and cost. As transportation decisions are inherently linked with lead-times and hence inventory related costs, a firm needs to optimize them jointly with inventory decisions to gain the full advantage
of low-cost sourcing. In this paper we analyze one such joint optimization for a firm that sources its product from a make-to-order supplier, and can ship inventories using two alternative freight modes differing in lead-time and costs.

The practice of using two transportation modes for order fulfillment has not been uncommon; firms that use slower ocean freight for shipping on a regular basis resort to faster and expensive air freight on an emergency basis. However, in recent years under the increased pressure of meeting demand on time to stay competitive, firms such as Kodak, Digital Equipment Corporation and Texas Instruments have increased the relative use of air freight (Bowman 1994, Sowinski 2004). The trend of combining different transportation modes is also evident in the variety of new multi-modal services offered by logistic solution providers and the growing demand for time critical expedited services (Krause 2000, Stanley 2004). Logistic solutions providers, such as UPS, FedEx, DHL and BAX Global offer single-source multi-modal logistic solutions to firms, which let firms decide the mode of transportation from the solution provider’s offerings of air, ocean, rail or road transportation modes. Recent advances in information technology (IT) have added a new dimension to how firms manage their logistics. There is now an increased emphasis on IT enabled logistics services that provide firms visibility across the supply chain, and facilitate transportation decisions based on the latest information. For example, using the IT logistic solution provided by TradeBeam, clothing retailer Liz Claiborne Inc. can monitor its shipments in the pipeline and expedite their delivery online. Similar solutions are adopted by power tools manufacturer Black & Dekker, and culinary and kitchen retailer Williams Sonoma to manage their inbound logistics (Enslow 2006).

Several papers in the Operations literature have addressed the issue of optimally managing inventory with two replenishment modes. Our work differs from this body of literature in two important aspects: First, most of these studies assume a simple quantity proportional cost of shipping, ignoring the transportation economies of scale. Consequently in the optimal solutions analyzed in these studies orders are allowed to be split in small quantities. This is contrary to the observation made from real life data, by Thomas and Tyworth (2006) in their recent review paper. In contrast, we consider a model in which in addition to the quantity proportional cost the firm incurs a fixed cost on each use of both freight modes. Our paper thus incorporates previously ignored transportation economies of scale. Second, most of the existing work assumes the transportation mode decision to be static in nature (i.e., the decision remains the same for each order), while in practice such decisions are increasingly being made based on the latest demand information. Our model captures this dynamic nature of the transportation mode decision by postponing it until the completion of manufacturing. Thus the optimal freight mode decision is based on the realized value of demand incurred during manufacturing. This also enables us to identify the value of information
visibility in the supply chain.

We restrict the analysis in this paper to \((Q,r)\) inventory systems. Often, in practice, it is not feasible to produce a quantity different from a fixed batch size. For example, products that require chemical processing have to be produced in a fixed batch size determined by the size of the vessel used for the chemical reaction, or production lot sizes are predetermined for scheduling efficiency. Similarly, in certain cases suppliers require firms to place orders larger than a minimum size, due to minimum economic scale required for production. A \((Q,r)\) inventory model is an accurate description of continuous review inventory management in such batch production environments, when the demand is not lumpy. Additionally, focusing on \((Q,r)\) policy allows us to employ a geometric representation of cost and optimal policy parameters, enabling a better understanding of an otherwise complex analysis.

We conclude this section by summarizing the organization of the rest of the paper: After a brief literature review in §2, we present our model and derive the cost function in §3. Next, we optimize our model for exogenous order size \(Q\): In section §4 we derive an optimal policy for shipping orders using the two freight modes; and in section §5, the optimal reorder point given that freight modes are used optimally for shipping each order. This problem is of independent interest, because often the order quantity cannot be varied arbitrarily, as discussed above. In §6 we address the question of determining the optimal order quantity for our model. §7 provides a discussion of properties of our model with the help of two special cases and numerically solved examples. In the same section, we also compare our model with single freight models and characterize the value of demand information that results from postponing the freight mode decision until the completion of manufacturing. Finally, §8 concludes the paper. All proofs and details of numerically solved example are provided in the online supplement of the paper.

2. Literature Review

There is a vast body of literature addressing the issue of optimal inventory control when orders can be placed with two independent sources differing in lead-times and costs. To our knowledge, ours is first paper to analyze the continuous review version of this problem in presence of transportation economies of scale, and involving dynamic choice of freight modes. We refrain from providing a detailed survey of the vast body of related literature, and focus on papers we deem most relevant to our work. For detailed literature surveys, we refer readers to Minner (2003), Thomas and Tyworth (2006) and references therein.

Among the papers that consider multiple deliveries from a single supplier, closely related to our work are Groenevelt and Rudi (2002), and Jain et al. (2005). Both papers consider this problem in a periodic review setting with costs accounted for continuously in time: Groenevelt and Rudi (2002)
in a base stock inventory policy setting, and Jain et al. (2005) with fixed costs of ordering and using freight modes. Similarly, in a periodic review setting Huggins and Olsen (2003) consider the problem of a manager who can use alternative sources such as production overtime and expedited delivery under the requirement of always meeting the complete demand. Another notable work in the same direction is Lawson and Porteus (2000), who extend the classical multi-stage model of Clark and Scarf (1960) to incorporate the possibility of expediting units between two consecutive stages. Muharremoğlu and Tsitsiklis (2003a) is a further generalization of Lawson and Porteus (2000). A few additional papers consider splitting an order from a single source into two shipments. The continuous review model of Chiang and Chiang (1996) and the periodic review inventory model of Chiang (2001) allow split deliveries of an order from a single source without any extra cost. Their analysis, however, is based on a heuristic, and the fixed ratio splitting considered in their work is a special case of our model, with no manufacturing lead-time. Some recent studies analyze continuous review inventory models with the possibility of expediting shipments without splitting orders: Bookbinder and Çakanyıldırım (1999) consider deterministic demand and stochastic lead-time, Gallego et al. (2003) consider stochastic demand and deterministic lead-times, and Çakanyıldırım and Luo (2005) consider delayed expediting that utilizes the latest demand information in making the lead-time decision.

The earliest works to consider inventory models with multiple replenishment sources are Barankin (1961), who studies a single-period model; Daniel (1962), Neuts (1964) and Fukuda (1964) extend this model to multiple periods and show that the optimal policy is of order-up-to nature in each period. In all these models, the order from the faster replenishment source is assumed to be delivered instantaneously. Whittemore and Saunders (1977) is amongst the first few papers that allow the lead-time of two supply modes to differ by an arbitrary number of review periods; the policy derived in their paper, however, is extremely complex and difficult to implement. Chiang and Gutierrez (1996), Tagaras and Vlachos (2001), and Vlachos and Tagaras (2001) provide variations of periodic review inventory models with two alternative supply sources. Among the recent works, Veeraraghavan and Scheller-Wolf (2003) propose a dual index policy that keeps track of two separate indices for placing orders with two different suppliers. Parallel to studies in the periodic review setting, Moinzadeh and Nahmias (1988) consider a continuous review inventory model with two supply modes differing in lead-times and costs and propose a heuristic reorder point-order quantity policy for both supply modes. Mohebbi and Posner (1999), and Johansen and Thorstenson (1998) propose variations of this policy.

Methodologically, our work is closely related to Axsäter (1990) and Zheng (1992). In his influential paper, Axsäter (1990) provides a novel perspective to traditional inventory management
problems. This perspective is analyzed in detail by Muharremoğlu and Tsitsiklis (2003b), who refer to it by the name “single unit decomposition approach”. In this paper, we use the single unit decomposition approach to derive a closed form expression for the average cost of our model. Zheng (1992) considers a continuous review \((Q, r)\) inventory model. In addition to presenting an optimal solution for the single lead-time model, Zheng (1992) offers an insightful geometric representation of the solution. We extend his approach to our more complex setting, and the insights offered from the geometry of the solution are even more pronounced, as they not only shed light on the optimal reorder point and order quantity, but also on the optimal use of freight modes.

3. Model and Cost Function

We consider a stochastic inventory system with a continuous review \((Q, r)\) inventory policy. Define inventory level as on-hand inventory minus back-orders, and inventory position as the sum of inventory level and outstanding orders. The inventory position is continuously monitored and as soon as it drops to the reorder point \(r\), an order of size \(Q\) is placed with a make-to-order supplier. Placing a manufacturing order incurs fixed cost \(K_1\) and triggers production at the manufacturing facility. In \(L_1\) time units (manufacturing lead-time) the order is completely manufactured and ready to be shipped. At that time, two freight modes are available for shipping the order: (i) Regular freight mode, which has transportation lead-time \(L_2\) (regular freight lead-time), and incurs fixed cost \(K_2\) per shipment. (ii) Express freight mode, which has transportation lead-time \(l_2\) (express freight lead-time, \(l_2 < L_2\)), and in addition to a fixed cost \(k_2\) per shipment, it incurs variable cost \(c_2\) per unit shipped. An order can be shipped completely by either of the freight modes, or can be split across the two in any proportion. We denote \(L = L_1 + L_2\) and \(l = L_1 + l_2\), the total lead-times for a unit shipped by regular and express freight modes, respectively. The system incurs holding cost \(h\) per time unit on each unit of physical inventory. The demand arriving to the system during stock-out is back-ordered, and each unit in back-order incurs penalty cost \(p\) per time unit.

We model the demand as a non-decreasing stochastic process with stationary and independent increments. \(D_{(t_1, t_2]}\) denotes cumulative stochastic demand incurred in the interval \((t_1, t_2]\) and \(F_{(t_1, t_2]}(\cdot)\) denotes its cumulative density function. The infinitesimal mean and variance of the demand process are given by \(\mu\) and \(\sigma^2\), respectively. \(\mathbb{E}\) denotes the expectation operator, and \(\mathbb{P}(\omega)\) and \(\mathbb{I}_{(\omega)}\) denote the probability and indicator functions, respectively, of an event \(\omega\).

Let \(q(x)\) be the number of units shipped via express freight, if a realized manufacturing lead-time demand is \(D_{[0, L_1]} = x\). The remaining \(Q - q(x)\) units are shipped via regular freight. The function \(q(\cdot)\) then defines a dual freight policy. We refer to the policy of always using the same freight mode as a pure freight policy, i.e., the pure regular freight policy is \(q(x) = 0, \forall x\) and the pure express freight policy is \(q(x) = Q, \forall x\).
To obtain an expression for the expected cost per time unit of our model with an arbitrary dual freight policy \( q(\cdot) \), we use the single-unit decomposition approach (see Axssäter 1990; Muharremoğlu and Tsitsiklis 2003b for the details of this approach), which requires the following assumption.

**Assumption 1.** Demand takes place one unit at a time and units are allocated to demand in the sequence in which they are ordered.

Assumption 1 implies that even if for some reason product units are delivered out of order sequence (which will be the case when orders cross in time due to the use of express freight shortly after the use of regular freight on the previous production order), they are provided to customers in sequence anyway. Hence under Assumption 1 it is possible that at a given point in time finished goods inventory and back-orders coexist. In practice one would reassign products to reduce both holding and penalty costs, so the cost expression derived below using the single unit decomposition approach is an upper bound to the true cost. When order crossings are rare, this upper bound will be tight (See also Remark 3 below). In what follows, we first provide a brief description of the single unit decomposition approach, and then derive an expression for average cost.

Consider our model as a two stage inventory system consisting of stage 1, the location of the inventory manager where the stochastic demand for single units of product arrives at a long-term average rate \( \mu \); and stage 2, the manufacturing facility. A “virtual” product unit enters the system at stage 2 when an order is placed for it by the inventory manager at stage 1. After spending \( L_1 \) time at stage 2, during which it becomes an “actual” product unit, the product unit is ready to be shipped to stage 1. Product units and demands are matched in the sequence in which they arrive. A product unit satisfies the matched demand as soon as the product unit and the demand both have reached stage 1. At that point, the product unit and the demand leave the system. A product unit incurs holding cost \( h \) for each time unit it waits for its matched demand at stage 1, and it incurs back-order penalty cost \( p \) for each time unit its matched demand waits for it at stage 1.

For \( v > 0 \), define \( t_v \) as the random time it takes for \( v \) units of demand to arrive. For \( v \leq 0 \), let \( \mathbb{P}(t_v = 0) = 1 \). Define the index of a product unit at an instant as 1 plus the total number of product units present in the system that entered the system before the unit in question minus the number of demands waiting to be satisfied at stage 1. The index of a product unit decreases as the system experiences demand, although the product unit may physically stay at the same location, and product units meet their demand in sequence of their indices at stage 1. The index of a product unit at an instant is an indicator of the total time for which either the product unit waits for its matched demand, or the matched demand waits for the product unit, at stage 1. Define \( g(y, L) \) to be the future expected cost of a product unit indexed \( y \), which at the current instant is getting shipped from stage 2 to stage 1 with deterministic lead-time \( L \). If \( y < 0 \), then the demand matched
to the product unit has already arrived in the system at stage 1 and will wait for the product unit for additional time $L$. If $y \geq 0$, then the demand matched to the product unit arrives at stage 1 after a random time $t_y$; and if $t_y > L$, then the product unit waits for the demand at stage 1 for $(t_y - L)$ time, otherwise, the demand waits for the product unit for $(L - t_y)$ time. Assigning holding and back-order penalty costs to these waiting times we, get the following expression for $g(y, L)$:

$$g(y, L) = \begin{cases} 
    pL & \text{if } y < 0, \\
    E(h(t_y - L)^+ + p(L - t_y)^+) & \text{if } y \geq 0.
\end{cases} \quad (1)$$

In our model with two freight modes, consider a product unit that enters the system at index $y$. The demand matched to the product unit has on average already waited for $E(t_{-y})$ time at stage 1. The product unit then spends $L_1$ time at stage 2 during which its matched demand spends $(L_1 - t_y)^+$ time waiting for it at stage 1. The product unit thus incurs expected cost $pE(t_{-y}) + pE(L_1 - t_y)^+$ before it is shipped. At the time of shipment the index of the product unit is $y - D_{(0,L_1)}$. The expected cost (conditional on the realized value of $D_{(0,L_1)}$ incurred on such a product unit after shipment is $g(y - D_{(0,L_1)}, L_2)$ if it is shipped by the regular freight, and $c_2 + g(y - D_{(0,L_1)}, L_2)$ if it is shipped by the express freight. Finally, in an $(Q, r)$ inventory policy, as soon as the highest index of a product unit in the system reduces to $r$, the inventory manager places an order of size $Q$, and a batch consisting of $Q$ (“virtual”) product units with indices $r+1$ to $r+Q$ then enters the system. The expected cost per time unit of our model $C_q(Q, r)$, with an arbitrary dual freight policy defined by $q(\cdot)$, can thus be obtained by taking the average of the above mentioned costs incurred on product units over indices $y = r + 1$ to $y = r + Q$, while amortizing the fixed costs over the batch.

$$C_q(Q, r) = \frac{K_1 + p \sum_{y=r+1}^{r+Q} \{ E(t_{-y}) + E(L_1 - t_y)^+ \} + E\Gamma(Q, r, q\{D_{(0,L_1)}\}, D_{(0,L_1)})}{Q/\mu}, \quad (2)$$

where the second term in the numerator is the expected cost incurred on a batch before it is shipped, and $\Gamma(Q, r, q, x)$ is the expected cost incurred on a batch after shipping, if $q$ units are shipped using express freight and the demand during the manufacturing lead time $D_{(0,L_1)}$ equals $x$. In calculating $\Gamma(Q, r, q, x)$, the $q$ product units with the smallest indices are assigned to express freight as the demands for these product units arrive earlier than the demands for the rest of the batch:

$$\Gamma(Q, r, q, x) = K_21_{q < Q} + K_21_{q > 0} + c_2q + \sum_{y=r+1}^{r+q} g(y - x, l_2) + \sum_{y=r+q+1}^{r+Q} g(y - x, L_2). \quad (3)$$

For a random variable $d$, define the cost rate function $G(y, d) \overset{\text{def}}{=} E(h(y - d)^+ + p(d - y)^+)$, where the expectation is taken only over the second argument $d$. Our next proposition simplifies the expression in (2), and states it in terms of cost rate function $G(\cdot, \cdot)$.
Proposition 1. The average cost of a \((Q,r)\) policy with an arbitrary dual freight policy \(q(\cdot) \in [0,Q]\) is

\[
C_q(Q,r) = \frac{E\{\mu K(Q,q(D_{0,L})) + S(Q,r,q(D_{0,L}),D_{0,L})\}}{Q},
\]

where

\[
K(Q,q) = K_1 + K_2 I_{q<Q} + k_2 I_{q>0},
\]

and

\[
S(Q,r,q,x) = \sum_{y=r+1}^{r+q} (\mu c_2 + G(y-x,D_{L_1,l})) + \sum_{y=r+q+1}^{r+Q} G(y-x,D_{L_1,l}).
\]

The numerator in the expression for \(C_q(Q,r)\) divided by the demand rate \(\mu\), is the expected aggregate cost incurred on each order, and it consists of aggregate fixed cost \(K(Q,q)\), and aggregate variable cost \(S(Q,r,q,x)/\mu\). Multiplying it by ordering frequency \(\mu/Q\), gives the expected cost per unit time. In the rest of the paper, we treat the decision parameters \(Q, r\) and \(q(\cdot)\) as continuous variables, and replace the expression for \(S(Q,r,q,x)\) in (6) by the following continuous approximation,

\[
S(Q,r,q,x) = \int_r^{r+q} (\mu c_2 + G(y-x,D_{L_1,l})) dy + \int_{r+q}^{r+Q} G(y-x,D_{L_1,l}) dy.
\]

This approximation is analogous to the continuous demand approximation commonly used in the analysis of continuous review \((Q,r)\) inventory models with single freight (see Zheng 1992 and Gallego 1998). Browne and Zipkin (1991) and Serfozo and Stidham (1978) discuss the conditions under which this approximation is exact for a demand process that has continuous sample paths.

Remark 1. The cost function in Proposition 1 is derived for a case where that inventory holding cost is incurred after the units have been received. Let \(h^p < h\) denote the cost incurred on each unit in transit per time unit, and \(c_2^p = c_2 - h(L_2 - l_2)\) the express freight cost adjusted for transit inventory. Then an expression for the average cost that accounts for transit inventory is,

\[
C^p_q(Q,r) = C_q(Q,r)\big|_{c_2=c_2^p} + \mu h^p L_2,
\]

where the first term on the right hand side is the same as in equation (4), but is calculated with \(c_2\) replaced by \(c_2^p\). All the results in this paper, therefore, are applicable when holding cost is incurred on transit inventory as well.

Remark 2. The cost function in expression (4) reduces to the cost function for the model with a single freight when \(q(x) = 0 \forall x\), and when \(q(x) = Q \forall x\). These expressions are the same as given by equation (2) in Zheng (1992) with appropriate lead-times and ordering costs.

Remark 3. When orders are allowed to cross, Assumption 1 is clearly a suboptimal way of satisfying demand. The expression \(C_q(Q,r)\) is, therefore, an upper bound on the actual average cost,
the tightness of which depends on how frequently orders cross. An upper bound on the probability of order crossing is
\[ P(D_{(0,L_2-l_2)} > Q) \]. Thus, when \( P(D_{(0,L_2-l_2)} > Q) \approx 0 \), orders rarely cross and \( C_q(Q,r) \) becomes a close approximation for the actual average cost. In §B of the online supplement, we verify that the probability of order crossing is indeed very small in all our numerical examples.

**Remark 4.** A final issue that is worth discussing briefly is whether \((Q,r)\) policies are optimal for the problem we consider. If there were just a single shipping mode in our model, there exists an optimal stationary \((Q,r)\) policy, since demands take place one unit at a time and orders cannot cross. For our model with two shipping modes, this result remains true as long as we assume that orders do not cross. In reality, since demand is stationary with independent increments, the probability of order crossings is not strictly zero, but when the fixed cost \( K_1 + \min\{k_2,K_2\} \) is large enough, order crossings are very rare, and our cost expression is a close approximation to the actual cost. From this it also follows that, even if the optimal policy for our problem were not a \((Q,r)\) policy, the best \((Q,r)\) policy would not deviate much in cost from the optimal policy.

## 4. Optimal Dual Freight Policy

In this section we derive the optimal dual freight policy \( q^*(\cdot) \) given arbitrary values of \( Q \) and \( r \), and characterize the cost function given the optimal policy is used for shipping each order. Define \( C(Q,r) \) as the expected cost per time unit of the model with the optimal dual freight policy. It follows from (4) that,

\[
C(Q,r) = \frac{E[ \min_{0 \leq q \leq Q} \{ \mu K(Q,q) + S(Q,r,q,D_{(0,L_1)}) \} ]}{Q}.
\]

The optimal dual freight policy \( q^*(\cdot) \) is the solution \( q^* \) to the minimization problem inside the square brackets as a function of the realized value of \( D_{(0,L_1)} \), given \( Q \) and \( r \). The objective function of the minimization problem is the sum of a discrete and a continuous function of \( q \), making a direct general analysis somewhat tedious. To facilitate insights about the solution we first consider the special case \( K_2 = k_2 = 0 \) in the next lemma. This result is similar to the solution of Groenevelt and Rudi (2002) for the use of express freight in the periodic review setting.

**Lemma 1.** If \( K_2 = k_2 = 0 \) then the optimal express freight quantity \( q^* \) is characterized by:

If \( c_2 \geq p(L_2 - l_2) \) then \( q^* = 0 \); otherwise \( q^* = \min \left\{ Q, \left( z^* - r + D_{(0,L_1)} \right)^+ \right\} \), where \( z^* \) solves

\[
\mu c_2 + G(z^*,D_{(L_1,l)}) = G(z^*,D_{(L_1,l)}).
\]

Notice that after the freight mode decision, \( r + q - D_{(0,L_1)} \) is the relevant inventory position for the cost incurred between the arrival of the express freight and that of the regular freight. Thus, the optimal dual freight policy in the absence of fixed costs of freight modes recommends shipping-up to \( z^* \) with express freight, if achievable. We geometrically illustrate the optimality of this policy.
Figure 1 \( \mu c_2 + G (u, D_{(L_1, \ell)}) \) and \( G (u, D_{(L_1, L)}) \) plotted as functions of \( u \), for \( c_2 < p(L_2 - l_2) \).

In Figure 1, where \( \mu c_2 + G (u, D_{(L_1, \ell)}) \) and \( G (u, D_{(L_1, L)}) \) are plotted functions of \( u \). In this plot, the aggregate variable cost \( S(Q, r, q, D_{(0,L_1)}) \) is the sum of the area under \( \mu c_2 + G (u, D_{(L_1, \ell)}) \) from \( u = r - D_{(0,L_1)} \) to \( u = r + q - D_{(0,L_1)} \), and the area under \( G (u, D_{(L_1, L)}) \) from \( u = r + q - D_{(0,L_1)} \) to \( u = r + Q - D_{(0,L_1)} \). This sum is minimized when \( r + q - D_{(0,L_1)} \) is the point \( z^* \) at which the two functions intersect. However, if \( c_2 \geq p(L_2 - l_2) \), then \( G (u, D_{(L_1, \ell)}) \) lies below \( \mu c_2 + G (u, D_{(L_1, L)}) \) for all values of \( u \), and hence it is optimal to use regular freight exclusively.

**Remark 5.** Note that \( K(Q, 0) = K_1 + K_2 \), \( K(Q, Q) = K_1 + k_2 \) and for \( 0 < q < Q \), \( K(Q, q) = K_1 + K_2 + k_2 \). Thus, if the optimal value of \( q \) (for the problem with non-zero \( K_2 \) and \( k_2 \)) is such that \( 0 < q < Q \), then it minimizes \( S(Q, r, q, D_{(0,L_1)}) \), and is given by Lemma 1.

**Remark 6.** For \( c_2 \geq p(L_2 - l_2) \), \( S(Q, r, q, D_{(0,L_1)}) \) is minimized at point \( q = 0 \). This, in conjunction with Remark 5, implies that for this case 0 and \( Q \) are the only candidates for the optimal value of \( q \). Additionally, if \( K_2 \leq k_2 \), then \( q = Q \) is also eliminated as a potential optimum.

In managerial terms, Remark 5 states that if it is optimal to split an order across the two freight modes for shipping, then it is optimal to ship-up-to \( z^* \) with express freight. Remark 6, on the other hand, states that if the maximum potential benefit of ordering a unit via express freight, namely \( p(L_2 - l_2) \), is less than its marginal cost \( c_2 \), then it is never optimal to split an order.

The next lemma is useful in deriving the optimal dual freight policy and the subsequent analysis.

**Lemma 2.** (a) \( G (u, D_{(0,L_2)}) - \mu c_2 - G (u, D_{(0,L_2)}) \) is non-increasing in \( u \),

(b) \( \int_r^{r+q} \left( G (u, D_{(0,L_2)}) - \mu c_2 - G (u, D_{(0,L_2)}) \right) du \) is non-increasing in \( r \), and

(c) \( -(h(L_2 - l_2) + c_2) \mu q \leq \int_r^{r+q} \left( G (u, D_{(0,L_2)}) - \mu c_2 - G (u, D_{(0,L_2)}) \right) du \leq (p(L_2 - l_2) - c_2) \mu q \).

The following proposition states the optimal dual freight policy with general fixed cost structure.

**Proposition 2.** Define,

\[
Q_e = \frac{k_2 - K_2}{p(L_2 - l_2) - c_2} \quad \text{and} \quad Q_r = \frac{K_2 - k_2}{h(L_2 - l_2) + c_2}.
\]
Define $z_p(Q)$ as the unique solution to

$$\int_{z_p(Q)}^{z_p(Q)+Q} \left( G(u, D_{(L_1,l)}) - \mu c_2 - G(u, D_{(L_1,l)}) \right) du = \mu(k_2 - K_2),$$

(11)

define $\bar{z}$ as the unique solution to

$$\int_{z}^{z*} \left( G(u, D_{(L_1,l)}) - \mu c_2 - G(u, D_{(L_1,l)}) \right) du = \mu k_2, \ z \leq z^*$$

(12)

and define $\bar{z}$ as the unique solution to

$$\int_{z^*}^{\infty} \left( \mu c_2 + G(u, D_{(L_1,l)}) - G(u, D_{(L_1,l)}) \right) du = \mu K_2, \ z \geq z^*.$$  

(13)

For any $r$ and $Q$, a value of the optimal express freight quantity $q^*$ that solves the minimization problem in (8) is given in Table 1 for different combinations of cost parameters.

<table>
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<th>Cases</th>
<th>$c_2 &lt; p(L - l)$</th>
<th>$c_2 \geq p(L - l)$</th>
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<tr>
<td>$K_2 \leq k_2$</td>
<td>If $Q \leq Q_c$, then $q^* = 0$. If $Q_c &lt; Q \leq \bar{z} - z$, then $q^* = 0$, if $D_{(0,L_1]} \leq r - z_p(Q)$; $q^* = Q$, otherwise. If $\bar{z} - z &lt; Q$, then $q^* = 0$, if $D_{(0,L_1]} \leq r - \bar{z}$; $q^* = Q$, if $D_{(0,L_1]} \geq r + Q - \bar{z}$; $q^* = z^* - r + D_{(0,L_1]}$, otherwise.</td>
<td>$q^* = 0$</td>
</tr>
<tr>
<td>$K_2 &gt; k_2$</td>
<td>If $y - x \leq Q_r$, then $q^* = Q$. If $Q_r &lt; Q \leq \bar{z} - z$, then $q^* = 0$, if $D_{(0,L_1]} \leq r - z_p(Q)$; $q^* = Q$, otherwise. If $\bar{z} - z &lt; Q$, then $q^* = 0$, if $D_{(0,L_1]} \leq r - \bar{z}$; $q^* = Q$, if $D_{(0,L_1]} \geq r + Q - \bar{z}$; $q^* = z^* - r + D_{(0,L_1]}$, otherwise.</td>
<td>If $Q \leq Q_r$, then $q^* = Q$. If $Q_r &lt; Q &lt; Q_c$, then $q^* = 0$, if $D_{(0,L_1]} \leq r - z_p(Q)$; $q^* = Q$, otherwise. If $Q_c \leq Q$, then $q^* = 0$.</td>
</tr>
</tbody>
</table>

Proposition 2 can be explained with the help of Figure 1. As noted earlier, when $c_2 \geq p(L - l)$, there is no marginal benefit in shipping a unit via express freight and only one or the other freight mode to be used for shipping the complete order depending the cost differences between the two. For the case $c_2 \leq p(L_2 - l_2)$, plotted in Figure 1: The aggregate variable cost savings earned in shipping an order by optimally splitting it over shipping it by regular freight is given by “Area 1”, which decreases with with $D_{(0,L_1]}$, and equals $\mu k_2$ for $D_{(0,L_1]} = r - \bar{z}$. Similarly, “Area 2” which
represents the aggregate variable cost savings earned in shipping an order by optimally splitting it over shipping it by express freight, decreases with $D_{(0,L_1)}$ and is equal to $\mu K_2$ for $D_{(0,L_1)} = r + Q - \bar{z}$. Since splitting an order incurs additional aggregate fixed costs $\mu k_2$ and $\mu K_2$, respectively, over use of only regular freight and only express freight, it is optimal if and only if Area 1 is at least as large as $\mu k_2$ and Area 2 is at least as large as $\mu K_2$. These two conditions can hold simultaneously only for $Q > \bar{z} - z^-$. In other words, for orders smaller than $\bar{z} - z^-$, the savings in inventory and variable shipping costs earned by splitting an order are never large enough to offset the higher fixed cost incurred. The aggregate variable cost savings earned in shipping by express freight, over shipping by regular freight is $\text{Area 1} - \text{Area 2}$, whereas the additional aggregate fixed cost incurred in doing so is $\mu(k_2 - K_2)$. Area 1 minus Area 2 is non-decreasing in $D_{(0,L_1)}$ and is equal to $\mu(k_2 - K_2)$ at $D_{(0,L_1)} = r - z_p(Q)$. Thus, for $D_{(0,L_1)} < r - z_p(Q)$ shipping by regular freight is optimal, and for $D_{(0,L_1)} > r - z_p(Q)$ shipping by express freight is optimal. For smaller order sizes, when $K_2 \leq k_2 (K_2 > k_2)$, $Q_e$ is the minimum order quantity needed to justifiy shipping of an order by express freight (regular freight), which has a higher fixed cost. And, for $K_2 > k_2$ and $c_2 \geq p(L - l)$, $Q_e$ is the minimum order quantity needed to justify use of only regular freight, which has a higher fixed but a smaller per unit cost.

It is worthwhile to note that the optimal dual freight policy derived in Proposition 2 is a significant departure from the results in existing literature (such as the policy derived in Lemma 1). First and foremost, it accommodates all four possible ways of combining the two freight modes: (i) $q^*$ dynamically takes values 0, $z^* - r + D_{(0,L_1)}$ and $Q$, (ii) $q^*$ dynamically takes values 0 and $Q$, (iii) $q^* = 0$, and (iv) $q^* = Q$. Of these one is chosen depending on the order size and model parameters. It is also the first to propose that splitting of orders be restricted only to orders larger than a threshold value. Finally, it does not prescribe shipment of arbitrarily small quantities via any of the freight modes. Specifically the minimum quantity limit for regular freight is $\min\{\bar{z} - z^*, Q\}$ and for express freight is $\min\{z^* - \bar{z}, Q\}$.

Next, we formulate an expression for $C(Q,r)$ (defined in (8)) that encompasses all the cases of optimal dual freight policy in Proposition 2. This approach enables us to carry out an integrative analysis (instead of case-by-case one) to determine the optimal values of $r$ and $Q$. Furthermore, it translates to a geometric representation that facilitates a better understanding of the optimal solution. To this end we require the following definitions.

**Definition 1.**

$$\tilde{G}(y,z) \overset{\text{def}}{=} \int_{-\infty}^{y-\bar{z}} G(y - x, D_{(L_1,L_1)}) dF_{(0,L_1)}(x) + \int_{y-\bar{z}}^{\infty} \left(\mu c_2 + G(y - x, D_{(L_1,l)})\right) dF_{(0,L_1)}(x).$$
Recall that in the single-unit decomposition approach described in §3, a product unit enters the system at index \( y \in (r, r + Q] \), and after \( L_1 \) time, is shipped via one of the freight modes. Given a realization \( x \) of \( D_{(0,L_1]} \), the expected inventory holding, back-order penalty and variable express shipping costs incurred on the unit is \( G(y - x, D_{(L_1,L_1}]) / \mu \) if it is shipped by regular freight, and \( c_2 + G(y - x, D_{(L_1,L]})) / \mu \) if it is shipped by express freight. Thus, \( \tilde{G}(y,z)/\mu \) is the expected cost incurred on a product unit that enters the system at index \( y \), and is shipped by express freight if \( D_{(0,L_1]} > y - z \), and by regular freight otherwise. Note that as \( z \downarrow -\infty \), \( \tilde{G}(y,z) \) becomes \( G(y,D_{(0,L_1]} \), the cost rate function with only regular freight; and as \( z \uparrow \infty \), \( \tilde{G}(y,z) \) becomes \( \mu c_2 + G(y,D_{(0,L]}), \) the cost rate function with only express freight.

**Definition 2.** Define functions \( z_1(\cdot) \) and \( z_2(\cdot) \) such that with the optimal dual freight policy an order is shipped by regular freight if \( D_{(0,L_1]} < r - z_1(Q) \); it is shipped by express freight if \( D_{(0,L_1]} > r + Q - z_2(Q) \); and it is optimally split across the two freight modes, otherwise.

In the mathematical expressions in the rest of the paper, we suppress the argument \( Q \) of functions \( z_1(\cdot) \) and \( z_2(\cdot) \) unless ambiguity would result.

**Remark 7.** It follows from Definition 2 and Proposition 2 that:

1. When \( q^* \) takes values from 0, \( z^* - r + D_{(0,L_1]} \) and \( Q \) then \( z_1(Q) = \tilde{z} \) and \( z_2(Q) = \bar{z} \).
2. When \( q^* \) takes values from 0 and \( Q \), then \( z_1(Q) = z_p(Q) \) and \( z_2(Q) = z_p(Q) + Q \).
3. When \( q^* = 0 \), then \( z_1(Q) = z_2(Q) \rightarrow -\infty \).
4. When \( q^* = Q \), \( z_1(Q) = z_2(Q) \rightarrow \infty \).

**Remark 8.** In each of the cases, except the one on upper-right quadrant of Table 1, the function \( z_p(Q) \) is well defined at points inside a convex set. It follows from the implicit function theorem on (11), that inside such a set, \( z_p(Q) \) is continuous and differentiable. Furthermore, it has the limiting values shown in Table 2.

<table>
<thead>
<tr>
<th>Cases</th>
<th>( e_2 &lt; p(L - l) )</th>
<th>( e_2 \geq p(L - l) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( K_2 \leq k_2 )</td>
<td>( \lim_{Q \uparrow Q_e, z \downarrow \bar{z}} z_p(Q) = \lim_{Q \uparrow Q_e, z \downarrow \bar{z}} z_p(\tilde{z} - \bar{z}) = \tilde{z} )</td>
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<td>( K_2 &gt; k_2 )</td>
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<td>( \lim_{Q \uparrow Q_e, z \downarrow \bar{z}} z_p(Q) = \lim_{Q \uparrow Q_e, z \downarrow \bar{z}} z_p(\tilde{z} - \bar{z}) = \tilde{z} )</td>
</tr>
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</table>

The next lemma provides a unified expression for the average cost \( C(Q,r) \) and its properties.

**Lemma 3.** (a) An expression for the average cost with the optimal dual freight policy is

\[
C(Q,r) = \frac{\mu K(Q,r) + \int_r^{r+Q} G_d(y|Q,r)dy}{Q},
\]

(14)
where $\mathcal{K}(Q,r) = K_1 + K_2 F_{[0,L]}(r + Q - z_2) + k_2(1 - F_{[0,L]}(r - z_1))$, and $G_d(y|Q,r)$ is a continuous and differentiable function of $y$ given by: When $c_2 < p(L_2 - L_1)$ then

$$G_d(y|Q,r) = \begin{cases} \hat{G}(y,y - r + z_1), & \text{if } y \leq r + (z^* - \bar{z}), \\ \hat{G}(y,y - Q + z_2), & \text{if } y \geq r + Q - (\bar{z} - z^*), \\ \hat{G}(y,z^*), & \text{otherwise}, \end{cases}$$

and $G_d(y|Q,r) = \tilde{G}(y,y - r + z_1)$, otherwise.

(b) $C(Q,r)$ is continuous and differentiable in $Q$ and $r$.

In the above lemma, $G_d(y|Q,r)/\mu$ is the sum of expected inventory holding, back-order penalty and variable express freight costs incurred on a product unit that enters the system at index $y \in (r, r + Q]$, and is shipped according to the optimal dual freight policy. Let $S(Q,r) \triangleq \int_{r}^{r+Q} G_d(y|Q,r)dy$, then $S(Q,r)/\mu$ is the expected aggregate variable cost, with the optimal dual freight policy. The total fixed cost incurred on an order of size $Q$ is $K_1 + K_2$ if it is shipped by regular freight (when $D_{[0,L]} < r - z_1$); $K_1 + k_2$, if it is shipped by express freight (when $D_{[0,L]} > r + Q - z_2$); and $K_1 + K_2 + k_2$, if it is split across the two freight modes for shipping (when $r - z_1 < D_{[0,L]} < r + Q - z_2$). $\mathcal{K}(Q,r)$ is thus the expected aggregate fixed cost incurred on an order shipped by the optimal dual freight policy. In comparing the cost function $C(Q,r)$ in (14) with the cost function of a single freight model (for example equation (1) in Zheng (1992)), we note that the expected aggregate fixed cost $\mathcal{K}(Q,r)$ is the dual freight model counterpart of fixed cost $K$, and the dual freight cost rate function $G_d(y|Q,r)$ is the counterpart of $G(y)$. In contrast to their single freight model counterparts, $\mathcal{K}(Q,r)$ and $G_d(y|Q,r)$ are functions of $Q$ and $r$.

5. Optimal Reorder Point

In this section we characterize the optimal reorder point for triggering manufacturing orders, given each order is shipped with the optimal use of the two freight modes. Let $r(Q) \triangleq \arg \min_r C(Q,r)$, denote the optimal reorder point for a given order size $Q$.

**Proposition 3.** $r(Q)$ is a solution to

$$G_d(r|Q,r) = G_d(r + Q|Q,r) \Leftrightarrow \hat{G}(r,z_1) = \hat{G}(r + Q,z_2). \quad (15)$$

The above optimality condition of $r$ is geometrically illustrated in Figure 2(a). In this plot, the form of the function $G_d(y|Q,r)$ depends on the chosen values of $Q$ and $r$. However, for a given $Q$, the loci of points $(r,G_d(r|Q,r))$ and $(r+Q,G_d(r+Q|Q,r))$, given by the functions $\hat{G}(y,z_1)$ and $\hat{G}(y,z_2)$ respectively, are fixed. Thus, the above optimality condition of $r$ selects a point $y = r$ on $\hat{G}(y,z_1)$, and a point $y = r + Q$ on $\hat{G}(y,z_2)$ such that the functions’ values are the same. The function $G_d(y|Q,r)$ then intersects the former at $y = r$ and the latter at $y = r + Q$. 
In Figure 2(b), the dual freight cost rate function \( G_d(y|Q,r) \) is plotted along with single freight cost rate functions \( \mu c_2 + G(r,D_{(0,l)}) \) and \( G(y,D_{(0,L)}) \) of the two freight modes. The geometric representation shown in Figure 2(a) and (b) effectively captures the dynamics of the optimal dual freight policy. In particular, it reflects the usage of each freight mode when they are optimally used for chosen \( r \) and \( Q \). When \( r \) is chosen such that \( r \) and \( r+Q \) take small values, \( \tilde{G}(r,z_1) \approx \mu c_2 + G(r,D_{(0,l)}) \) and \( \tilde{G}(r+Q,z_2) \approx \mu c_2 + G(r+Q,D_{(0,l)}) \). The dual freight cost rate, which passes through these points is \( G_d(y|Q,r) \approx \mu c_2 + G(y,D_{(0,l)}) \), reflecting a dominant use of express freight for shipping. Intuitively, small values of \( r \) and \( r+Q \) result in a large number of expected back-orders, making it optimal to ship most of the units via express freight. Similarly, for large values of \( r \) and \( r+Q \), \( G_d(y|Q,r) \approx G(y,D_{(0,L)}) \), reflecting that at such \( Q \) and \( r \), shipping most of the units with regular freight is optimal as it saves expected inventory holding and variable express freight costs without substantial risk of back-orders. For intermediate values of \( r \) and \( r+Q \), such as the case shown in Figure 2(a), for each \( y \) \( G_d(y|Q,r) \) closely follows the smaller of \( \mu c_2 + G(r,D_{(0,l)}) \) and \( G(y,D_{(0,L)}) \), reflecting that with the optimal dual freight policy a significant fraction of units is shipped by each freight mode.

There may exist multiple values of \( r \) satisfying the necessary optimality condition in (15). All such solutions, however, lie inside a finite interval (as shown in Lemma 4). The optimal value of \( r \) can thus be found by a bounded search. Moreover, the geometric representation of the optimality condition explains when its multiple solutions can be ruled out. Consider \( c_2 < p(L_2 - l_2) \): Then for small values of the order quantity \( Q \), \( z_1 \) and \( z_2 \) are both either very large or very small numbers, and \( \tilde{G}(r,z_1) \) and \( \tilde{G}(r+Q,z_2) \) are both approximately equal to one of the single freight mode cost rate functions. In such cases the solution to (15) is unique, and in fact approximately equal to the optimal value of \( r \) for the equivalent single freight model with the same \( Q \). On the other hand, for large values of \( Q \), the optimal value of \( r \) is such that \( \tilde{G}(r,z_1) \approx \mu c_2 + G(r,D_{(0,L+1+l_2)}) \)
and $\tilde{G}(r + Q, z_2) \approx G(r, D(0, L_1 + L_2))$. This implies that $\tilde{G}(y, z_1) \geq \tilde{G}(r, z_1)$ for $y < r$ and $\tilde{G}(y, z_2) \geq \tilde{G}(r + Q, z_1)$ for $y > r + Q$, and the solution to (15) is unique. For intermediate values of $Q$, an extensive numerical investigation indicates that there may exist at most three value of $r$ satisfying (15). Of these three, the smallest and the largest are local minima corresponding to the greater use of express and regular freight, respectively, and the third is a local maximum. The optimal reorder point in such cases can be found by a simple comparison of costs at the two local minima.

Also noteworthy is the fact that $r(Q)$ is not a continuous function of $Q$ in general. When there exist multiple solutions to the necessary optimality condition (15), each solution corresponds to a different mix of the freight modes. As the cost of freight modes exhibit economies of scale, sometimes as $Q$ increases it becomes beneficial to switch from a solution that prescribes greater use of one freight to another solution that prescribes greater use of the other. This results in discontinuity of $r(Q)$ at the point where the switch takes place. Such discontinuity in $r(Q)$ exists only at the values of $Q$, for which there exist multiple solutions to (15). We clarify this further with the help of a numerical example in §B.1 of the online supplement.

The next corollary provides a managerial interpretation of the optimality condition of $r$.

**Corollary 1.** Let $I(Q, r)$ denote the inventory level at an arbitrary point in time with the optimal dual freight policy. Then $r(Q)$ satisfies

$$
P(I(Q, r) > 0) = \frac{p}{p + h}.
$$

The left-hand side of (16) is the time average probability of being in-stock, and is a measure of service level for the inventory system. According to Corollary 1, for a given order quantity the optimal reorder point sets the value of this service level to $p/(h + p)$. Analogous observations have been made by Gallego (1998) for a continuous review inventory model with single freight, and by Groenevelt and Rudi (2002) for a base stock inventory model with single as well as two freights.

The next lemma compares $r(Q)$ with its single freight counterparts. In addition, it provides bounding values for $r(Q)$, that are easier to compute. Let $r_s(Q) \equiv \arg \min_r ES(Q, r, 0, D(0, L_1))$ and $r_f(Q) \equiv \arg \min_r ES(Q, r, Q, D(0, L_1))$, the optimal reorder points for a given order quantity $Q$ with pure regular freight and pure express freight policies, respectively.

**Lemma 4.** $r_f(Q) \leq r(Q) \leq r_s(Q)$.

Intuitively, for the same values of $Q$ and $r$, the time average probability of being in-stock with the optimal dual freight policy is greater than that with the pure regular freight policy, and is less than that with the pure express freight policy. However, in all three cases, at the optimal reorder point for a given order size, the time average probability of being in-stock is equal to the ratio $p/(h + p)$. 
Further, for a given $Q$, the value of the time average probability of being in-stock increases with the reorder point, implying the result in Lemma 4.

6. Optimal Order Quantity

The analysis in §4 and §5 is carried out for exogenous value of order quantity $Q$, and is applicable to batch production environments where batch size cannot be changed in short term. This section addresses the usually longer term issue of determining the optimal order size. Let $S(Q) \equiv S(Q, r(Q))$ and $K(Q) \equiv K(Q, r(Q))$, then the average cost as a function of the order quantity, given that the jointly optimal reorder point and dual freight policy for that order quantity are used, is

$$C(Q) \equiv \min_r C(Q, r) = \frac{\mu K(Q) + S(Q)}{Q},$$

Recall that when $Q$ is not very large $r(Q)$ may be discontinuous for some values of $Q$, implying non-differentiability of $C(Q)$ at such points. This makes the determination of optimal $Q$ somewhat complex. The next proposition provides the optimality condition of $Q$ for the cases where $r(Q)$ is a continuous function.

**Proposition 4.** Let $A(Q) \equiv Q \tilde{G}(r(Q), z_1) - S(Q)$. If $r(Q)$ is continuous in $Q$, then the optimal order quantity $Q^*$ is a solution to,

$$A(Q) = \mu K(Q).$$

When $C(Q)$ is not differentiable at a point, it is because at that point the optimal reorder point $r(Q)$ switches between the two locally optimal values of $r$ satisfying (15). As illustrated in §B.1 of the online supplement, these two locally optimal $r$ constitute continuous functions of $Q$, one of which vanishes as $Q$ becomes large. This implies that average cost $C(Q, r)$ evaluated at each locally optimal $r$ is continuous and differentiable in $Q$, wherever the locally optimal $r$ exists. Thus, when $r(Q)$ is not continuous, $C(Q)$ is essentially the minimum of two continuous functions. Clearly, any local minimum of $C(Q)$ has to be a local minimum of at least one of these continuous functions, and hence has to satisfy the first order condition. Any optimal value of $Q$, therefore has to satisfy the first order condition stated in Proposition 4.

The second derivative of $C(Q)$ evaluated at a point satisfying the first order condition of $Q$ is,

$$\frac{d^2C(Q)}{dQ^2} = \frac{1}{Q} \frac{dG(r(Q), z_1)}{dQ} = \frac{1}{Q} \left( \left. \frac{\partial \tilde{G}(y, z_1)}{\partial y} \right|_{y=r(Q)} \frac{dr(Q)}{dQ} + \left. \frac{\partial \tilde{G}(r(Q), z)}{\partial z} \right|_{z=z_1} \frac{dz_1(Q)}{dQ} \right).$$

Let $\tilde{G}'(y, z^*) = \partial \tilde{G}(y, z^*)/\partial y$, then for large values of $Q$, $r(Q)$ is uniquely characterized by (15), and satisfies $\tilde{G}'(r(Q), z_1) < 0$ and $\tilde{G}'(r(Q) + Q, z_2) > 0$, implying that

$$\frac{dr(Q)}{dQ} = -\frac{\tilde{G}'(r(Q) + Q, z_2)}{\tilde{G}'(r(Q) + Q, z_2) - \tilde{G}'(r(Q), z_1)}.$$
is well defined and non-positive. Thus, the first term inside the parenthesis is positive, while the second term vanishes for large values of \(Q\) (as \(z_1\) takes a constant value). Consequently, for large values of \(K_1\) (leading to a larger solution \(Q\) to the first order condition), the optimality condition in (18) characterizes the unique minimum.

For smaller values of \(K_1\), there may exist multiple local minima satisfying (18). Intuitively, each local minimum satisfying (18) is an order quantity that matches the rate at which average fixed costs \(\mu K(Q)/Q\) decrease with \(Q\), with the rate at which the average variable cost \(S(Q)/Q\) (consisting of inventory holding, back-order penalty and variable cost of express freight) increase with \(Q\). Due to economies of scale in transportation costs these rates are not uniformly monotone. More specifically, when \(Q\) becomes large enough so that orders can be split more often, \(S(Q)/Q\) does not increase with \(Q\) as rapidly as it does when one of the freight modes is dominantly used.

This results in multiple ways of achieving the trade-off between the average fixed and average variable costs by mixing the two freight modes in different ways. Numerical investigation suggests that there may exist at most two local minima, of which typically the smaller one corresponds to the dominant use of one of the freight modes. In §B.1 of the online supplement we provide one such example in which there are two local minima: (i) A smaller order size shipped dominantly with regular freight, and (ii) a larger order size which can be split more often, resulting in significant use of both freights. Note that when \(K_1\) is too small, the latter is not feasible (i.e., order quantity can not be sufficiently large to justify splitting order), and when \(K_1\) is too large the former is not optimal (i.e., orders are always large enough to allow splitting).

Revisiting Figure 2(a), when \(r\) is chosen optimally the chord connecting the points \((r, G_d(y|Q,r))\) and \((r+Q, G_d(y+Q|Q,r))\) is a horizontal line, and the function \(A(Q)\) is the area enclosed between the chord and the cost rate function \(G_d(y|Q,r)\). Proposition 4 therefore states that at the optimal \(Q\), this area is equal to the expected aggregate fixed cost \(K(Q)\) times the demand rate \(\mu\). Although, this geometric representation of the optimality conditions of \(r\) and \(Q\) is similar to the one shown in Zheng (1992) for a single freight model, its dynamics is more complex in the current context. Indeed, the result of Zheng (1992) is a special case of our result. When the express freight is prohibitively expensive, then \(z_1, z_2 \downarrow -\infty\) implying that \(G_d(y|Q,r) = G(y, D_{(0,L)})\). Similarly, when the regular freight is prohibitively expensive, then \(z_1, z_2 \uparrow \infty\) implying that \(G_d(y|Q,r) = \mu c_2 + G(y, D_{(0,L)})\). In both these cases, our solution recommends using the cheaper freight mode all the time, and its geometric representation reduces to the one shown in Zheng (1992) for the model with a single freight mode. However, for a less extreme case of freight mode costs, when the optimal solution allows significant use of both freights, the geometric representation becomes a combination of the two geometric representations corresponding to the two single freight models.
7. Discussion

This section illustrates the properties of our model with the help of two special cases and a computational study. In §7.1, where we analyze the special case with negligible fixed costs of freight modes, highlights that our model has properties similar to a single freight model for large values of $K_1$, and provides analytical bounds on saving resulting from using two freight modes. §7.2 provides insights into the value of information resulting from delaying the freight mode decision and its effects on the optimal cost and order size. Finally, in §7.3 we discuss the key observations from a computational study of our model.

7.1. Special Case without Freight Mode Fixed Costs

When $c_2 < p(L_2 - l_2)$ and the freight mode fixed costs are negligible, i.e., $K_2 = k_2 = 0$, the expected aggregate fixed cost $K(Q, r) = K_1$ and the dual freight cost rate function $G_d(y|Q, r) = \tilde{G}(y, z^*)$ are considerably simpler. Note that for very large values of $K_1$ relative to $K_2$ and $k_2$, the dual freight cost rate function asymptotically approaches the cost rate function of this special case. Hence this special case is a good approximation for the general case, when $K_1$ is very large. Groenevelt and Rudi (2002) provide a comprehensive analysis of this special case for a base stock inventory model.

**Remark 9.** Let $y_s^* = \arg \min_y \{G(y, D(0, l_1))\}$ and $y_f^0 = \arg \min_y \{\mu c_2 + G(y, D(0, l))\}$, then the condition $P(D(0, L_2 - l_2) > Q) \approx 0$ for no order crossing is equivalent to $Q \gg y_s^* - y_f^0$. Further, it follows from Lemma 5(c) that $\tilde{G}'(y, z^*) < 0$ for $y < y_f^0$, and $\tilde{G}'(y, z^*) > 0$ for $y > y_s^*$. Thus, for the special case $K_2 = k_2 = 0$ and $c_2 < p(L_2 - l_2)$, if $Q \gg y_s^* - y_f^0$ then the optimality condition for $r$ (equation (15)) is satisfied by a unique value which satisfies $y_s^* - Q < r(Q) < y_f^0$.

The above remark leads to the following properties of the optimal solution for the special case.

**Lemma 6.** For $K_2 = k_2 = 0$, $c_2 < p(L_2 - l_2)$ and $Q \gg y_s^* - y_f^0$,

(a) $r(Q)$ is a continuous and differentiable function of $Q$.

(b) $-1 < r'(Q) < 0$.

(c) $\lim_{Q \to \infty} \left( r(Q) - \left( \frac{\mu pl + hL + c_2}{h + p} - \frac{h}{h + p}Q \right) \right) = 0$.

(d) $A(Q)$ is a continuous, differentiable and increasing function of $Q$, and

(e) the optimal order quantity is increasing in $K_1$.

The above lemma illustrates that the special case of our model has similar properties to the model with single freight mode as shown by Zheng (1992), and various managerial interpretations of the solution remain unchanged.
Next, we provide a cost comparison of the optimal dual freight policy with the pure freight policies. Let \( S_s(Q) \overset{\text{def}}{=} \min_r,\mathbb{E}S(Q,r,0,D_{(0,L_1)}) \) and \( S_f(Q) \overset{\text{def}}{=} \min_r,\mathbb{E}S(Q,r,Q,D_{(0,L_1)}) \). Then \( C_s(Q) \overset{\text{def}}{=} (\mu(K_1 + K_2) + S_s(Q))/Q \) and \( c_2(Q) \overset{\text{def}}{=} (\mu(K_1 + k_2) + S_f(Q))/Q \) are the average costs of the pure regular and the pure express freight policies, respectively, given the reorder points are chosen optimally for order quantity \( Q \).

**Lemma 7.** Define \( \delta_r \overset{\text{def}}{=} \frac{-b}{h+p}(p(L_2 - l_2) - c_2) \), \( \delta_f \overset{\text{def}}{=} \frac{p}{h+p}(h(L_2 - l_2) + c_2) \), \( C^* \overset{\text{def}}{=} \min_Q C(Q) \), \( C^*_r \overset{\text{def}}{=} \min_Q C_s(Q) \) and \( C^*_f \overset{\text{def}}{=} \min_Q c_2(Q) \). Then for \( K_2 = k_2 = 0 \) and \( c_2 < p(L_2 - l_2) \),

(a) \( \max\{C_s(Q) - \mu\delta_r, c_2(Q) - \mu\delta_f\} \leq C(Q) \leq \min\{C_s(Q), c_2(Q)\} \), and

(b) \( \max\{C^* - \mu\delta_r, C^*_f - \mu\delta_f\} \leq C^* \leq \min\{C^*, C^*_f\} \).

### 7.2. Value of Information

Recall that the decision of allocating product units to the freight modes is postponed until \( L_1 \) time after placing an order. We refer to the cost reduction due to this postponement as the value of information. To better understand the value of information we parameterize the manufacturing lead-time as \( \Delta \), while keeping the total regular freight lead-time \( L \) and the total express freight lead-time \( l \) fixed (where \( \Delta \) satisfies \( 0 \leq \Delta \leq l \)). For simplicity we focus on the cases with \( K_2 = k_2 = 0 \) and \( c_2 < p(L - l) \). The expression for the average cost of this model is

\[
C_\Delta(Q,r) = \frac{\mu K_1 + \int_r^{r+Q} G_\Delta(y)dy}{Q},
\]

where the cost rate function \( G_\Delta(\cdot) \) is given by

\[
G_\Delta(y) = \min_z \left\{ \int_{-\infty}^{y-z} G(y-x,D_{(\Delta,L_1)}) \, dF_{(0,\Delta)}(x) + \int_{y-z}^{\infty} (\mu c_2 + G(y-x,D_{(\Delta,L)})) \, dF_{(0,\Delta)}(x) \right\}.
\]

When \( \Delta = L_1 \) and \( G_{L_1}(y) = \tilde{G}(y,z^*) \), this is equivalent to the model studied in §7.1. In Figure 3 the cost rate function \( G_\Delta(y) \) is plotted for \( \Delta = 0 \) and \( \Delta = L_1 \). In this plot, the area between the cost rate functions \( G_0(y) \) and \( G_{L_1}(y) \) represents the value of information. In the next proposition, we characterize the effect of parameter \( \Delta \) on expected cost and order quantity.

**Proposition 5.** Let \( C^*_\Delta \) denote the minimum value of the cost function in (19), and \( r^*_\Delta \) and \( Q^*_\Delta \) the optimal values of reorder point and order quantity, respectively. Given \( \Delta_1 < \Delta_2 \),

(a) \( G_{\Delta_1}(y) \geq G_{\Delta_2}(y), \forall y \),

(b) \( C^*_{\Delta_1} \geq C^*_{\Delta_2} \), and

(c) \( Q^*_{\Delta_1} \geq Q^*_{\Delta_2} \).

The intuitive explanation of parts (a) and (b) of the above proposition is quite straightforward: When \( \Delta \) is large, the freight mode decision is made based on more information and consequently the
cost is smaller. However, the effect of $\Delta$ on the optimal order quantity is more subtle: In our model with no postponement of freight mode decision, i.e., $\Delta = 0$, the system experiences the highest cost of demand variance (i.e., the difference in stochastic and deterministic cost rates), when the inventory level is close to 0 (in Zheng 1992, this portion of the cost is referred to as “uncontrollable cost”). In order to incur this situation less frequently the optimal order quantity with stochastic demand and zero manufacturing lead-time $Q^*_0$ is higher than its deterministic counterpart. However, with a larger value of $\Delta$, the optimal dual freight policy allocates the manufacturing order across the two freight modes more effectively, thus decreasing the cost of uncertainty around inventory level 0. Therefore, with the larger value of $\Delta$, it does not cost as much to place more frequent orders and the optimal order quantity $Q^*_\Delta$ is smaller.

7.3. Observations From Numerical Study
We carried out an extensive computational study to supplement the analytical results in this paper. We provide a detailed description of the computational work in §B of the online supplement and discuss only the key observations in this subsection.

The first part of the study focuses on the effects of freight mode costs $K_2$ and $k_2$ and $c_2$ on the optimal solution and cost. The observations from these numerically solved examples suggest that when the fixed cost of placing an order $K_1$ is smaller than or comparable in magnitude to the freight mode fixed costs $K_2$ and $k_2$, the optimal solution is sensitive to fixed costs $K_2$ and $k_2$. On the other hand, when $K_1$ is large as compared to $K_2$ and $k_2$, the solution becomes more sensitive to the variable cost of express freight $c_2$. When $K_1$ is small, we notice that the optimal solution recommends shipping most of the units via the freight mode with smaller fixed costs, while using the other freight mode only in extreme cases. The optimal order quantity, reorder point and cost in such cases are very similar to the single freight model with smaller cost. Similarly, in response to changes in express freight variable cost $c_2$, the optimal solution can typically be characterized by three phases: For very small values of $c_2$ the use of express freight dominates; for very large values of $c_2$ the use of regular freight dominates; and for intermediate values of $c_2$, significant fractions
of the order quantity are shipped by each freight mode. The intermediate phase vanishes when $K_1$ takes small values.

The numerical study also sheds light on the value of having two freight modes available for shipping. We found that whenever the difference in the optimal costs of the two single freight models (i.e., $C^s$ and $C^f$) is large, the optimal cost with two freight model $C^*$ is marginally smaller than $\min\{C^s, C^f\}$. In such cases the optimal solution with two freight modes corresponds to the dominant use of one of the freight modes, and is very similar to that of the single freight model with smaller cost. However, when the difference between $C^s$ and $C^f$ is small, each freight mode is used often enough and $C^*$ is substantially smaller than $\min\{C^s, C^f\}$. These cases correspond to parameter values $K_1 \gg \max\{K_2, k_2\}$ and $c_2 < p(L_2 - l_2)$. In such cases we also observe that $Q^* \geq \max\{Q^s, Q^f\}$. With parameter value $K_2 = k_2 = 0$ and $c_2 < p(L_2 - l_2)$, this phenomenon always occurred in our numerical examples. We discuss this further in §B.4 of the online supplement. Intuitively, when orders are split between the two freight modes, the average variable cost (consisting of inventory holding, back-order penalty and variable cost of express freight) does not increase with $Q$ as rapidly as it does when each order is shipped in single shipment. Thus, with the greater possibility of splitting orders, the optimal trade-off between $\mu K(Q)/Q$ and $S(Q)/Q$ is attained at a larger value of $Q$. With the optimal dual freight policy, the orders are split with higher probability when $c_2 < p(L_2 - l_2)$ and $K_1 \gg \max\{K_2, k_2\}$, hence in such cases $Q^*$ is larger than $Q^s$ and $Q^f$.

As noted previously, it is often not feasible to change the production batch size. A firm can nevertheless ship orders optimally using the two freight modes and choose reorder point optimally for the given batch size. An issue of interest therefore is the sensitivity of the cost function $C(Q)$ to the order quantity $Q$. To establish insensitivity of the cost to the order quantity for a single freight model, Zheng (1992) has analytically shown that using the optimal order quantity determined assuming deterministic demand, when the demand is actually stochastic, leads to an increase in cost no greater than $1/8$ of the optimal cost. In his computational study the increase never exceeded 2.9%. In a similar test of our model, over a large set of examples solved (described in §B.3 of the online supplement) we found that the cost of our model is similarly insensitive to the order quantity. Establishing this finding analytically remains a subject for future study.

8. Concluding Remarks

Our paper contributes to the rich literature on inventory management with multiple replenishment modes by incorporating economies of scales in transportation costs and modeling the demand-responsive nature of the logistic decisions. In contrast to the existing results in which orders are always split, the optimal dual freight policy derived in this paper allows for four different ways of mixing the two freight modes for shipping an order. One of these four ways is chosen based on the
order size and cost parameters.

While the effect of freight mode fixed costs on the optimal usage of the freight modes for a given order is relatively straightforward, its overall impact on the optimal reorder point and order quantity is not so. Our approach to analyze this optimization problem integrates all four possible cases of the optimal freight mode decision as well as its contingent nature. To elucidate this somewhat complex analysis involving non-convex functions, we employ a geometric representation of the cost function and the optimality conditions. The geometric representation of our analytical solution enables easier understanding of the effects of various cost parameters and decision postponement on the optimal solution, and allows a comparison of our model with single freight models. Our model illustrates that the fixed costs of freight modes play a significant role in determining freight mode mix for shipping inventory when the fixed cost of placing an order is relatively small. However, at higher values of the fixed ordering cost, the variable cost of express freight mode becomes more important. The availability of two freight modes is most beneficial when the fixed costs of freight modes are smaller as compared to the fixed ordering cost, and the per unit cost of express freight mode is not too large. Using extensive numerical examples, we show that the cost of our model is relatively insensitive to the order quantity used.

Our work has several interesting extensions. While in our paper the freight modes differ in lead-times and costs, one can analyze situations where freight modes differ in reliability and costs (i.e., the firm can ship using a cheaper freight mode which is prone to random delays, or an expensive one that arrives on time). In addition to providing decision rules for mixing two such freight modes, such analysis would provide valuable insights into the premium for lead-time reliability.

The fixed and variable cost assumed for the freight modes in this paper are typically part of a contract negotiated between the shipper firm and its logistics service provider(s). Our model can be used as a building block for analyzing such contracts. A game theoretic model developed along these lines could answer several interesting questions such as: What is the nature of equilibrium contract choice (i.e., the combination of fixed and variable cost of using each freight mode), and how does this depend on various model parameters? If both freight modes are offered by a single logistic service provider, then should the firms try to negotiate a contract with smaller freight mode fixed costs, compensating that by either a higher variable cost or a higher fixed cost on each order irrespective of how two freight modes are employed for shipping it? Equally interesting would be the effects of contract choices on fleet utilization of the logistic service provider and the total supply chain costs.
References


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Online Supplement to
Continuous Review Inventory Model with Dynamic Choice of Two Freight Modes with Fixed Costs

This document contains the supplementary material for the paper, “Continuous Review Inventory Model with Dynamic Choice of Two Freight Modes with Fixed Costs”, and follows the notation introduced there.

A. Proofs of Propositions and Lemmas

Proof of Proposition 1. The equivalence between expressions (2), stated in terms of expected waiting times, and (4), stated in terms of expected inventory levels, follows from Little’s law. To show this, we first apply Little’s law to a special case of our inventory model, and then extend it to the general case.

Consider the simple case of our model facing the same stochastic demand process, with an arbitrary single lead-time \( L \), and a one-for-one continuous review inventory policy with order-up-to level \( y \). In this special case of the model, upon the arrival of each demand, a product unit indexed \( y \) enters the system and reaches stage 1 after \( L \) time. For this system, the expected waiting time for a demand is \( \mu \mathbb{E}(L - t_y) + \mu \mathbb{E}(t_y) \), and the expected number of back-orders or, equivalently, the number of demands waiting for fulfillment is \( \mathbb{E}(D(0,L] - y) \). Applying Little’s law to this system, we get

\[
\mu \mathbb{E}(L - t_y) + \mu \mathbb{E}(t_y) = \mathbb{E}(D(0,L] - y). \tag{A.1}
\]

Since Little’s law holds irrespective of batch size, as long as the demand process remains the same, the above equality holds for the general case of our model as well. Now consider the expression for \( C_q(Q,r) \) in (2): First, note that when the demands arrive in single units \( \mu \mathbb{E}(t_y) = (-y) \). This along with (A.1) and the definitions of \( g(y,L) \) and \( G(y,D(0,L]) \), gives \( \mu g(y,L) = G(y,D(0,L]) - p(-y) \). Substituting for \( g(y,D(0,L]) \) and \( g(y,D(0,L],L_2) \) in (3) and taking expectation over \( D(0,L] \) gives the following expression for \( \mathbb{E}(Q,r,q(D(0,L],D(0,L])) \):

\[
\mathbb{E}
\left(
K_2 \mathbb{1} \{q(D(0,L]) < Q\} + k_2 \mathbb{1} \{q(D(0,L]) > 0\} + c_2 q(D(0,L])
\right)
+ \frac{1}{\mu} \mathbb{E}
\left(
\sum_{y=r+1}^{r+q(D(0,L_1])} G(y-D(0,L_1],D(L_1,l)) + \sum_{y=r+q(D(0,L_1]) + 1}^{r+Q} G(y-D(0,L_1],D(L_1,l))
\right)
- \frac{1}{\mu} \left(\sum_{y=r+1}^{r+Q} p \mathbb{E}(D(0,L_1] - y)^+\right).
\]

Substituting the above in (2), the last term in the above cancels off with the second term in the
that the objective function is non-decreasing for all values of \( r \).

**Proof of Lemma 1.** When \( K_2 = 0 \) and \( k_2 = 0 \), \( K(Q, q) = K_1 \) and consequently minimizing \( K(Q, q) + S(Q, r, q, D_{(0,L_1)}) \) is equivalent to minimizing \( S(Q, r, q, D_{(0,L_1)}) \), which can be rewritten as

\[
S(Q, r, q, D_{(0,L_1)}) = \int_{-\infty}^{r+q-D_{(0,L_1)}} (\mu c_2 + G(u, D_{(L_1,L)})) \, du + \int_{r+q-D_{(0,L_1)}}^{\infty} G(u, D_{(L_1,L)}) \, du
-
\int_{-\infty}^{r-D_{(0,L_1)}} (\mu c_2 + G(u, D_{(L_1,L)})) \, du - \int_{r+Q-D_{(0,L_1)}}^{\infty} G(u, D_{(L_1,L)}) \, du.
\]

In the above expression only the first two terms depend on \( q \). The unconstrained problem of determining the optimal order-up-to level \( z^* \) for the express freight is thus

\[
\min_z \left\{ \int_{-\infty}^{z} (\mu c_2 + G(u, D_{(L_1,L)})) \, du + \int_{z}^{\infty} G(u, D_{(L_1,L)}) \, du \right\}.
\]

It can be shown that the objective function in this problem is convex in \( z \). For \( c_2 \geq p(L_2 - l_2) \), the objective function is non-decreasing for all values of \( z \) and hence \( z^* \rightarrow -\infty \). For \( c_2 < p(L_2 - l_2) \), \( z^* \) is the unique solution to the first order condition (9). The unconstrained optimal value of \( q \) is such that \( r + q - D_{(0,L_1)} = z^* \). In presence of constraint \( q \in [0, Q] \), when \( r - D_{(0,L_1)} > z^* \), \( q^* = 0 \) and when \( r + Q - D_{(0,L_1)} < z^* \), \( q^* = Q \).

**Proof of Lemma 2.** The first derivative of \( G(u, D_{(0,L_2)}) - \mu c_2 - G(u, D_{(0,L_2)}) \) with respect to \( u \) is

\[
(h + p) \left( F_{(0,L_2)}(u) - F_{(0,L_2)}(u) \right) \leq 0,
\]

where the inequality follows as \( D_{(0,L_2)} \) is stochastically greater than \( D_{(0,L_2)} \). This inequality implies (a), while (b) follows from differentiating \( \int_r^{r+q} (G(u, D_{(0,L_2)}) - \mu c_2 - G(u, D_{(0,L_2)})) \, du \) with respect to \( r \) and using (a). Finally, since the function is non-increasing in \( r \), its infimum is obtained by letting \( r \rightarrow \infty \) and its supremum is obtained by letting \( r \rightarrow -\infty \). Taking these limits we get (c).

**Proof of Proposition 2.** It follows from Lemma 1 and Remark 5 that regardless of the parameters, the optimal value of \( q \) is one of 0, \( z^* - r + D_{(0,L_1)} \) and \( Q \), where the second one is achievable only when it falls between 0 and \( Q \) and can be excluded when \( c_2 \geq p(L_2 - l_2) \). The values of the objective function for these three cases are

\[
\mu(K_1 + K_2) + \int_{r-D_{(0,L_1)}}^{r+Q-D_{(0,L_1)}} G(u, D_{(L_1,L)}) \, du,
\]

\[
\mu(K_1 + K_2 + k_2) + \int_{r-D_{(0,L_1)}}^{z^*} (\mu c_2 + G(u, D_{(L_1,L)})) \, du + \int_{z^*}^{r+Q-D_{(0,L_1)}} G(u, D_{(L_1,L)}) \, du,
\]

and

\[
\mu(K_1 + k_2) + \int_{r-D_{(0,L_1)}}^{r+Q-D_{(0,L_1)}} (\mu c_2 + G(u, D_{(L_1,L)})) \, du,
\]
respectively. First consider \( c_2 < p(L_2 - l_2) \). Then all three expressions need to be compared. The difference between expressions (A.2) and (A.3) is

\[
-\mu k_2 + \int_{z^*}^{\infty} \left( G(u, D_{(L_1, l_1)}) - \mu c_2 - G(u, D_{(L_1, l_1)}) \right) du. \tag{A.5}
\]

The integrand in the above expression is non-negative for \( u \leq z^* \) (Lemma 2(a) and the definition of \( z^* \) in (9)). Consequently, the integral is non-negative for \( D_{(0, L_1)} \geq r - z^* \) and is non-decreasing in \( D_{(0, L_1)} \). Hence (A.5) is non-decreasing in \( D_{(0, L_1)} \) and vanishes at \( D_{(0, L_1)} = r - \bar{z} \geq r - z^* \) (where \( \bar{z} \) solves (12)). Clearly, between the choices \( q = z^* - r + D_{(0, L_1)} \) and \( q = 0 \), the former is optimal when \( D_{(0, L_1)} > r - \bar{z} \), and the latter is optimal when \( D_{(0, L_1)} < r - \bar{z} \). Next, the difference between expressions (A.3) and (A.4) is

\[
\mu K_2 + \int_{z^*}^{\infty} \left( G(u, D_{(L_1, l_1)}) - \mu c_2 - G(u, D_{(L_1, l_1)}) \right) du. \tag{A.6}
\]

Using Lemma 2(a) and the definition of \( z^* \) in (9), the integrand in the second term is non-positive for \( u \geq z^* \). The overall expression in (A.6) is thus non-increasing in \( D_{(0, L_1)} \) and vanishes at \( D_{(0, L_1)} = r + \bar{z} \leq r + Q - z^* \), (where \( \bar{z} \) solves (13)). Thus, between the choices \( q = Q \) and \( q = z^* - r + D_{(0, L_1)} \), the former is optimal when \( D_{(0, L_1)} > r + Q - \bar{z} \), and the latter is optimal when \( D_{(0, L_1)} < r + Q - \bar{z} \). In summation, if \( D_{(0, L_1)} < r - \bar{z} \) then \( q^* = 0 \), if \( D_{(0, L_1)} > r + Q - \bar{z} \) then \( q^* = Q \) and if \( r - \bar{z} < D_{(0, L_1)} < r + Q - \bar{z} \) then \( q^* = z^* - r + D_{(0, L_1)} \). However, when \( Q \leq \bar{z} - \bar{z} \), the last case never occurs and either \( q = 0 \) or \( q = Q \) is optimal. The difference between expressions (A.2) and (A.4) is

\[
\mu K_2 - \mu k_2 + \int_{z^*}^{\infty} \left( G(u, D_{(L_1, l_1)}) - \mu c_2 - G(u, D_{(L_1, l_1)}) \right) du. \tag{A.7}
\]

Using Lemma 2(b) the integral in the above expression is non-decreasing in \( D_{(0, L_1)} \) and so is the overall expression. Also, for a given value of \( Q \) the overall expression vanishes at \( D_{(0, L_1)} = r - z_p(Q) \), where \( z_p(Q) \) is defined in (11). We conclude that for \( Q \leq \bar{z} - \bar{z} \), if \( D_{(0, L_1)} < r - z_p(Q) \) then \( q^* = 0 \), and if \( D_{(0, L_1)} > r - z_p(Q) \) then \( q^* = Q \).

The expression in (A.7) is bounded below by \( \mu K_2 - \mu k_2 - (h(L_2 - l_2) + c_2)\mu Q \) and bounded above by \( \mu K_2 - \mu k_2 + (p(L_2 - l_2) - c_2)\mu Q \) (Lemma 2(c)). Thus, when \( K_2 < k_2 \) and \( Q \leq Q_e \), the expression is non-positive for all values of \( D_{(0, L_1)} \), implying \( q^* = 0 \), and when \( K_2 > k_2 \) and \( Q \leq Q_r \), the expression is non-negative for all values of \( D_{(0, L_1)} \), implying \( q^* = Q \). This completes the proof of the proposition for the cases in the first column of Table 1.

For \( c_2 \geq p(L_2 - l_2) \), using Remark 6, \( q^* \) can take only the two values 0 or \( Q \). Also, if \( c_2 \geq p(L_2 - l_2) \), then \( S(Q, r, Q, D_{(0, L_1)}) > S(Q, r, 0, D_{(0, L_1)}) \). Further if \( K_2 < k_2 \) (or, equivalently, \( K(Q, Q) > K(Q, 0) \)) then clearly \( q^* = 0 \). This completes the proof for the case in the upper-right quadrant of Table 1. For \( K_2 > k_2 \), the comparison of the two choices follows from analyzing expression (A.7):
When \( D_{[0,L_1]} < r - z_p(Q) \) then \( q^* = 0 \) and when \( D_{[0,L_1]} > r - z_p(Q) \) then \( q^* = Q \), and when \( Q \leq Q_r \) then \( q^* = Q \). Additionally, as the expression in (A.7) is bounded above by \( \mu K_2 - \mu k_2 - \mu c_2( \mu c_2 - hL_2 - \mu L_2) \mu Q \), when \( Q \geq Q_r \) then \( q^* = 0 \) for all values of \( D_{[0,L_1]} \). This covers the lower-right quadrant of Table 1 and completes the proof. □

**Proof of Lemma 3.** (a) Equation (8) can be written as,

\[
C(Q,r) = \frac{\mu \mathbb{E}K(Q, q^*(D_{[0,L_1]})) + \mathbb{E}S(Q, r, q^*(D_{[0,L_1]}), D_{[0,L_1]})}{Q}.
\]

Let \( K(Q, r) = \mathbb{E}K(Q, q^*(D_{[0,L_1]})) \), then it follows from equation (5), that

\[
K(Q, r) = K_1 + K_2 \mathbb{P}(q^*(D_{[0,L_1]}) < Q) + K_2 \mathbb{P}(q^*(D_{[0,L_1]}) > 0).
\]

This, along with Definition 2 of \( z_1(Q) \) and \( z_2(Q) \), lead to the desired expression for \( K(Q, r) \). Substituting for \( q^*(D_{[0,L_1]}) \) from Proposition 2 in the expression for \( S(Q, r, q^*(D_{[0,L_1]}), D_{[0,L_1]}) \) in (7), taking expectation over all values of \( D_{[0,L_1]} \), and subsequent algebraic manipulation of the resulting expression, lead to the following:

\[
\mathbb{E}S(Q, r, q^*(D_{[0,L_1]}), D_{[0,L_1]}) = \int_{r}^{r+Q} G_d(y|Q, r) \, dy,
\]

where, (i) if \( q^*(D_{[0,L_1]}) \) takes values 0, \( z^* - r + D_{[0,L_1]} \) and \( Q \) then

\[
G_d(y|Q, r) = \begin{cases} \left( \int_{-\infty}^{y-z} G(y-x, D_{[L_1,L_1]}) \, dF_{[0,L_1]}(x) \right) & \text{if } y \leq r + (z^* - \bar{z}), \\ \left( \int_{-\infty}^{y-Q-z} G(y-x, D_{[L_1,L_1]}) \, dF_{[0,L_1]}(x) \right) & \text{if } y \geq r + Q - (\bar{z} - z^*), \\ \left( \int_{-\infty}^{\infty} G(y-x, D_{[L_1,L_1]}) \, dF_{[0,L_1]}(x) \right) & \text{otherwise}, \end{cases}
\]

(ii) if \( q^*(D_{[0,L_1]}) \) takes values 0 and \( Q \) then

\[
G_d(y|Q, r) = \int_{-\infty}^{r-z_p(Q)} G(y-x, D_{[L_1,L_1]}) \, dF_{[0,L_1]}(x) + \int_{r-z_p(Q)}^{\infty} \left( \mu c_2 + G(y-x, D_{[L_1,L_1]}) \right) \, dF_{[0,L_1]}(x) ,
\]

(iii) if \( q^*(D_{[0,L_1]}) = 0 \) then \( G_d(y|Q, r) = G(y, D_{[0,L_1]}) \), and (iv) if \( q^*(D_{[0,L_1]}) = Q \) then \( G_d(y|Q, r) = \mu c_2 + G(y, D_{[0,L_1]}) \). Using Definition 1 of \( \tilde{G}(y, z) \), Definition 2 of \( z_1(Q) \) and \( z_2(Q) \) and Remark 7, we can express \( G_d(y|Q, r) \) for these four cases as it is stated in the lemma.
(b) Equation (8) can be alternatively expressed as

$$QC(Q, r) = \mu K_1 + E_s \left( r - D_{[0,L_1]}, r + Q - D_{[0,L_1]} \right),$$

where, for a given realization $x$ of $D_{[0,L_1]}$,

$$s(r - x, r + Q - x) = \min_{r-x \leq z \leq r+Q-x} \left\{ \mu K_2 I_{\{z < r+Q-x\}} + \mu k_2 I_{\{z > r-x\}} \right. + \int_{r-x}^{r} (\mu c_2 + G(u, D_{[L_1,L]})) du + \int_{r}^{r+Q-x} G(u, D_{[L_1,L]}) du \right\}. $$

It follows from this expression that $s(v, w)$ is continuous and differentiable in $v$ and $w$ almost everywhere. This implies for non-deterministic and continuous $D_{[0,L_1]}$, that $E_s \left( r - D_{[0,L_1]}, r + Q - D_{[0,L_1]} \right)$ is continuous and differentiable in $r$ and $r + Q$, and so is $C(Q, r)$ in $Q$ and $r$. □

**Proof of Proposition 3.** In the expression for $C(Q, r)$ in (14), only the term in the numerator depends on $r$. As $QC(Q, r)$ is a continuous and differentiable function of $r$, which $\uparrow \infty$ as $|r| \to \infty$, the first order condition is a necessary condition for the optimal value of the reorder point. Denoting $f_{[0,L_1]}(x) = \partial F_{[0,L_1]}(x) / \partial x$, the probability density function of $D_{[0,L_1]}$, the first derivative of $QC(Q, r)$ with respect to $r$ is,

$$\frac{\partial QC(Q, r)}{\partial r} = G_d(r + Q|Q, r) - G_d(r|Q, r) + \int_{r}^{r+Q} \frac{\partial G_d(y|Q, r)}{\partial r} dy$$

$$+ \mu K_2 f_{[0,L_1]}(r + Q - z_2) - \mu k_2 f_{[0,L_1]}(r - z_1). \quad (A.8)$$

Substituting for $G_d(y|Q, r)$ (from Lemma 3) in the third term of the above expression and calculating the resultant expression for each case in Remark 7, while applying equations (11), (12) and (13), it follows that in all cases, the sum of the last three terms of the right hand side of (A.8) vanishes. This leads to the first order condition $G_d(r|Q, r) = G_d(r + Q|Q, r)$, which can be alternately expressed as $\hat{G}(r, z_1) = \hat{G}(r + Q, z_2)$. The existence of at least one value of $r$ satisfying (15) is ensured as $\hat{G}(r + Q, z_2) - \hat{G}(r, z_1)$ is continuous in $r$, $\hat{G}(r + Q, z_2) - \hat{G}(r, z_1) < 0$ for $r \to -\infty$, and $\hat{G}(r + Q, z_2) - \hat{G}(r, z_1) > 0$ for $r \to \infty$. □

**Proof of Corollary 1.** Let $T(Q, r)$ denote the time spent by an arbitrary product unit in inventory (i.e., available to meet its demand). Further, let $Y$ denote a uniformly distributed random number in $(r, r + Q]$, representing the index of an arbitrary product unit when it enters the system. Then, given a realization $x$ of demand $D_{[0,L_1]}$, $T(Q, r)$ takes value $(t_{Y-x} - L_2)^+$ if the unit is shipped via regular freight and $(t_{Y-x} - l_2)^+$ if the unit is shipped by express freight. Thus, with the optimal dual freight policy, $T(Q, r)$ satisfies
In the above equation, substituting \( \mu_c \) or stated as,

Finally, as \( P(T(Q,r) > 0) = 1 \). \( Q \)

\[ P(T(Q,r) > 0) = \frac{1}{Q} \left( \int_{-\infty}^{r-z_1} \int_{r}^{r+Q} P(t_{y-x} > L_2) dy dF_{(0,L_1)}(x) \right. \]

\[ + \int_{r-z_1}^{r+Q-z_2} \left( \int_{r}^{r+Q} P(t_{y-x} > l_2) dy + \int_{r+Q}^{r+Q+Q-z_2} P(t_{y-x} > l_2) dy \right) dF_{(0,L_1)}(x) \]

\[ \left. + \int_{r+Q-z_2}^{\infty} \int_{r}^{r+Q} P(t_{y-x} > l_2) dy dF_{(0,L_1)}(x) \right) . \]

Further, the optimality condition of \( r \) stated in (15) can alternatively be expressed as

\[ \left( \int_{-\infty}^{r-z_1} (G(r + Q - x, D_{(L_1,L_1)}) - G(r - x, D_{(L_1,L_1)})) dF_{(0,L_1)}(x) \right) \]

\[ + \int_{r-z_1}^{r+Q-z_2} (G(r + Q - x, D_{(L_1,L_1)}) - \mu c_2 - G(r - x, D_{(L_1,L_1)})) dF_{(0,L_1)}(x) \]

\[ + \int_{r+Q-z_2}^{r+Q} (G(r + Q - x, D_{(L_1,L_1)}) - G(r - x, D_{(L_1,L_1)})) dF_{(0,L_1)}(x) \]

\[ = 0. \]

In the above equation, substituting \( \mu c_2 = G(z^*, D_{(L_1,L_1)}) - G(z^*, D_{(L_1,L_1)}) \) from (9), and \( G(r + Q, D_{(0,L)}) - G(r, D_{(0,L)}) = (h + p) \int_{r}^{r+Q} F_{(0,L)}(y) dy - pQ \), we get

\[ \frac{1}{Q} \left( \int_{-\infty}^{r-z_1} \int_{r}^{r+Q} F_{(L_1,L_1)}(y-x) dy dF_{(0,L_1)}(x) \right) \]

\[ + \int_{r-z_1}^{r+Q-z_2} \left( \int_{r}^{r+Q} F_{(L_1,L_1)}(y-x) dy + \int_{r+Q}^{r+Q+Q-z_2} F_{(L_1,L_1)}(y-x) dy \right) dF_{(0,L_1)}(x) \]

\[ + \int_{r+Q-z_2}^{\infty} \int_{r}^{r+Q} F_{(L_1,L_1)}(y-x) dy dF_{(0,L_1)}(x) \]

Since \( P(t_0 > L) = F_{(0,L)}(v) \), the right-hand side of (A.9) is equivalent to the left-hand side of (A.10), or

\[ P(T(Q,r) > 0) = \frac{p}{h+p} . \]

Finally, as \( P(T(Q,r) > 0) = P(T(Q,r) > 0) \), the desired result follows. \( \square \)

**Proof of Lemma 4.** The first order condition \( \tilde{G}(r + Q, z_2) - \tilde{G}(r, z_1) = 0 \) can be alternatively stated as,

\[ G(r + Q, D_{(0,L_1)}) - G(r, D_{(0,L_1)}) \]

\[ + \int_{r+Q-z_2}^{\infty} \left\{ \mu c_2 + G(r + Q - x, D_{(L_1,L_1)}) - G(r + Q - x, D_{(L_1,L_1)}) \right\} dF_{(0,L_1)}(x) \]

\[ - \int_{r-z_1}^{\infty} \left\{ \mu c_2 + G(r - x, D_{(L_1,L_1)}) - G(r - x, D_{(L_1,L_1)}) \right\} dF_{(0,L_1)}(x) \]

\[ = 0. \]

As \( r + Q - z_2 \geq r - z_1 \), the term inside the square brackets can be rearranged to

\[ \int_{r+Q-z_2}^{\infty} \left\{ \mu c_2 + G(r + Q - x, D_{(L_1,L_1)}) - G(r + Q - x, D_{(L_1,L_1)}) \right\} \]
\[-\left\{ \mu c_2 + G \left( r - x, D_{(L_1,l)} \right) - G \left( r - x, D_{(L_1,l)} \right) \right\} dF_{(0,L_1]}(x)\]
\[-\int_{r-z_2}^{r+Q-z_2} \left\{ \mu c_2 + G \left( r - x, D_{(L_1,l)} \right) - G \left( r - x, D_{(L_1,l)} \right) \right\} dF_{(0,L_1]}(x).\]

The integrand of the first integral in the above expression is always non-negative (Lemma 2(a)), and so is the integral. The second integral vanishes for the cases \((z_1, z_2) = (z_p(Q), z_p(Q) + Q), (z_1, z_2) \rightarrow (-\infty, -\infty)\) and \((z_1, z_2) \rightarrow (\infty, \infty)\). Finally, when \((z_1, z_2) = (\bar{z}, \bar{z})\) the integrand in the second integral is non-positive between the limits of integration (Lemma 2(a) and the definitions of \(\bar{z}\) and \(\bar{z}\)). It follows that the whole expression is always non-negative. Thus \(r(Q)\) satisfies

\[G \left( r + Q, D_{(0,L_1]} \right) - G \left( r, D_{(0,L_1]} \right) \leq 0,\]

where the expression on the left-hand side is non-decreasing in \(r\) and is equal to 0 when \(r = r_s(Q)\), implying \(r_s(Q) \geq r(Q)\). Similarly, it can be shown that \(r(Q)\) satisfies

\[G \left( r + Q, D_{(0,L_1]} \right) - G \left( r, D_{(0,L_1]} \right) \geq 0,\]

where the expression on the left-hand side is non-decreasing in \(r\) and is equal to 0 for \(r = r_f(Q)\), implying \(r_f(Q) \leq r(Q)\). \(\square\)

**Proof of Proposition 4.** Given that \(r(Q)\) is continuous in \(Q\), \(C(Q)\) is also continuous and differentiable in \(Q\). Also, as \(\lim_{Q \downarrow 0} C(Q) \rightarrow \infty\) and \(\lim_{Q \uparrow \infty} C(Q) \rightarrow \infty\), the first order condition is a necessary condition for the optimal order quantity. The first derivative of \(C(Q)\) with respect to \(Q\) is

\[\frac{dC(Q)}{dQ} = -\frac{1}{Q^2} \left( \mu K(Q) - A(Q) \right),\]

which implies the optimality condition in (18). \(\square\)

**Proof of Lemma 5.** Two alternative expressions for \(\tilde{G}(y,z^*)\) are

\[\tilde{G}(y,z^*) = G \left( y, D_{(0,L_1]} \right) - \int_{y-z^*}^{\infty} \left( G \left( y - x, D_{(L_1,l)} \right) - \mu c_2 - G \left( y - x, D_{(L_1,l)} \right) \right) dF_{(0,L_1]}(x), \quad (A.11)\]

and

\[\tilde{G}(y,z^*) = \mu c_2 + G \left( y, D_{(0,L_1]} \right) - \int_{-\infty}^{y-z^*} \left( \mu c_2 + G \left( y - x, D_{(L_1,l)} \right) - G \left( y - x, D_{(L_1,l)} \right) \right) dF_{(0,L_1]}(x). \quad (A.12)\]

The inequality in (a) follows from noting that the two integrals on the right-hand sides of (A.11) and (A.12) are non-negative (Lemma 2(a) and the definition of \(z^*\) in equation (9)). The two limits of (b) follow from letting \(y \rightarrow -\infty\) in (A.12) and letting \(y \rightarrow \infty\) in (A.11), respectively. Taking the derivative with respect to \(y\) of both sides of (A.11) and (A.12) gives,

\[\frac{\partial \tilde{G}(y,z^*)}{\partial y} - \frac{\partial G \left( y, D_{(0,L_1]} \right)}{\partial y} = -\int_{y-z^*}^{\infty} \left( F_{(L_1,l]}(y-z^*) - F_{(L_1,l]}(y-z^*) \right) dF_{(0,L_1]}(x) \geq 0, \quad (A.13)\]
and
\[
\frac{\partial \tilde{G}(y, z^*)}{\partial y} - \frac{\partial G(y, D_{0, l})}{\partial y} = - \int_{-\infty}^{y-z^*} (F_{(l, 1)}(y-z^*) - F_{(l, 1)}(y-z^*)) \, dF_{(0, l)}(x) \leq 0, \quad (A.14)
\]
where the inequalities of (c) follow as \(D_{(l, 1, l)}\) is stochastically greater than \(D_{(l, 1, l)}\). □

Proof of Lemma 6. From Remark 9, if \(Q \gg y_0^b - y_0^f\) then the optimal reorder point \(r(Q)\) is the unique solution to \(\tilde{G}(r(Q), z^*) = \tilde{G}(r(Q) + Q, z^*)\) and it satisfies \(y_0^b - Q < r(Q) < y_0^f\). Let
\[
\tilde{G}'(y, z^*) = \frac{\partial \tilde{G}(y, z^*)}{\partial y},
\]
then taking the first derivative of the first order condition of \(r\), we get
\[
\frac{d r(Q)}{dQ} = \frac{\tilde{G}'(r(Q) + Q, z^*)}{\tilde{G}'(r(Q), z^*) - \tilde{G}'(r(Q) + Q, z^*)}.
\]
Since the above derivative is well defined for \(y_0^b - Q < r(Q) < y_0^f\), part (a) follows. These bounds on \(r(Q)\) imply that \(\tilde{G}'(r(Q), z^*) < 0\) and \(\tilde{G}'(r(Q) + Q, z^*) > 0\), hence part (b). By letting \(Q \to \infty\) on both sides of \(\tilde{G}(r(Q), z^*) = \tilde{G}(r(Q) + Q, z^*)\) and using Lemma 5(b), part (c) follows. Continuity and differentiability for \(A(Q)\) follow from part (a). The first derivative of \(A(Q)\) with respect to \(Q\) is
\[
\frac{dA(Q)}{dQ} = Q \tilde{G}'(r(Q), z^*) \frac{d r(Q)}{dQ} > 0,
\]
where the inequality follows from \(\tilde{G}'(r(Q), z^*) < 0\) and part (b), resulting in part (d). The optimality condition (18) along with part (d) implies part (e). □

Proof of Lemma 7. The inequalities \(C(Q) \leq \min\{C_s(Q), C_2(Q)\}\) and \(C^* \leq \min\{C^*, C^f\}\) follow immediately from noting that both pure freight policies are also feasible dual freight policies. Comparing the pure regular freight policy and the optimal dual freight policy, the maximum cost saving incurred on a unit (for which demand would otherwise be back-ordered) due to the availability of express freight is \(p(L_2 - l_2) - c_2\). When the reorder point is chosen optimally, in the long term the demands are back-ordered at the rate \(\mu h/(h + p)\). Thus the maximum reduction in the inventory holding and penalty cost over the pure regular freight policy, if the freight modes are optimally used, is \(\delta_s\) per unit of demand. Similarly, the maximum reduction in inventory holding and penalty costs over the pure express freight policy, if freight modes are used optimally (delaying a part of replenishment and thus saving on holding and shipping cost), is \(\delta_f\) per unit of demand. This proves the first inequality in (a). Considering the cost functions in (a) with their respective optimal order quantities gives (b). □

Proof of Proposition 5. Let \(z^*_\Delta\) denote the optimal ship-up-to level for express freight for the model in which the freight mode decision is postponed by \(\Delta\) time units. It follows from Lemma 1 that \(z^*_\Delta\) solves \(\mu c_2 + G(z, D_{(\Delta, l)}) = G(z, D_{(\Delta, l)})\). Then,
\[
G_{\Delta_1}(y) = \int_{-\infty}^{y-z^*_\Delta_1} G(y - x, D_{(\Delta_1, l)}) \, dF_{(0, \Delta_1)}(x) + \int_{y-z^*_\Delta_1}^{\infty} (\mu c_2 + G(y - x, D_{(\Delta_1, l)}) \, dF_{(0, \Delta_1)}(x),
\]
follows as

can observe that for all values of \( Q \) cost \( C \) is minimizes the cost functions and using part (a). The cost rate functions \( \int_{y}^{\infty} G(y - x) dF_D(x) + \int_{0}^{y} (\mu c + G(y - x)) dF_D(x) \)

for all values of \( Q \), maximizes the cost for all values of \( Q \) minimizing the cost functions and using part (a). The cost rate functions \( \int_{y}^{\infty} G(y - x) dF_D(x) + \int_{0}^{y} (\mu c + G(y - x)) dF_D(x) \).

Because of the definition of \( z \). The second equality is obtained by conditioning over random demand \( D \) and changing the order of integrals. The inequality then follows from the definition of \( z^* \). The subsequent equality follows as \( D_{\{0,1\}} + D_{\{1,2\}} = D_{\{0,2\}} \), and completes the proof of part (a). Part (b) follows by minimizing the cost functions and using part (a). The cost rate functions \( G_{\Delta_1}(y) \) and \( G_{\Delta_2}(y) \) each other approach asymptotically, and the latter has a smaller value of \( \min_r \left\{ \int_r^{r+Q} G_{\Delta}(y) \right\} \) (this follows from Part (a) of the lemma) for the same \( Q \). Thus it follows that \( G_{\Delta_2}(y) \) has a larger area enclosed between the cost rate function and the horizontal chord connecting the cost rate function at points \( r \) and \( r + Q \) than \( G_{\Delta_1}(y) \) has. As the optimal \( r \) and \( Q \) are characterized by this area becoming equal to \( \mu K_1 \), the optimal \( Q \) is smaller for \( G_{\Delta_2}(y) \), or \( Q_{\Delta_1} \geq Q_{\Delta_2} \), which proves part (c). □

B. Numerical Illustrations

B.1. Multiple Solutions to First Order Condition of \( r \)

For the optimal reorder point decision discussed in §5, we describe an example in which there exist multiple solutions to the first order condition of \( r \) in equation (15). The values of parameters in this instance of our model are \( L_1 = 0.3, L_2 = 0.7 \) and \( l_2 = 0.2, h = 1, p = 9, c_2 = .05, K_1 = 50, K_2 = 37.5 \) and \( k_2 = 37.5 \). The lead-time demands are approximated by Normal distribution with \( \mu = 50, \sigma = 10 \). In this example, for \( Q < 130.5 \), there exist multiple solutions \( r \) to (15). These three solutions (labeled \( r_1(Q) \), \( r_2(Q) \) and \( r_3(Q) \), in the order of their magnitudes) along with the optimal reorder point \( r(Q) \) are plotted in Figure B.1(a) as a function of \( Q \). Corresponding values of average cost \( C(Q,r) \) and fraction of units shipped with express freight are plotted in Figure B.1(b). We can observe that for all values of \( Q \), \( \max \{ C(Q,r_1(Q)), C(Q,r_3(Q)) \} < C(Q,r_2(Q)) \). Clearly, the smallest solution \( r_1(Q) \) and the largest solution \( r_3(Q) \) are local minima, while the intermediate solution \( r_2(Q) \) is a local maximum. The optimal reorder point \( r(Q) \) is therefore the more economical
of $r_1(Q)$ and $r_3(Q)$. Also note that for smaller values of $Q$, at with reorder point $r_1(Q)$ the fraction of units shipped at reorder point is close to 100%, while with reorder point $r_3(Q)$ it is close to 0%. In other words, the local minima $r_1(Q)$ and $r_3(Q)$ correspond to dominant use of express freight and regular freight, respectively, with the latter being more cost effective hence optimal. However as $Q$ increases, inventory cost benefits resulting from splitting orders outweigh the fixed costs incurred in using both freights simultaneously. This results in a significant use of both freight modes at $r_1(Q)$, leading to $r_1(Q)$ becoming more economical than $r_3(Q)$ (hence optimal) for $Q > 109.05$. This switch between the local minima at $Q = 109.05$, makes $r(Q)$ discontinuous and $C(Q,r(Q))$ non-differentiable at that point.

Observe that as $Q$ increases, one of the local minima, ($r_3(Q)$ in this case) and the local maximum converge and vanish. Thus for larger value of $Q$ ($Q > 130.5$ in this case) there exists a unique solution to the first order condition of $r$. When a local minimum and a local maximum converge to a point, that point becomes an inflection point, which is neither a local maximum nor a local minimum. Note that function $C(Q,r(Q))$ can be discontinuous in $Q$ only if the optimal reorder point switches from a local minimum to the other at a point where the former vanishes. Clearly, this cannot happen, as at such a point the former is not a local minimum. Thus $C(Q,r(Q))$ is continuous at all points.

Finally, in this example two different values of $Q$ satisfy the first condition of Proposition 4: (i) $Q = 102.97$ at which $r(Q) = r_3(Q) = 40.26$, $C(Q,r(Q)) = 93.25$ and fraction of units shipped via express freight is 0.41%, and (ii) $Q = 126.95$, at which $r(Q) = r_1(Q) = 15.49$, $C(Q,r(Q)) = 92.47$ and fraction of units shipped via express freight is 28.06%. The optimal value of $Q$ is clearly the latter.
B.2. Effects of Cost Parameters on the Optimal Solution

The results of numerically solved examples are displayed in Figures B.2 and B.3. The plots in these figures are presented in two columns: In the first column the optimal values of order size $Q^*$ and reorder point $r^*$ ($\overset{\text{def}}{=} r(Q^*)$) are plotted, whereas in the second column the optimal values of average cost $C^*$ and the fraction of units shipped by express freight $E_{Q^*}/Q^*$ are plotted. To enable a comparison, in each plot, the optimal values of order quantity, reorder point and cost for the single freight models are plotted using dotted lines. To obtain these plots, computations are performed with the lead-time demands approximated by normally distributed random variables with infinitesimal mean 50 and standard deviation 15; lead-times $L_1 = 0.3$, $L_2 = 0.7$ and $l_2 = 0.2$; cost parameters $h = 1$ and $p = 9$.

Figure B.2, where the optimal policy parameters and cost are plotted as a function of $k_2 \in [0, 100]$
Figure B.3  Sensitivity of optimal $Q^*$, $r^*$, $C^*$ and $Eq^*/Q^*$ to $c_2$.

(a) $Q^*$ and $r^*$ for $K_1 = 250$, $K_2 = 50$, $k_2 = 25$

(b) Optimal cost and $\frac{Eq^*}{Q^*}$% for $K_1 = 250$, $K_2 = 50$, $k_2 = 25$

with $k_2 + K_2 = 100$ and $c_2 = 0.05$ illustrate the sensitivity of the optimal solution to our model with two freight modes for different values of $K_1$. In these plots observe that at the left and right extremes, (i.e., near $K_2 = 100$, $k_2 = 0$ and $K_2 = 0$, $k_2 = 100$) the optimal policy resembles the optimal policy for the single freight model with smaller fixed cost. At these points, the expected fraction of order shipped by express freight is 1 on the left side, and 0 on the right side, indicating that in these cases the freight mode with smaller fixed cost is almost always used and the other freight mode is used only in extreme cases of the manufacturing lead-time demand. On the other hand, for intermediate values of $K_2$ and $k_2$, both freight modes are used for shipping a significant fraction of ordered units, and the optimal cost with dual freight model $C^*$ is much smaller than optimal costs with both single freight models, $C^s$ and $C^f$. Also note that the intermediate region is larger when $K_1$ is larger, implying that the dual freight model performs better than the best of the single freight models for wider combinations of $K_2$ and $k_2$, and is less sensitive to the values of $K_2$ and $k_2$.

Figure B.3, where $K_1 = 250$, $K_2 = 50$, $k_2 = 25$ and $c_2$ is varied from $-0.25$ to 2, indicates that the performance of the model with the optimal dual freight policy is very sensitive to the value of $c_2$, and can typically be characterized by three phases: For very small values of $c_2$, $Eq^*/Q^* \approx 1$ and the use of express freight dominates; for very large values of $c_2$, $Eq^*/Q^* \approx 0$ and the use of regular freight dominates; and for intermediate values of $c_2$, significant fractions of the order quantity are shipped by each freight mode. For larger values of $k_2$, the transitions between these phases are sharper and occur at smaller values of $c_2$. The phase with dominant use of express freight does not occur if $k_2$ is of comparable magnitude to or larger than $K_2$. 
B.3. Sensitivity of Average Cost $C(Q)$ to Order Quantity $Q$

This part of our computational study is aimed towards evaluating the sensitivity of the dual freight model cost $C(Q)$ to the order quantity $Q$. To this end we solved numerical examples for a Poisson demand process with $\mu \in \{5, 25, 50\}$, lead-times $L_1 = 0.3$, $L_2 = 0.7$ and $l_2 = 0.2$, holding cost $h \in \{0.1, 0.5, 1, 2, 5\}$, penalty cost $p$ such that $h + p = 10$, $K_2, k_2 \in \{10, 25, 50, 100\}$, $K_1 \in \{0, 10, 25, 100, 500, 1000\}$ and $c_2 \in \{0.05, 0.10, 0.25, 0.50, 0.75\}$. Following Zheng (1992) for each of the 7200 problems solved, we calculate the relative cost increase $R = (C(\hat{Q}) - C^*)/C^*$ incurred by using the optimal order quantity determined assuming deterministic demand $\hat{Q}$, instead of $Q^*$. Of these 7200 instances of our model, we found that the value of $R$ exceeds 0.1% in 1190 (16.53%) instances and exceeds 1% in only 192 (2.67%) instances, with its highest value at 5.85%. We also calculated $R^*$, the counterpart of $R$ for a regular freight model for 1650 unique regular freight models obtained from the same set of parameters. The values of $R^*$ are also of a similar order of magnitude, exceeding 0.1% in 255 (17.71%) instances and exceeding 1% in 40 (2.78%) instances, with its highest value at 4.71%. We did not find any specific ordering of $R$ and $R^*$. Our numerical findings thus suggest that the model with the optimal dual freight policy is similarly insensitive to the order quantity as the single freight model.

An upper bound on the probability that the orders cross in a replenishment cycle is $P(D_{(0,L_2-l_2)} > Q^*)$. This upper bound is exact only when both freight modes are used for shipping the manufacturing order in each replenishment cycle. We calculate this probability at optimal order quantity for all 7200 instances in our study and find that it exceeds 0.001 in 515 (7.15%) instances and exceeds 0.01 in 369 (5.5%) instances. However, in all cases where $P(D_{(0,L_2-l_2)} > Q^*) > 0.001$, $Eq^*/Q^* \approx 0$ or $Eq^*/Q^* \approx 1$, indicating that in these cases, one of the freight modes is rarely used. Thus, for these cases the actual probability of order crossing is much smaller than the upper bound $P(D_{(0,L_2-l_2)} > Q^*)$. Hence, for all numerically solved examples, $C_q(Q,r)$ in (4) is a close approximation of the actual average cost of our model.

B.4. Comparing $Q^*$ with $Q^*$ and $Q^f$ when $K_2 = k_2 = 0$ and $c_2 < p(L_2 - l_2)$

We provide an heuristic argument, based on the geometric representation of the optimal solution, for $Q^* \geq \max \{Q^*, Q^f\}$. Let $A_s(Q)$ ($A_f(Q)$) denote the single regular (express) freight model counterpart of the function $A(Q)$ defined in Proposition 4. First, we argue that when $K_2 = k_2 = 0$, $c_2 < p(L_2 - l_2)$ and $Q$ sufficiently large, $A(Q) \leq \min \{A_s(Q), A_f(Q)\}$. This inequality then implies to the desired result. We present a geometric argument for $A(Q) \leq A_s(Q)$. An analogous argument can be constructed for $A(Q) \leq A_f(Q)$. Define the cost rate function

$$G_{\delta_s}(y) \equiv G \left( y + \frac{\mu \delta_s}{h}, D_{(0,L_1)} \right) - \mu \delta_s,$$
and let \( r_{s_\delta}(Q) = \min_r \left\{ \int_r^{r+Q} G_{s_\delta}(y)dy \right\} \), the optimal \( r \) for given \( Q \) with the cost rate function \( G_{s_\delta}(y) \). First, note that \( G_{s_\delta}(y) \) is obtained by shifting the cost rate function for the pure regular freight policy \( G(y,D_{(0,L)}) \) right by \( \mu \delta_s / h \) and down by \( \mu \delta_s \), so that \( G_{s_\delta}(y) \) asymptotically approaches \( \tilde{G}(y,z^*) \). Thus, for a sufficiently large value of \( Q \), \( G_{s_\delta}(r_{s_\delta}(Q)) \approx \tilde{G}(r(Q),z^*) \). Second, in Lemma 7 we established that \( S_s(Q) - \mu \delta_s Q = \min_r \left\{ \int_r^{r+Q} G_{s_\delta}(y)dy \right\} \leq \min_r \left\{ \int_r^{r+Q} \tilde{G}(y,z^*)dy \right\} \). These two facts imply that for sufficiently large values of \( Q \), with \( r \) chosen optimally for \( Q \), \( G_{s_\delta}(y) \) has a greater area enclosed between the cost rate function and the chord connecting the cost rate function at points \( r \) and \( r + Q \), than \( \tilde{G}(y,z^*) \) has. In other words, the counterpart of \( A(Q) \) for \( G_{s_\delta}(y) \), which is equal to \( A_s(Q) \), is greater than \( A(Q) \). Figure B.4, where typical plots of \( \tilde{G}(y,z^*) \) and \( G_{s_\delta}(y) \) are shown, illustrates our argument. Finally, note that \( A_s(Q) \) and \( A_f(Q) \) are increasing in \( Q \), \( Q^* \) satisfies \( A_s(Q) = \mu K_1 \) and \( Q^f \) satisfies \( A_f(Q) = \mu K_1 \) (Lemma 6 of Zheng 1992). Given \( A(Q) \leq \min \{ A_s(Q), A_f(Q) \} \), these facts immediately imply \( Q^* \geq \min \{ Q^s, Q^f \} \).