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A VARIATIONAL APPROACH TO THE STEINER NETWORK PROBLEM

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Abstract

Suppose n points are given in the plane. Their coordinates form a $2n$-vector X. To study the question of finding the shortest Steiner network S connecting these points, we allow X to vary over a configuration space. In particular, the Steiner ratio conjecture is well suited to this approach and short proofs of the cases $n = 4, 5$ are discussed. The variational approach was used by us to solve other cases of the ratio conjecture ($n = 6$, see [11] and for arbitrary n points lying on a circle). Recently, Du and Hwang have given a beautiful complete solution of the ratio conjecture, also using a configuration space approach but with convexity as the major idea. We have also solved Graham's problem to decide when the Steiner network is the same as the minimal spanning tree, for points on a circle and on any convex polygon, again using the variational method.

1. Introduction

Let $x_1, x_2, \ldots, x_n$ denote n points, not necessarily distinct, in the plane $\mathbb{R}^2$. A Steiner minimal tree S is a network of shortest possible length connecting these points. There is an algorithm, due to Melzak [8], for finding such an S (which may not be unique). However, determining S has been shown to be an NP-complete problem [3] and is usually referred to as the Steiner network problem.

In this paper, we introduce a new approach to Steiner networks. Let $X = (x_1, x_2, \ldots, x_n)$ be the $2n$-tuple of coordinates in $\mathbb{R}^{2n}$. We will call X the configuration. By perturbing X, simple ideas from the calculus of variations can be applied to various questions relating to S.

Let T be a shortest tree connecting the points of X but with vertices only at $x_1, x_2, \ldots, x_n$. T is called a minimal spanning tree and there is a well-known algorithm due to Prim [10] and Kruskal [6] for finding T in polynomial time in n. Let L_S, L_T denote the (total) lengths of S, T, respectively, and let $\rho = L_S/L_T$. ρ is called the Steiner ratio. Gilbert and Pollak [4] conjectured that $\rho \geq \sqrt{3}/2$, and this has been shown to be true for $n = 3$ [4], 4 [9,2], and 5 [1].

In section 2, we formulate this Steiner ratio conjecture as a variational problem. The case $n = 3$ is proved and some useful variations are established. In section 3, a short proof of the case $n = 4$ is given and, similarly, the argument for $n = 5$ is discussed in section 4. In a subsequent paper [11], we prove the ratio conjecture for $n = 6$ by similar techniques.

A variational method for attacking the *Steiner ratio conjecture* is promising for a number of reasons, of which two principal reasons are the following:

(i) If you have a Steiner tree, and move an end point along it, the length of the corresponding minimal tree (without Steiner points) changes in a way which is very easily computable. In fact, other kinds of motions of a vertex also lead to easy control of the way in which the Steiner ratio changes.

(ii) In the process of minimising ρ, one ends up with a multiplicity of minimal trees with many line segments of equal length. The set of all edges in these minimal trees is a very nice object to work with.

2. Variations and the Steiner ratio

To solve the Steiner ratio conjecture, it suffices to look at the case where S is a *full* Steiner tree, i.e. has $2n-2$ vertices (cf. [4]). In this case, n of the vertices are the points of X and the other $n-2$ vertices have 3 incident edges meeting at 120° angles. The latter are called Steiner vertices.

A choice of S is often called a *topology*. It is easy to check that we can parametrize the configuration X by the lengths $(y_1, y_2, \ldots, y_{2n-3})$ of the $2n-3$ edges of S. This corresponds to the fact that rigid motions (translations, rotations and reflections) of the plane reduce the number of coordinates (degrees of freedom) of the configuration by 3. Also, by a homothety (change of scale) it can be assumed, if convenient, that $\sum_{i=1}^{2n-3} y_i = 1$. Consequently, the configuration space

$$\Delta = \{Y = (y_1, y_2, \ldots, y_{2n-3}) : \sum_{i=1}^{2n-3} y_i = 1,\ y_i \geq 0 \text{ for all } i\}$$

is a $(2n-4)$-dimensional simplex and is compact.

Our objective is to study the behaviour of the Steiner ratio ρ on Δ. The Steiner ratio conjecture is equivalent to showing $\rho \geq \sqrt{3}/2$ on Δ for all choices of topology for S. Let $T_1, T_2, \ldots, T_k$ be all possible spanning trees for X. For the remainder of the paper, we will fix the topology of S. Then $\rho = L_S/\min L_T$ is continuous and Gâteaux differentiable, i.e. possesses a differential $D\rho(v)$ in the direction of any vector v at Y. (Y is usually clear from the context.) In fact,

$$D\rho(v) = \sum_{h \to 0} \frac{1}{h} \left(L_S(Y+hv)/L_{T_j}(Y+hv) - L_S(Y)/L_{T_j}(Y) \right),$$

where j is chosen so that $L_{T_j} = \min\{L_{T_1}, L_{T_2}, \ldots, L_{T_k}\}$ for all points $Y + hv$, for h sufficiently small. In other words, T_j is the minimal spanning tree with length decreasing fastest (or increasing slowest) in the direction of v at Y. $D\rho(v)$ is continuous in Y and v.