Strong structural properties of unidirectional star graphs

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Abstract

Day and Tripathi [K. Day, A. Tripathi, Unidirectional star graphs, Inform. Process. Lett. 45 (1993) 123–129] proposed an assignment of directions on the star graphs and derived attractive properties for the resulting directed graphs; an important one is that they are strongly connected. In [E. Cheng, M.J. Lipman, On the Day–Tripathi orientation of the star graphs: Connectivity, Inform. Process. Lett. 73 (2000) 5–10] it is shown that the Day–Tripathi orientations are in fact maximally arc-connected when \( n \) is odd; when \( n \) is even, they can be augmented to maximally arc-connected digraphs by adding a minimum set of arcs. This gives strong evidence that the Day–Tripathi orientations are good orientations. In [E. Cheng, M.J. Lipman, Connectivity properties of unidirectional star graphs, Congr. Numer. 150 (2001) 33–42] it is shown that vertex-connectivity is maximal, and that if we delete as many vertices as the connectivity, we can create at most two strong connected components, at most one of which is not a singleton. In this paper we prove an asymptotically sharp upper bound for the number of vertices we can delete without creating two nonsingleton strong components, and we also give sharp upper bounds on the number of singletons that we might create.

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1. Introduction

Unidirectional interconnection networks are important in the area of parallel and distributed computing. Some recent research in this area includes [3–10]. In particular, Chern, Jwo, and Tuan [6] and Chou and Du [7] provided applications for directed interconnection networks such as high-speed networking. [6] also includes a comparison of the diameters among some known unidirectional interconnection networks.

The star graph \( S_n \), proposed by [1], is one of the most popular and most well-studied interconnection networks. It has many advantages over the hypercube such as lower degree and smaller diameter. Day and Tripathi first proposed an orientation of the star graph in [9]. Every vertex of their orientated star graph \( \vec{S}_n \) has in-degree and out-degree equal to \( \left\lceil \frac{n-1}{2} \right\rceil \) and \( \left\lfloor \frac{n-1}{2} \right\rfloor \) (not necessarily respectively). They gave an efficient, near-optimal distributed routing algorithm for \( \vec{S}_n \); we refer the readers to [9] for the details.
An important criterion of a good graph topology is that it is “highly connected.” By the same token, one should also require any unidirectional graph topology to be “highly connected.” Indeed Jwo and Tuan [10] showed that the unidirectional hypercube proposed in [7] has this property. Since the star graph was introduced as a competitive alternative to the hypercube, it is desirable to have an orientation having the same property. The (arc-)connectivity of $\overrightarrow{S}_n$ was studied in [3,5] and it was shown that in addition to $\overrightarrow{S}_n$ being maximally connected, it is also loosely and tightly super-connected. This means that when one deletes as many vertices as its connectivity, the resulting graph can have at most two strongly connected components, and when we do get two components, one of them must be a singleton.

In this paper we examine what happens if we delete more vertices, and we prove an asymptotically sharp upper bound for the number of vertices we can delete from $\overrightarrow{S}_n$ without getting two nonsingleton strong components, and we also give sharp upper bounds on the number of possible singletons that we can obtain.

2. Preliminaries

The star graph, $S_n$ ($n \geq 3$), introduced by Akers, Harel, and Krishnamurthy [1], is a graph with vertex set being the set of permutations on $n$ vertices. Two permutations $a_1a_2a_3\ldots a_n$ and $b_1b_2b_3\ldots b_n$ are adjacent if and only if there exists an $i \neq 1$ such that $a_1 = b_i$, $a_i = b_1$, and $a_j = b_j$ for $j \notin \{1, i\}$. In other words, given two permutations, $\pi_a$ and $\pi_b$, they are adjacent if and only if one of them can be obtained from the other by exchanging the symbols in position 1 and position $i$ for some $i \neq 1$. Since this change is through position $i$, we denote this change by $q(\pi_a, \pi_b) = i$, and refer to such an edge as an $i$-edge. Fig. 1 shows a picture of $S_4$. For example, 2413 and 1423 are adjacent through a 3-edge in $S_4$, and $q(2413, 1423) = 3$.

We now describe the orientation of star graphs given in [9]. Let $\pi_a$ and $\pi_b$ be adjacent in $S_n$ through an $i$-edge. We may assume $\pi_a$ is even and $\pi_b$ is odd. Then the edge is orientated from $\pi_a$ to $\pi_b$ if $i$ is even, while it is orientated from $\pi_b$ to $\pi_a$ if $i$ is odd. The resulting graph is denoted by $\overrightarrow{S}_n$. Fig. 2 shows $\overrightarrow{S}_4$. The in-degree and out-degree of a vertex $\pi \in \overrightarrow{S}_n$ are $\left\lceil \frac{n-1}{2} \right\rceil$ and $\left\lfloor \frac{n-1}{2} \right\rfloor$, respectively, if $\pi$ is odd, and vice versa if $\pi$ is even. This orientation of the star graph $S_n$ will be called the Day–Tripathi orientation.

We use basic terminology in graph theory, see e.g. West [15]. Unless otherwise specified, a graph (respectively, directed graph) has no multiple edges (respectively, arcs) and no loops (respectively, directed loops).

Since undirected interconnection networks are regular, [3] augments $\overrightarrow{S}_n$ into a regular digraph (i.e. the in-degree is equal to the out-degree at every vertex) by adding a directed matching if $n$ is even. Moreover, it was shown that for these graphs, the arc-connectivity is equal to the regularity. However, one may want to study the connectivity in $\overrightarrow{S}_n$.

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1 A permutation is even (odd) if it can be written as a product of an even (odd) number of transpositions.
directly without the augmentation. It is shown in [5] that the connectivity of $\overrightarrow{S}_n$ is $\left\lfloor \frac{n-1}{2} \right\rfloor$, which is best possible. The following theorem gives some known properties of star graphs and unidirectional star graphs:

**Theorem 2.1.** Let $n \geq 3$:

1. $S_n$ is an $(n-1)$-regular undirected graph on $n!$ vertices.
2. $S_n$ is vertex-transitive and edge-transitive.
3. The edge-connectivity and connectivity of $S_n$ is $n-1$.
4. The girth (that is, the length of the shortest cycle) in $S_n$ is 6.
5. Let $H_i$ be the subgraph of $S_n$ $(n \geq 4)$ with $i$ in the last position for $1 \leq i \leq n$. Then $H_i$ is isomorphic to $S_{n-1}$. Moreover, there are $(n-2)!$ independent\(^4\) edges between $V(H_i)$ and $V(H_j)$ for $1 \leq i < j \leq n$. Given any $i$ with $1 \leq i \leq n$, every vertex in $H_i$ has exactly one neighbour not in $H_i$ via the $n$-edge.
6. Let $H_i$ be the subgraph of $\overrightarrow{S}_n$ $(n \geq 4)$ with $i$ in the last position for $1 \leq i \leq n$. Then $H_i$ is isomorphic to $\overrightarrow{S}_{n-1}$. Moreover, if $1 \leq i < j \leq n$, then of the $(n-2)!$ independent edges between $V(H_i)$ and $V(H_j)$ in $\overrightarrow{S}_n$, exactly half of them are directed from $V(H_i)$ to $V(H_j)$ in $\overrightarrow{S}_n$.

**Proof.** The first two and the last three parts are obvious.

The edge-connectivity portion of part 3 follows from the following result whose proof can be found in [11]: If $G = (V, E)$ is a connected, vertex-transitive, $r$-regular graph, then $G$ has edge-connectivity $r$. The connectivity portion of part 3 follows from the following result of Watkins [14]: A connected simple graph with an edge-transitive automorphism group with all degrees at least $r$ is $r$-connected. (Of course, the edge-connectivity portion also follows from the connectivity portion.)

Note that the last part is not correct if we consider the subgraph of $\overrightarrow{S}_n$ that has $i$ in the $j$th position with $j$ and $n$ of different parity ($2 \leq j < n$). □

Although having a “maximal” connected interconnection network seems to be the best possible feature in terms of connectivity, it is important to study the resulting disconnected directed graph when a sufficient number of vertices have been deleted. A graph (respectively, directed graph) with connectivity $r$ is called loosely super-connected if for any $r$ vertices deleted, the resulting graph (respectively, directed graph) is either still connected (respectively, strongly connected) or has at most one component (respectively, strong component) of size greater than one.

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\(^2\) Two vertices $u$ and $v$ are equivalent if the graph has an automorphism $\phi$ such that $\phi(u) = v$. A graph is vertex-transitive if every pair of vertices are equivalent.

\(^3\) Two edges $(u, v)$ and $(x, y)$ are equivalent if the graph has an automorphism $\phi$ such that $\phi(u) = x$ and $\phi(v) = y$, or $\phi(u) = y$ and $\phi(v) = x$. A graph is edge-transitive if every pair of edges are equivalent.

\(^4\) A set of edges are independent if no two of them are incident to the same vertex.
The notion of “superness” was first introduced by Bauer et al. [2]. One can immediately see that this is an important concept in the study of interconnection networks, as this provides an insight into the severity of a disconnection. “Superness” suggests that the “core” of the network remains intact if the network is minimally disconnected. However, if we delete a vertex in $S_n$, then the resulting directed graph is a directed path of length five, hence it has five singleton strong components (which we will simply call singletons). So a stronger property is needed: A graph (respectively, directed graph) with connectivity $r$ is tightly super-connected if for any $r$ vertices deleted, the resulting graph (respectively, directed graph) is either still connected (respectively, strongly connected) or has exactly two components (respectively, strong components), one of which is a singleton. In the literature sometimes the terms super-connected and hyper-connected are used for loosely and tightly super-connected. A related concept is Vosperian [12], which, in most cases, is equivalent to being tightly super-connected.

In [5] the following result was established:

**Theorem 2.2.** Let $n \geq 3$.

1. $S_n$ is maximally connected, that is, it has connectivity $\left\lfloor \frac{n-1}{2} \right\rfloor$.
2. $S_n$ is loosely super-connected.
3. $S_n$ is tightly super-connected unless $n = 3$.

This result on the loosely and tightly super-connectedness of $S_n$ raises natural related questions: As we increase the number of deleted vertices, when will two nonsingleton strongly connected components appear? And how many singletons can we have if there is still only one nonsingleton strong component?

In the following section we will prove an asymptotically sharp upper bound for the number of vertices we can delete from $S_n$ without getting two nonsingleton strong components, and we also give sharp upper bounds on the number of possible singletons that we can obtain.

3. **Beyond tightly super-connected**

In $S_n$, every in- and out-degree is at least $\left\lfloor \frac{n-1}{2} \right\rfloor$, so if we pick $m$ pairwise nonadjacent vertices of $S_n$ and delete all their predecessors (or successors), altogether $m \left\lfloor \frac{n-1}{2} \right\rfloor$ vertices (not necessarily different), we will get at least $m$ singletons in the resulting graph. Similarly, since the girth of $S_n$ is six, the smallest nonsingleton strong component that we can obtain is a directed 6-cycle, which can be achieved if we delete all its predecessors (or successors), altogether $3n - 9$ vertices ($n \geq 3$). Our main result is that both of these examples are asymptotically sharp (the first only for the relevant case of $m \leq 6$):

**Theorem 3.1.** Part I: Let $T$ be a set of vertices of $S_n$ with $|T| \leq 3n - 16$. Then in $S_n - T$ we have exactly one nonsingleton strong component and at most five singletons.

Part II:

1. If $|T| \leq 2\left\lfloor \frac{n-1}{2} \right\rfloor - 2$ with $n \geq 6$, then $S_n - T$ has at most one singleton.
2. If $|T| \leq 3\left\lfloor \frac{n-1}{2} \right\rfloor - 4$ with $n \geq 6$, then $S_n - T$ has at most two singletons.
3. If $|T| \leq 4\left\lfloor \frac{n-1}{2} \right\rfloor - 5$ with $n \geq 8$, then $S_n - T$ has at most three singletons.
4. If $|T| \leq 5\left\lfloor \frac{n-1}{2} \right\rfloor - 7$ with $n \geq 12$, then $S_n - T$ has at most four singletons.

Before proving Theorem 3.1 we summarize some properties that a general digraph has if it has only one nonsingleton strong component:

**Lemma 3.2.** If a directed graph $G$ has only one strongly connected component $X$ that is not a singleton, then for every singleton $v \in V(G)$ either every predecessor of $v$ is a singleton or every successor of $v$ is a singleton.

**Proof.** If vertex $v$ has a predecessor that is not a singleton and a successor that is not a singleton, then both of these vertices belong to $X$, so there is an arc from $v$ to $X$ and another arc from $X$ to $v$, so $v$ belongs to the strongly connected component $X$, and cannot be a singleton. □
Corollary 3.3. Let $G$ be a digraph and let $T \subseteq V(G)$. If $G - T$ has only one nonsingleton strong component, then either every predecessor or every successor of every singleton of $G - T$ must also be singletons.

Now using the fact that $\overrightarrow{S_n}$ is bipartite and has no cycles of length at most five we get the following:

Lemma 3.4. Let $T$ be a subset of the vertices of $\overrightarrow{S_n}$. If $\overrightarrow{S_n} - T$ has exactly $m$ singletons and one nonsingleton strong component, then $|T| \geq m\left\lfloor\frac{n-1}{2}\right\rfloor - \binom{m}{2}$.

Proof. By Corollary 3.3, for every singleton $v$ in $\overrightarrow{S_n} - T$ either every predecessor or every successor of $v$ must be a singleton itself. If there is a directed path from the nonsingleton strong component to $v$ in $\overrightarrow{S_n} - T$, then count its successors, otherwise count its predecessors in $\overrightarrow{S_n}$ (these must be all singletons or vertices in $T$). Since in $\overrightarrow{S_n}$ the in- and out-degrees are at least $\left\lfloor\frac{n-1}{2}\right\rfloor$, overall we counted at least $m\left\lfloor\frac{n-1}{2}\right\rfloor$ vertices (they are not necessarily different), every one of which is either a singleton or belongs to $T$. Some of $v$’s counted predecessors or successors may have been counted unnecessarily (due to being singletons in $\overrightarrow{S_n} - T$) or several times (being predecessors or successors of several singletons in $\overrightarrow{S_n} - T$), call them overcounted vertices. Each overcounted vertex corresponds to two singletons, the possibilities are shown in Fig. 3, where $v$ and $w$ are the singletons, and their counted predecessors/successors are boxed. Since $\overrightarrow{S_n}$ has no triangles and 4-cycles, we get that in the first four cases this pair of singletons can’t correspond to any other overcounted vertices. In the last case we count the successors of one of the singletons, say $v$, so there is a directed path from the strong nonsingleton component to $v$. Then we also have a directed path from the strong nonsingleton component to the other singleton, $w$, so we chose to count the successors of $w$ as well, hence this case never occurs.

Thus each pair of singletons corresponds to at most one overcounted vertex, hence at least $m\left\lfloor\frac{n-1}{2}\right\rfloor - \binom{m}{2}$ of the counted vertices must be different and actually belong to $T$. \(\square\)

Note that for a fixed $n$ this bound first gets better then worse as $m$ increases, and for $m \geq n$ the bound is actually negative. Hence this lemma is useful only if we have an upper bound on the number of singletons.

We will also need the possible orientations of 6-cycles in $\overrightarrow{S_n}$:

Lemma 3.5. Every 6-cycle in $S_n$ is oriented in one of the two ways shown in Fig. 4.

Proof. It is easy to see that every 6-cycle in $S_n$ is generated by alternating $i$-edges and $j$-edges starting from a vertex for some $2 \leq i < j \leq n$. When $i$ and $j$ have the same parity, the edges are orientated away from the vertices on the 6-cycle having the same parity, so we get the orientation on the right in Fig. 4. When $i$ and $j$ have different parity, the two edges incident to a vertex in the 6-cycle are orientated differently (one towards this vertex, one away), so we get an orientated 6-cycle shown on the left in Fig. 4. \(\square\)
Using the structure of 6-cycles in $\overline{S_n}$ we can improve Lemma 3.4 for $m \geq 4$:

**Lemma 3.6.** Let $T$ be a subset of the vertices of $\overline{S_n}$ such that $\overline{S_n} - T$ has only one nonsingleton component and exactly $m$ singletons, where $m \geq 4$. Then $|T| \geq m\left\lfloor \frac{n-1}{2} \right\rfloor - \left\lceil \frac{m^2}{4} \right\rceil$.

**Proof.** From Lemma 3.4 we know that for $m = 3$ we have $|T| \geq 3\left\lfloor \frac{n-1}{2} \right\rfloor - 3$. This can only happen if for any two of the three singletons there is an overcounted vertex among their predecessors/successors, which implies that these three singletons lie on a 6-cycle of $\overline{S_n}$ (the other possibilities in Fig. 3 lead to shorter cycles). Notice that on any 6-cycle of $\overline{S_n}$ two types of edges alternate ($i$- and $j$-edges for some $i \neq j$), thus if two vertices of $\overline{S_n}$ have a common predecessor/successor, then there is a unique 6-cycle containing them.

Now let $\overline{S_n} - T$ have exactly four singletons. As in the proof of Lemma 3.4, the number of predecessors/successors of these singletons that must be either in $T$ or singletons themselves is at least $4\left\lfloor \frac{n-1}{2} \right\rfloor$ (with overcount), and there is at most one overcounted vertex for any pair of singletons. If there are three singletons (say $v$, $w$, and $x$) among which there are three overcounted vertices, then they must lie on a 6-cycle in $\overline{S_n}$ as observed above, and then the fourth singleton ($z$) can only have an overcounted vertex with one of them, otherwise we get two 6-cycles containing the same two singletons (see Fig. 5, vertices of $T$ are boxed). Hence there are at most four overcounted vertices overall. On the other hand, if there are at most two overcounted vertices among the predecessors/successors/any three singletons, then counting the number of overcounted vertices for all possible choices of three singletons we get at most $2 \cdot \binom{4}{3} = 8$ overcounted vertices, and each of them was counted exactly twice (there are exactly two ways to pick the third singleton), hence the number of overcounted vertices is at most $\frac{8}{2} = 4$. Thus $|T| \geq 4\left\lfloor \frac{n-1}{2} \right\rfloor - 4$.

Now let $a_m$ denote the bound we have on the number of overcounted vertices when $\overline{S_n} - T$ has $m$ singletons for $m \geq 4$, and assume that $\overline{S_n} - T$ has $m + 1$ singletons. Among any $m$ singletons there can be at most $a_m$ overcounted vertices, giving us a total overcount of at most $a_m \cdot (m + 1)$ if we consider all $m + 1$ possible ways to pick the $m$ singletons. Since every overcounted vertex corresponds to a pair of singletons, and there are $m - 1$ ways to choose the remaining $m - 2$ singletons out of the other $m - 1$ singletons, each overcounted vertex is counted exactly $\binom{m-1}{m-2} = m - 1$ times. Hence the number of overcounted vertices is at most $a_{m+1} = a_m \cdot \frac{m+1}{m-1}$. Since $a_4 = 4$, the solution for this recurrence is easily seen to be $a_m = \lceil \frac{m^2}{4} \rceil$ (see [13]), so we get that $|T| \geq m\left\lfloor \frac{n-1}{2} \right\rfloor - \left\lceil \frac{m^2}{4} \right\rceil$ when $\overline{S_n} - T$ has $m \geq 4$ singletons. \(\square\)

In the proof of our main theorem we will use the following upper bounds implied by these lemmas:

\[
|T| \geq \begin{cases} 
2 \left\lfloor \frac{n-1}{2} \right\rfloor & - 1 \text{ if } m = 2, \\
3 \left\lfloor \frac{n-1}{2} \right\rfloor & - 3 \text{ if } m = 3, \\
4 \left\lfloor \frac{n-1}{2} \right\rfloor & - 4 \text{ if } m = 4, \\
5 \left\lfloor \frac{n-1}{2} \right\rfloor & - 6 \text{ if } m = 5, \\
6 \left\lfloor \frac{n-1}{2} \right\rfloor & - 9 \text{ if } m = 6.
\end{cases}
\]
Again we emphasize that these bounds first get better then worse as \( m \) increases, hence we can apply them only if we have an upper bound on the number of singletons.

All these bounds are easily seen to be sharp for \( \overrightarrow{S_n} \) for \( n \geq 7 \). An example is shown in Fig. 6, which shows a subgraph of \( \overrightarrow{S_7} \). Six vertices and each edge type is indicated (\( i \) on a directed edge means it is an \( i \)-edge). If we choose the six labeled vertices to be singletons, and choose \( T \) to be all their successors, then there will be exactly nine overcounted vertices (shown boxed), so \( |T| = 6\lceil \frac{n-1}{2} \rceil - 9 \) (choosing fewer singletons will give the other bounds). Thus (1) shows that the (i)–(iv) are all sharp in Theorem 3.1.

Now we are able to prove the first part of our main theorem.

**Theorem 3.7 (Theorem 3.1, Part I).** If we delete at most \( 3n - 16 \) vertices from \( \overrightarrow{S_n} \), then the resulting graph will have at most six strongly connected components, and all but one of the components will be singletons.

**Proof.** We use induction on \( n \). The case \( n \leq 5 \) is trivial, since we are not deleting any vertices, and \( \overrightarrow{S_n} \) is strongly connected. For \( n = 6 \) we delete at most two vertices, so the claim follows from the fact that \( \overrightarrow{S_6} \) is tightly super-connected. Now assume the claim for \( \overrightarrow{S_{n-1}} \) and prove it for \( \overrightarrow{S_n} \) with \( n \geq 7 \). Assume we delete the vertices of \( T \) from \( \overrightarrow{S_n} \), where \( |T| \leq 3n - 16 \). Recall that \( H_i \) is the subgraph of \( \overrightarrow{S_n} \) with vertices having \( i \) in the last position, and define \( T_i = V(H_i) \cap T \). We consider several cases:

**Case 1:** \( |T_i| \leq 3n - 19 \) for every \( i, 1 \leq i \leq n \).

Since \( H_i \) is isomorphic to \( \overrightarrow{S_{n-1}} \), from the induction hypothesis we can conclude that every \( H_i \) has at most six strongly connected components, among which at most one is not a singleton. We claim that \( \overrightarrow{S_n} \) has at most one strongly connected component that is not a singleton.

First prove that the nonsingleton components of \( H_i - T_i \) for \( i = 1, 2, \ldots, n \) belong to a strongly connected component in \( \overrightarrow{S_n} - T \). Recall that originally there are \( (n-2)! \) edges between \( H_i \) and \( H_j \) for any \( i \neq j \), half of them going from \( H_i \) to \( H_j \), and the other half the other way round. As long as \( \frac{(n-2)!}{2} > 3n - 19 + 2 \cdot 5 \), at least one of these edges remain between the nonsingleton vertices of \( H_i - T_i \) and \( H_j - T_j \) in each direction, so they are strongly connected. Since for \( n \geq 7 \) this inequality holds, we get that the nonsingleton components of \( H_i - T_i \) for \( i = 1, 2, \ldots, n \) belong to one strongly connected component, call it \( Y \), of \( \overrightarrow{S_n} - T \).

Now look at a singleton \( v \) of \( H_i - T_i \), where \( 1 \leq i \leq n \). If \( v \) is not a singleton of \( \overrightarrow{S_n} - T \) and does not belong to the strong component \( Y \), then every other vertex in \( v \)'s strong component must be a singleton in one of the subgraphs \( H_j - T_j \) for some \( 1 \leq j \leq n \). This strong component must contain a cycle of even length with at least six vertices. Not all of these vertices come from the same subgraph \( H_j - T_j \), since it has at most five singletons. There is only one arc between a vertex of \( H_j \) and \( \overrightarrow{S_n} - V(H_j) \) (via an \( n \)-edge), which is directed from or towards \( H_j \) depending on the parity of the singleton, so each \( H_j - T_j \) contains an even number of vertices of this cycle. Hence either at least one \( H_j - T_j \) has four singletons and another has at least two singletons, or at least three \( H_j - T_j \)'s have at least two singletons each, and since each \( H_j \) is isomorphic to \( \overrightarrow{S_{n-1}} \), using (1) and checking all the cases we get that we deleted at least \( 6\lceil \frac{n-2}{2} \rceil - 6 \geq 3n - 19 \) vertices, so this case is impossible.

Hence singletons in \( H_i - T_i \) stay singletons in \( \overrightarrow{S_n} - T \). If overall we get at least six of them, then again we can use (1) to check the possibilities for the number of singletons in each \( H_i - T_i \) and conclude that we must have deleted...
at least $6\left\lfloor \frac{n-2}{3} \right\rfloor - 6 > 3n - 16$ vertices, contradiction (the worst case is when one $H_i - T_i$ contains five singletons and another contains one). Hence we can have at most five singletons in $\overline{S}_n - T$ apart from the strong component $Y$, proving the claim.

**Case 2:** $|T_i| \geq 3n - 18$ for some $i$.

Without loss of generality we may assume that $i = 1$. We delete at most two vertices of $\overline{S}_n - V(H_1)$, so it will stay strongly connected, let $Y$ denote the strong component in $\overline{S}_n - T$ containing $\overline{S}_n - (V(H_1) \cup T)$. We first show that any nonsingleton component $C$ of $H_1 - T$ belongs to $Y$. Since $C$ is not a singleton, it has at least six vertices, among them at least three odd and three even. In $\overline{S}_n$ each of these vertices has one arc going to or from $\overline{S}_n - V(H_1)$. We deleted at most two vertices of $\overline{S}_n - V(H_1)$, so at least one arc remains in each direction, so $C$ is part of $Y$.

Hence the only strong components that $\overline{S}_n - T$ can have apart from $Y$ are singletons of $H_1 - T_1$. We will show that we can have at most five such singletons.

Let $a \in V(H_1 - T_1)$ be a singleton in $\overline{S}_n - T$. Assume that $a$ is even, and there is a vertex $y \in V(\overline{S}_n - H_1)$ such that there is an arc from $a$ to $y$ (the case when $a$ is odd or the arc goes from $y$ to $a$ is similar and will be omitted). We will describe the structure of singletons in $H_1 - T_1$. Since $a$ does not belong to the strong component $Y$, there are two possibilities:

**Case 2a:** There is no directed path from $Y$ to $a$ in $\overline{S}_n - T$.

Since $Y$ is the only nonsingleton component in $\overline{S}_n - T$, any predecessor of $a$ in $H_1$ must either belong to $T_1$ or be another singleton itself. If $b \in V(H_1 - T_1)$ is a predecessor of $a$, then since $b$ is odd, there is a vertex $z \in \overline{S}_n - V(H_1)$ such that we have an arc from $z$ to $b$. Since there is no directed path from $Y$ to $a$ in $\overline{S}_n - T$, the vertex $z$ must be in $T$. Since $|T - T_1| \leq 2$, we get that all but at most two predecessors of $a$ are in $T_1$.

First assume that exactly two such predecessors are not in $T_1$, say $b$ and $c$, so their predecessors in $\overline{S}_n - V(H_1)$ both belong to $T$ (call them $z$ and $w$, these must be different, otherwise we get a cycle of length $4$). The remaining predecessors of $b$ and $c$ are all in $H_1$, and each of them must either belong to $T_1$ or be a singleton itself (otherwise it belongs to $Y$, and we get a directed path from $Y$ to $a$). If $d_1$ is a predecessor of $b$ or $c$ that is also a singleton, then since $a$ is even, $d_1$ is even, too, and we claim that all its predecessors must be in $T_1$. If not, say $e$ is a predecessor of $d_1$ in $H_1$ and $e \notin T_1$, then since $e$ is odd, it has a predecessor in $\overline{S}_n - V(H_1)$, which cannot be $z$ or $w$, since that would create a 4-cycle or a wrongly oriented 6-cycle (e.g. $e \rightarrow d_1 \rightarrow b \rightarrow a \rightarrow c \rightarrow w \rightarrow e$). Hence this predecessor of $e$ is not in $T$, so it belongs to $Y$, and we get a directed path from $Y$ to $a$ (through $e \rightarrow d_1 \rightarrow b \rightarrow a$), contrary to our assumption. Thus the structure of singletons is similar to the graph shown in Fig. 7 (vertices in $T$ are drawn in boxes, vertices of $\overline{S}_n - V(H_1)$ are on the right, $b$ and $c$ may have more or fewer singleton predecessors in $H_1 - T_1$). Notice that all of the vertices drawn in the picture must be different, otherwise we get either a wrongly oriented 6-cycle or a cycle whose length is less than six (the longest possible cycle is obtained if e.g. $d_1$ and $d_3$ have a common predecessor, say $e$, then we get the cycle $e \rightarrow d_1 \rightarrow b \rightarrow a \rightarrow c \rightarrow d_3 \rightarrow e$).

Second, assume that there is exactly one predecessor of $a$ that is not in $T_1$, call it $b$. Again, $b$ has one predecessor $z$ in $\overline{S}_n - V(H_1)$, which must be in $T$, and each of $b$'s other predecessors in $H_1$ belongs to $T_1$ or is a singleton itself. If $c_1$ is such a singleton, then since it is even, all its predecessors are in $H_1$, so each must belong to $T_1$ or be a singleton itself. If $d$ is a predecessor of $c_1$ not in $T_1$, then since $d$ is odd, it has a predecessor $w$ in $\overline{S}_n - V(H_1)$, which must be in $T$, otherwise we get a directed path from $Y$ to $a$ ($w \neq z$ otherwise we get a 4-cycle). Since now we have two vertices of $T$ in $\overline{S}_n - V(H_1)$, we can continue the same way as in the previous case and conclude that every predecessor of $d$ in $H_1$ is either in $T_1$ or must be a singleton in $H_1 - T_1$, and if $c_1 \in V(H_1 - T_1)$ is a predecessor of $d$, then all predecessors of $c_1$ must be in $T_1$. So the structure of singletons is similar to the graph shown in Fig. 8 ($b$ and $d$ may have more or fewer singleton predecessors in $H_1 - T_1$). Again it is easy to see that all vertices shown must be different, otherwise we either get a wrongly oriented 6-cycle or a cycle of length at most four.
Case 2b: There is no directed path from \( a \) to \( Y \) in \( \overrightarrow{S_n} - T \).

Here we immediately get \( y \in T \). Now every other successor of \( a \) (they are all in \( H_1 \)) must belong to \( T_1 \) or be a singleton itself. If \( b_1 \) is such a singleton, it is odd, so all its successors are in \( H_1 \), thus each of them either belongs to \( T_1 \) or is a singleton itself. If \( c \in V(H_1 - T_1) \) is a successor of \( b_1 \), then it has a successor in \( \overrightarrow{S_n} - V(H_1) \), call it \( z \), which must be in \( T \), otherwise we get a directed path from \( a \) to \( Y \). Now we have two vertices of \( T \) in \( \overrightarrow{S_n} - V(H_1) \), so by continuing the same way we can conclude that every successor of \( c \) in \( H_1 \) is either in \( T_1 \) or must be a singleton in \( H_1 - T_1 \), and if \( d_1 \in V(H_1 - T_1) \) is a successor of \( c \), then all its successors must be in \( T_1 \). So the structure of singletons is similar to the graph shown in Fig. 9 (\( a \) and \( c \) may have more or fewer singleton successors in \( H_1 - T_1 \)).

The vertices shown again must be different, otherwise we get a wrongly oriented 6-cycle or a cycle of length at most four.

Now pick a singleton \( a \) in \( H_1 - T_1 \) and build the structure of predecessors or successors as in Figs. 7 and 8, or Fig. 9 (if we can use both the predecessors and the successors, chose one of them arbitrarily). Call such a structure the block of \( a \) (this includes their predecessors/successors indicated in the pictures). Note that the blocks of two singletons may have singletons in common. Let \( k \) denote the number of predecessors of any vertex in \( H_1 \) for Figs. 7 and 8, or the number of successors in \( H_1 \) for Fig. 9 (we have either \( k = \lfloor \frac{n-2}{2} \rfloor \) or \( k = \lceil \frac{n-2}{2} \rceil + 1 \), and consider one of these structures with exactly \( m \) singletons. Since in the block of \( a \) the vertices shown are all different, we get at least \( mk - (m-1) = m(k-1) + 1 \) different vertices of \( T_1 \) (count each successor/predecessor of each singleton to get \( mk \) different vertices, out of which \( m - 1 \) vertices are singletons, the rest are in \( T_1 \)). Since \( T_1 \leq 3n - 16 \leq 6k - 10 \), we get \( m(k-1) + 1 \leq 6k - 10 \), so \( m \leq \frac{6k - 11}{k-1} \leq 6 \). Thus each block of a singleton can have at most five singletons in \( H_1 - T_1 \).

Assume now that we have at least six singletons in \( H_1 - T_1 \). Choose blocks of singletons one by one until overall they contain at least six singletons. Each time we choose a new block of a singleton, stop at the singletons that already
Lemma 3.6
Theorem 3.1
If 
and 
with the upper bounds for 
2.2
Theorem 3.7
improve as the number of singletons increases up to five, so 
is sharp. The only gap
Fig. 4
Theorem 3.1 implies that
Theorem 3.1
(1)
Theorems 2.1
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Proof. It is easy to check that in each case we have \(|T| \leq 3n - 16\), so Theorem 3.7 implies that \(\overline{S_n} - T\) has at most five singletons. Notice that for \(n \geq 6\) the bounds from (1) improve as the number of singletons increases up to five, so combining (1) with the upper bounds for \(|T|\) we immediately get the results for the number of possible singletons for each case. 

Finally we prove the second part of Theorem 3.1:

Theorem 3.8 (Theorem 3.1, Part II). Let \(T\) be a set of vertices of \(\overline{S_n}\).
(i) If \(|T| \leq 2\lfloor \frac{n-1}{2} \rfloor - 2\) with \(n \geq 6\), then \(\overline{S_n} - T\) has at most one singleton.
(ii) If \(|T| \leq 3\lfloor \frac{n-1}{2} \rfloor - 4\) with \(n \geq 6\), then \(\overline{S_n} - T\) has at most two singletons.
(iii) If \(|T| \leq 4\lfloor \frac{n-1}{2} \rfloor - 5\) with \(n \geq 8\), then \(\overline{S_n} - T\) has at most three singletons.
(iv) If \(|T| \leq 5\lfloor \frac{n-1}{2} \rfloor - 7\) with \(n \geq 12\), then \(\overline{S_n} - T\) has at most four singletons.

Proof. Note that when \(n\) is even, we have \(6\lfloor \frac{n-1}{2} \rfloor - 10 = 3n - 16\), so the first part of Theorem 3.1 is sharp. The only gap occurs when \(n\) is odd, and although we believe that the bound \(6\lfloor \frac{n-1}{2} \rfloor - 10 = 3n - 13\) is sharp instead of our proven bound of \(3n - 16\), any proof is likely to be very complicated. And even though six singletons can appear if we delete \(6\lfloor \frac{n-1}{2} \rfloor - 9\) vertices, it is not clear if that is enough to create two nonsingleton strong components; one may need to delete at least \(3n - 9\) vertices as mentioned earlier.

Also notice that Theorem 3.1(i) for \(n \geq 6\) is a stronger result than the tightly super-connectedness of \(\overline{S_n}\) in Theorem 2.2.

4. Conclusion

We examined several questions coming from the loosely and tightly super-connectedness of \(\overline{S_n}\). We gave an asymptotically sharp bound on the number of vertices that we can delete without getting two nonsingleton strongly connected components and obtained sharp upper bounds for the number of singletons that we can get.

Note that we only used the general properties of \(\overline{S_n}\), namely the properties listed in Theorems 2.1 and 2.2, the possible orientations of 6-cycles shown in Fig. 4, and the fact that two 6-cycles can have at most two consecutive vertices in common. Hence our results can be applied for any graph having these properties.

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References