Abstract—The multiplicity of the fractional Fourier transform (FRT), which is intrinsic in any fractional operator, has been claimed by several authors, but never systematically developed. The paper starts with a general FRT definition, based on eigenfunctions and eigenvalues of the ordinary Fourier transform, which allows us to generate all possible definitions. The multiplicity is due to different choices of both the eigenfunction and the eigenvalue classes. A main result, obtained by a generalized form of the sampling theorem, gives explicit relationships between the different FRT’s.

Index Terms—Fourier transform, fractional Fourier transform, sampling theorem.

I. INTRODUCTION

Since its inception in 1980 [1], [2] and rediscovery in 1993 [3]–[5], the fractional Fourier transform (FRT) has become a fundamental tool for optical information processing and has seen various developments in fiber optics [6], bulk optics [4], [7], [8], and signal processing [9]. As expected from any fractional operator definition, various forms of the FRT have emerged from the literature, but apart from some isolated attempts, no systematic treatment of the inherent multiplicity can be found.

The customary FRT of a continuous-time signal \( s(t) \) has the form

\[
S_\alpha(f) = \frac{1}{\sqrt{1 - \alpha^2}} K_\alpha \sum_{n=-\infty}^{\infty} s(t) e^{-j2\pi \alpha n^2 f^2} dt
\]

where \( \alpha \) is the “fraction,” and \( B_\alpha = \cos((\pi/2)\alpha) \), \( C_\alpha = \cos((\pi/2)\alpha) \), \( K_\alpha = \sqrt{1 - \alpha^2} \), where \( z^2 \) denotes the complex fourth root \( z \), with \(-\pi/4 < \arg z < \pi/4\). In particular, for \( \alpha = 1 \), the FRT becomes the ordinary Fourier transform (FT)

\[
S(f) = \int_{-\infty}^{\infty} s(t) e^{-j2\pi ft} dt = S(f)
\]

and for \( \alpha = 0 \), the kernel in (1) degenerates into the delta function \( \delta(f - t) \) so that the FRT is the signal itself \( S_0(f) = s(f) \).
FRT’s with the aim of showing how broad the class of FRT’s is and how they all ultimately relate to the standard forms. Finally, the dependence on the eigenfunctions is discussed in Section VI.

II. GENERAL FRT DEFINITION

In this section, we apply the general definition given in [16] to continuous-domain signals. Let \( \mathcal{F}^a \) be an operator generating an FRT, i.e., an operator that for any given “fraction” \( \alpha \in \mathbb{R} \) maps a signal \( s(t) \), \( t \in \mathbb{R} \) into an FRT \( S_\alpha(f) \), \( f \in \mathbb{R} \). \( \mathcal{F}^a \) will be an FRT operator if it

1) is linear;
2) verifies the FT condition \( \mathcal{F}^a = \mathcal{F} \);
3) has the additive property \( \mathcal{F}^{a+b} = \mathcal{F}^a \mathcal{F}^b \) for every choice of \( a \) and \( b \).

As stated in [16], these constraints assure certain fundamental properties for the operator \( \mathcal{F}^a \), such as satisfying the marginal conditions \( \mathcal{F}^0 = \mathcal{F}^1 = \mathcal{I} \), \( \mathcal{F}^2 = \mathcal{I}^- \), \( \mathcal{F}^3 = \mathcal{F}^- \), where

\( \mathcal{I} \) identity operator;
\( \mathcal{I}^- \) reflection operator;
\( \mathcal{F}^- \) ordinary inverse Fourier transform.

The constraints also imply periodicity in \( a \), with period 4 \( \mathcal{F}^{a+4} = \mathcal{F}^a \). From 1), the FRT operator can be written, with a suitable kernel \( \psi_0(f, t) \), as

\[
\mathcal{F}^a: \quad S_\alpha(f) = \int_{-\infty}^{+\infty} \psi_\alpha(f, t)s(t) \, dt. \tag{4}
\]

In terms of the kernel, the FT property 2) is expressed as

\[
\psi_\alpha(f, t) = e^{-i2\pi \alpha f}. \]

Moreover, the kernel becomes

\[
\delta(f - t) \quad \text{when} \quad a = 0;
\delta(f + t) \quad \text{when} \quad a = 2;
\]

and is periodic in \( a \) with period 4. Both the CFRT (1) and the WFRT (2) belong to this class as their respective kernels are

\[
\psi_\alpha(f, t) = \begin{cases} 
K_\alpha e^{-i\pi(B_\alpha(f^2 + t^2) - 2C_\alpha ft)} & \text{if } \alpha \text{ is even}; \\
K_\alpha e^\pi(B_\alpha(f^2 + t^2) - 2C_\alpha ft) & \text{if } \alpha \text{ is odd};
\end{cases} \tag{5a}
\]

\[
\psi_\alpha(f, t) = i_0(a)\delta(f - t) + i_3(a)e^{-i2\pi \alpha f t} + i_3(a)\delta(f + t) + i_0(a)e^{i2\pi \alpha f t}. \tag{5b}
\]

Both satisfy the additive property and the marginal conditions: the former through the structure of the parameters \( K_\alpha, B_\alpha, \) and \( C_\alpha \) and the latter through the properties of the weights \( i_n(a) \).

We shall see that the whole kernel class can be generated by the eigenfunctions of the ordinary FT, which we now examine in detail.

A. Basis of Eigenfunctions and Diagonalization of Operator \( \mathcal{F} \)

Following the approach of [16], we consider a complete orthonormal basis of FT eigenfunctions \( \{\varphi_n(t)\}_{n \in \mathbb{N}} \) with \( \int_{-\infty}^{+\infty} \varphi_n(t)\varphi_m(t) \, dt = \delta_{nm} \). We assume the basis to be ordered so that we can write the corresponding FT eigenvalues as

\[
\mu_n = e^{-\pi(\pi/2)^n} \in \{1, -j, j, -1\}. \tag{6}
\]

Such a basis permits a signal expansion of the form

\[
s(t) = \sum_{n=0}^{\infty} S_n \varphi_n(t), \quad t \in \mathbb{R} \tag{7}
\]

whose coefficients \( S_n \) are given by

\[
S_n = \int_{-\infty}^{+\infty} s(t)\varphi_n(t) \, dt. \tag{8}
\]

Considering that the FT of \( \varphi_n(t) \) is \( \mu_n \varphi_n(f) \), from (7), we get the expansion

\[
S(f) = \sum_{n=0}^{\infty} \mu_n S_n \varphi_n(f), \quad f \in \mathbb{R}. \tag{9}
\]

Inserting (8), we find

\[
S(f) = \sum_{n=0}^{\infty} \mu_n \varphi_n(f) \int_{-\infty}^{+\infty} s(t)\varphi_n(t) \, dt \tag{10}
\]

which, compared with (4), yields the expansion for the FT kernel

\[
\psi(f, t) = \sum_{n=0}^{\infty} \mu_n \varphi_n(f)\varphi_n(t). \tag{11}
\]

The above relationships lead to the decomposition of the FT operator as in

\[
\mathcal{F} = \mathcal{U} \mathcal{D} \mathcal{U}^*, \tag{12}
\]

where

\[
\mathcal{U}^*: \quad S_n = \int_{-\infty}^{+\infty} s(t)\varphi_n^*(t) \, dt
\]

\[
\mathcal{D}: \quad Y_n = \mu_n \varphi_n
\]

\[
\mathcal{U}: \quad S(f) = \sum_{n \in \mathbb{N}} Y_n \varphi_n(f). \tag{13}
\]

A “signal theory” interpretation of (12) is given in Fig. 1. The operator \( \mathcal{U}^* \) yields the coefficients \( S_n \) for the signal expansion into a series of eigenfunctions. As it works on a continuous-argument signal and produces a discrete sequence, it can be thought of as a sampling-like operator. The operator \( \mathcal{D} \) is diagonal and performs a simple multiplication of the sequence \( \{S_n\} \) by the eigenvalues \( \{\mu_n\} \). Finally, \( \mathcal{U} \) converts the sequence \( \{Y_n\} \) into the continuous-argument transform \( S(f) \) and can therefore be thought of as an interpolation-like operator. It can be easily seen that the operators \( \mathcal{U} \) and \( \mathcal{U}^* \) are adjoints of each other (hence justifying our notation). Moreover, since the eigenfunctions \( \varphi_n(t) \) are orthonormal, they are also the inverse of each other, i.e., they are unitary operators, in symbols \( \mathcal{U}^* \mathcal{U} = \mathcal{U} \mathcal{U}^* = \mathcal{I} \), with \( \mathcal{I} \) the identity operator. Since all the eigenvalues \( \mu_n \) have unit amplitude, the operator \( \mathcal{D} \) is unitary as well.

Since \( \mathcal{F} \) is unitary, (12) represents the diagonalization of the FT operator. From the theory of linear operators in Hilbert
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Fig. 1. Decomposition of the FT operator $F$ on $R$.

spaces [17], [18], we know that a unitary operator with only a point spectrum can be decomposed in the form $UDU^a$, where $U$ is determined by the eigenfunctions and $D$ by the corresponding eigenvalues. Moreover, the decomposition is unique if, and only if, all the eigenvalues are distinct. This is not the case with the operator $F$, which has only four distinct eigenvalues. The decomposition (12) is thus not unique, and we can find other orthogonal bases $\{\phi_n\} \rightarrow U$ such that $F = UDU^a$.

The best known basis for $F$ is given by the Hermite–Gauss (HG) functions [1], [19]

$$\phi_n(t) = \frac{\phi_\sqrt{a}}{\sqrt{\pi^2 n!}} H_n \left(\sqrt{2\pi} t \right) e^{-\pi t^2}, \quad n \in \mathbb{N}$$ (14)

where $H_n(t) = (-1)^n e^t d^n e^{-t^2}/dt^n$ is the $n$th-order Hermite polynomial. Recently, another basis has been identified [20], whose lowest order functions are

$$\begin{align*}
\phi_0(t) &= \text{sech} \pi t, \\
\phi_1(t) &= \pi \text{sech}^2 \pi t (2t \cosh \pi t + \sinh \pi t), \\
\phi_2(t) &= \text{sech}^3 \pi t \left[ \left( \frac{n}{2} - \pi + \frac{n^2+2}{2} \right) (1 + \cosh 2\pi t) + 2\pi \sinh \pi t - n^2 \right].
\end{align*}$$ (15)

Different bases can be obtained by linear combinations of a given basis. Specifically (see Section VI and Appendix E), we have the following theorem.

**Theorem 1:** Given an orthonormal basis of FT eigenfunctions $\{\phi_n(t)\}_{n \in \mathbb{N}}$ with associated eigenvalues $(-j)^n$, every other orthonormal basis of FT eigenfunctions $\{\phi_n(t)\}_{n \in \mathbb{N}}$ is obtained by

$$\tilde{\phi}_{3n+h}(t) = \sum_{m \in \mathbb{N}} \alpha_{3n+h,4m+h} \phi_{4m+h}(t)$$

$$n \in \mathbb{N}, \quad h = 0, 1, 2, 3$$ (16)

where the coefficients $\alpha_{3n+h,4m+h} \in \mathbb{C}$ satisfy the four-block unitary condition

$$\sum_{m \in \mathbb{N}} \alpha_{3n+h,4m+h}^* \alpha_{3m+h,4n+h} = \delta_{kn}$$

$$k, n \in \mathbb{N}, \quad h = 0, 1, 2, 3.$$ (17)

**B. Construction of a General FRT**

Starting from the decomposition $F = UDU^a$, we can construct an FRT operator by replacing the diagonal operator $D$ by its $a$th power, namely

$$F^a = UDU^a$$ (18)

where $D^a$ is the diagonal operator [see (13b)] $D^a: Y_n = \mu_n^a s_n$. The operator $F^a$ can be interpreted by the scheme of Fig. 1. Substituting $D^a$ for $D$, the scheme produces the FRT $S_a = F^a s$ of the input signal $s(t)$. We can see immediately that the $F^a$ defined by (18)

1) is linear;  
2) verifies the FT condition;  
3) has the additive property.

In fact, considering that $\mu_n^{a+b} = \mu_n^a \mu_n^b$, we have

$$\begin{align*}
F^{a+b} &= UDU^a D^b U^a = UDU^a U^a U^a = F^{a+b}
\end{align*}$$

where $U^a = I$. Moreover, we have the following.

a) Since $F^{a} F^{a} = I$, the identity operator, the inverse FRT operator is $F^{-a}$.

b) Consider that $|\mu_n^a| = 1$, $D^a$ is unitary; therefore, $F^a$ is unitary.

c) Equation (18) is a diagonalization of the unitary operation $F^a$.  
d) Since $F^{a}$ is unitary for every $a$, the Parseval relation $\int_{-\infty}^{\infty} |s(f)|^2 df = \int_{-\infty}^{\infty} |s(t)|^2 dt$ holds for every $a$.

e) The basis $\{\phi_n(t)\} \rightarrow U$ consists of eigenfunctions of the FRT, with eigenvalues $\{\mu_n^a\} \rightarrow D^a$.

To find an FRT kernel, it is thus sufficient to replace the eigenvalues $\mu_n$ in (11) by their $a$th power, to give

$$F^a: \quad \psi_a(f, t) = \sum_{n=0}^{\infty} \mu_n^a \phi_n(f) \phi_n^a(t), \quad f, t \in \mathbb{R}.$$ (19)

**C. FRT Multiplicity, Generating and Perturbing Sequences**

The multiplicity of the FRT is twofold: First, a real power of a complex number $\mu_n$ is not unique. Considering (6), the possible values of $\mu_n^a$ are given by

$$\mu_n^a = e^{-j(\pi/2)(n+q_n)a}, \quad q_n \in \mathbb{Z}$$ (20)

where $q_n$ is an arbitrary sequence of integers. Different choices of $q_n$ lead to different kernels and, hence, to different FRT definitions. We shall call the function

$$g_n = n + 4q_n, \quad n = 0, 1, 2, \cdots$$ (21)

the generating sequence (GS) of the FRT. A GS can be expressed in the alternative form:

$$g_n = (n)_4 + 4p_n$$ (22)

where the $p_n$ are also arbitrary integers, and the two forms are related by $p_n = [n/4] + q_n$. Table I collects examples of GS’s, some of which may be of doubtful interest for applications, but are introduced to illustrate the wide variety of FRT’s.

The second multiplicity arises because, as stated by Theorem 1, the basis is not unique. Its effect on the FRT depends on the fraction $a$ and on the GS. This will be discussed in Section VI, where we will take the HG basis defined by (14) for reference and determine any other basis with the infinite-dimensional matrix $\{\alpha_{m,n}\}$ of Theorem 1. We will also call $\{\alpha_{m,n}\}$ a perturbing sequence (PS), with respect to the HG basis, for which $\alpha_{m,n} = \delta_{mn}$.
In conclusion, an FRT operator $\mathcal{F}$ is unambiguously determined by choosing a GS $g$ and a PS $\alpha$, which may be indicated by rewriting (18) in the form

$$\mathcal{F} = \mathcal{U}(\alpha) \mathcal{D}(g) \mathcal{P}(\alpha).$$  \hfill (23)

Once the ambiguity of $\mu_{\alpha}^a$ has been resolved by choosing a specific GS and the basis has been fixed, by choosing a specific PS, the FRT expression becomes unique, and the kernel is determined by means of (19). In Namias [1], the implied choices were the HG basis and $g_n = n$, which lead to the CFRT (see Section IV). The choice $g_n = (n)_A$, leads to the WFRT (2), which does not depend on the basis. We will see in Section IV that the choice $g_n = (n)_A$ permits generalizing the WFRT to any order $4L$.

### D. Properties of the General FRT

It is well known that many properties of the FRT are quite different from those of the ordinary transform [3]. Some of the properties that hold for every FRT were listed above in 1)–3) and a)–e). Further properties can be achieved with specific PS's and GS's. If the basis consists of real eigenfunctions (such as HG functions), the kernel satisfies (22); for a real signal $s(t)$, this gives

$$S_m(f) = S^*_m(f),$$

are independent of the fraction $\alpha$. As a consequence, the expression of the FRT becomes

$$S_{\alpha}(f) = \sum_{m \in G} X_m(f) e^{-j(\pi/2)ma}$$

where the terms

$$X_m(f) = \int_{-\infty}^{+\infty} s(t) U_m(f, t) dt$$

are also independent of $\alpha$.

The main problem is to find a closed-form solution of (26), i.e., a result expressed with a finite number of terms, whereas the above expression has infinite terms. A numerical evaluation would be of doubtful interest as it would give no insight. Note that the problem of finding a closed-form evaluation persists, even when the cardinality $|G|$ is finite. In fact, although (26) has a finite number of terms, it turns out that some subkernels (27) are expressed as infinite series. Further problems are the interrelationships between FRT's generated by different GS's and different PS's. In the next three sections, we assume that the basis is given. The effect of changing the basis is discussed in Section VI.

### III. ANALYSIS IN THE DOMAIN OF THE FRACTION

We solve two of the above problems by investigating the behavior of the kernel $\psi_{\alpha}(f, t)$ in the domain of the fraction $\alpha$ for the different GS's $g_n$. For this, we write the kernel (24) in the abbreviated form

$$u(\alpha) \triangleq \psi_{\alpha}(f, t) = \sum_{n=0}^{\infty} B_n e^{-j2\pi g_n F_\alpha}$$

(30)

where $B_n = \varphi_n(f) \varphi^*_n(t)$ and $f$ and $t$ are considered fixed. For clarity, we denote the general period of $u(\alpha)$ by $T_\alpha$ and its general frequency by $F$, keeping in mind that $T_\alpha = 4$ and $F = 1/4$. The periodic "signal" (30) can also be written as a Fourier series [see (26)]

$$u(\alpha) = \sum_{n=-\infty}^{\infty} U_m e^{-j2\pi m F_\alpha} = \sum_{m \in G} U_m e^{-j2\pi m F_\alpha}$$

(31)
whose Fourier coefficients are given by

\[ U_m = \sum_{n \in \mathcal{N}_m} B_n, \quad \mathcal{N}_m = \{n | g_n = m\} \]  

(32)

and \( U_m = 0 \) when \( \mathcal{N}_m \) is empty, i.e., for \( m \notin \mathcal{G} \). Note that the coefficients \( B_n \) are independent of the GS \( g_n \), whereas the Fourier coefficients \( U_m \) depend on \( g_n \).

Inspection of (31) reveals the role of the cardinality \( |\mathcal{G}| \); it gives the number of degrees of freedom of the dependence on the fraction \( a \), i.e., the number of harmonics that are present in the signal \( u(a) \). In other words, the kernel (26), with respect to the fraction \( a \), is bandlimited for a WFRT and not bandlimited for a CFRT.

### A. Sampling Theorem

It is well known that a periodic signal containing only \( N \) consecutive harmonics can be perfectly recovered from \( N \) equally spaced samples within a period \( T_p \). To formulate a more general version of this theorem, we need the following preliminaries.

The harmonic support of a periodic signal of the form (31) is defined by \( \sigma(u) = \{m | U_m \neq 0\} \). The support can be obtained starting from the kernel form (30) of the periodic signal and, more specifically, is given by the image of the GS \( \sigma(u) = \{g_n | h_n \in \mathbb{N}\} = \mathcal{G} \). For instance, with \( g_n = (n)_{\mathbb{N}} \), we find \( \sigma(u) = \{0, 1, \ldots, 7\} = \mathcal{G} \).

A unit cell of \( \mathbb{Z} \) of size \( N \) is any subset \( \mathcal{C} \) of \( \mathbb{Z} \) with \( N \) elements such that the sets \( \mathcal{C} + hN \subseteq \{c + kN | c \in \mathcal{C}\} \) represent a partition of \( \mathbb{Z} \), i.e.,

\[ \bigcup_{h=-\infty}^{+\infty} \{c + hN\} = \mathbb{Z}, \quad \{c + hN\} \cap \{c + kN\} = \emptyset, \quad h \neq k. \]  

(33)

For example, \( \mathcal{C} = \{0, 1, \ldots, 7\} \) is a unit cell of size 8, and \( \mathcal{C} + m_0 = \{m_0, m_0 + 1, \ldots, m_0 + 7\} \) are unit cells of size 8 for every choice of \( m_0 \in \mathcal{C} \). A unit cell may consist of nonconsecutive integers, e.g., \( \mathcal{C} = \{0, 2, 3, 5, 6, 7, 9, 48\} \) is again a unit cell of size 8.

Bandlimitation of a periodic signal is stated in terms of its harmonic support \( \sigma(u) \), but the formulation of the sampling theorem requires the choice of a specific unit cell containing the support.

**Theorem 2:** Let \( u(a) \) be a periodic signal of period \( T_p \) with a limited harmonic support \( \sigma(u) \), and let \( \mathcal{C} \) be a unit cell of size \( N \) containing \( \sigma(u) \), i.e., \( \mathcal{C} \supseteq \sigma(u) \). Then, the signal can be perfectly recovered from \( N \) samples \( u(\alpha T) \), with \( T = T_p/N \), according to

\[ u(a) = \sum_{\nu=0}^{N-1} u(\nu T) \hat{c}(\alpha - \nu T) \]  

(34)

where the interpolating function is

\[ \hat{c}(\alpha) = \frac{1}{N} \sum_{m \in \mathcal{C}} e^{-j2\pi m Fa}. \]  

(35)

The proof is given in Appendix A. In practice, \( C \) is always chosen as the smallest unit cell \( \mathcal{C}_0 \) containing the support \( \sigma(u) \). When \( C \) consists of \( N \) consecutive integers, i.e., \( C = \{m_0, m_0 + 1, \ldots, m_0 + N - 1\} \), the interpolating function takes the standard form

\[ \hat{c}(\alpha) = \frac{1}{N} \sum_{m = m_0}^{m_0+N-1} e^{-j2\pi m Fa} = e^{j\pi(2m_0 + N-1)F} \sin(NF \alpha) \]  

(36)

where \( \sin(x) \equiv (1/N) \sin(\pi x)/\sin(\pi x/N) \) is the periodic version of the sinc function.

Three examples of interpolating functions \( \hat{c}(\alpha) \), obtained with three different unit cells \( \mathcal{C} \) of size \( N = 8 \), are illustrated in the complex plane in Fig. 2.

A marginal consequence of the sampling theorem is the possibility of evaluating the nonzero Fourier coefficients from the samples, namely (see Appendix A)

\[ u_m = \frac{1}{N} \sum_{\nu=0}^{N-1} u(\nu T) e^{j2\pi m \nu/N} \quad m \in \mathcal{C} \]  

(37)

which states that the \( U_m \) are obtainable from the DFT of the signal samples.

### B. Sampling Relationship

Different GS’s lead to different kernels. However, the kernels may have some samples in common.

**Theorem 3:** Let \( u(a) \) and \( \hat{u}(a) \) satisfy the sampling conditions \( \hat{u}(\alpha T) = u(\alpha T), \quad T = T_p/N \), if and only if

\[ \hat{g}_h = g_h \quad (\text{mod } N). \]  

(39)

The proof of the “if” part is trivial; for the “only if,” part see Appendix B.

The preceding result can be generalized to sequences that are periodic (mod \( N \))

\[ g_{h+N \equiv h} = g_h \quad (\text{mod } N), \quad \forall h \in \mathbb{Z}. \]  

(40)

For instance, both the sequences \( g_n = n \) and \( g_n = n + 4n^3 \) have this property for any \( N \).

**Theorem 4:** Given two GS’s \( g_n \) and \( \hat{g}_h \), if both are periodic (mod \( N \)) and the values of the first sequence \( \{0, \hat{g}_0, \hat{g}_1, \ldots, g_{N-1}\} \) are distinct (mod \( N \)), then the samples of the \( \hat{u}(a) \) can be obtained from the samples of \( u(a) \) as

\[ \hat{u}(\alpha T) = \sum_{\nu=0}^{N-1} u(\nu T) \beta(r, p), \quad T = T_p/N \]  

(41)

where \( \beta(r, p) = (1/N) \sum_{m=0}^{m=N-1} e^{j2\pi (m_0 - m_0) \nu/N} \).

The proof is given in Appendix B.
The proof is given in Appendix C. Note that if\( \tilde{g}_m = g_m + h_m N \), as in Theorem 3, then\( \beta(r, p) = (1/N) \sum_{m=0}^{N-1} e^{j2\pi g_m (r-p)/N} = \delta_{rp} \), and\( \tilde{u}(rT) = u(rT) \). Hence, Theorem 4 generalizes Theorem 3.

C. Other Relationships

We continue the comparison of two kernels\( u(a) \) and\( \tilde{u}(a) \) generated by different sequences\( g_n \) and\( \tilde{g}_n \), as in (38). The following results are direct consequences of expressions (30) and (31).

Shifting: If\( \tilde{g}_n = g_n + 4m_0 \), then\( \tilde{u}(a) = u(a)e^{-j2\pi a m_0} \). Scaling: If\( \tilde{g}_n = m_0 g_n \) (and it is necessary that\( m_0 = 4p+1 \) in order for\( \tilde{g}_n \) to be itself a GS), then\( \tilde{u}(a) = u(m_0 a) \).

Perturbation of a GS: If the two GS’s are equal,\( \tilde{g}_n = g_n \), except for a point\( n_0 \), where\( \tilde{g}_{n_0} = g_{n_0} + 4m_0 \), then

\[
\tilde{u}(a) = u(a) + B_{n_0} e^{-j2\pi a m_0} (e^{-j2\pi a m_0} - 1).\] (42)

This rule is easily extended to an arbitrary number of perturbations. If the perturbation is periodic, that is,\( \tilde{g}_n = g_n + 4m_0 \) for\( n = 4Lh + n_0 (\forall h \) and for\( m_0 \) fixed), then

\[
\tilde{u}(a) = u(a) + v_{n_0}(a) \left( e^{-j2\pi a m_0} - 1 \right)
\]

with

\[
v_{n_0}(a) = \sum_{h=0}^{\infty} B_{4h+n_0} e^{-j2\pi a Lh+n_0}.
\] (43)

IV. STANDARD FRT FORMS

In this section, we assume that the basis is given by HG functions and investigate the FRT’s generated by the sequences

1) \( g_n = n \)
2) \( g_n = (n)_{4L} \)

which yield the familiar forms of the FRT.

A. Standard Chirp FRT

The FRT generated by\( g_n = n \) has the kernel

\[
u^{(c)}(a) = \psi^{(c)}(f, t) = \sum_{n=0}^{\infty} \varphi_n(f) \varphi_n^*(t) e^{-j\pi(2/4)m \phi_n}.
\] (44)

When the\( \varphi_n(t) \) are the HG functions (14), the sum of the series can be derived in closed form using Mehler’s expansion

\[
\sum_{n=0}^{\infty} \varphi_n(f) \varphi^*_n(t) = \frac{\sqrt{2}}{\sqrt{1-p^2}} \exp\left[ \pi(f^2 + t^2) - \frac{2\pi[(f^2 + t^2) - 2\rho ft]}{1-p^2} \right]
\] (45)

and yields the expression

\[
u^{(c)}(a) = \psi^{(c)}(f, t) = K_a e^{j\pi(B_a(f^2 + t^2) - 2C_a ft)}
\] (46)

where\( K_a, B_a, \) and\( C_a \) are the parameters of (1).
We note from (44) that $u^{(c)}(a)$ has an infinite number of harmonics so that it is not bandlimited. Fig. 3 illustrates $u^{(c)}(a)$ as a function of $a$ for $f$ and $t$ fixed. Note the “turbulent” behavior around the points $a = 0$ and $a = 2$, where the kernel is represented by the distributions $\delta(f - t)$ and $\delta(f + t)$. This causes problems in the numerical evaluation of the CFRT for $a$ around $2k$. These can be overcome [13] by decomposing the FRT of order $2k + \epsilon$ into the cascade of an ordinary FT and an FRT of order $2k - 1$ since the kernel is smooth around odd integers.

As an example of closed-form evaluation, we consider the standard CFRT of the rectangular signal $s(t) = \text{rect}(t/T)$, which can be expressed in terms of the Fresnel integral $E(x) = \int_0^x e^{i(\pi/2)y^2} dy$ as

$$S_a(f) = \frac{K_a}{\sqrt{2Ba}} e^{ijf^2 \sec(\pi/2)a} \left[ E\left(f \sec \frac{\pi}{2}a + \frac{T}{2}\right) - E\left(f \sec \frac{\pi}{2}a - \frac{T}{2}\right) \right]$$

for $0 < a < 1$

and can be obtained for a general $a \in \mathbb{R}$ by using symmetries and periodicity. This transform is illustrated in Fig. 4 for different values of the fraction $a$ with $T = 1$.

### B. Standard Weighted FRT of Order $4L$

For the GS $g_n = (n)_{4L}$, the kernel (26) becomes

$$u^{(4L)}(a) = \psi^{(4L)}_a(f, t) = \sum_{m=0}^{4L-1} U_m^{(4L)}(f, t) e^{-i\pi/2m}$$

where the subkernels are given by (27) as

$$U_m^{(4L)}(f, t) = \sum_{h=0}^{\infty} \varphi_{4Lh+m}(f) \varphi^*_a e^{i\pi/2m}$$

$$= \sqrt{\frac{2\pi}{4Lh+m}} \sum_{h=0}^{\infty} \frac{1}{2^{4Lh}(4Lh+m)!} \cdot H_{4Lh+m}( \sqrt{2\pi} f ) H_{4Lh+m}( \sqrt{2\pi} t ).$$

The sum of this series is only known for $L = 0$ (Mehler’s expansion) but not for the present case, where $L > 1$. Inspection of (47) shows that $u^{(4L)}(a)$ has finite harmonic support.

3 This example was considered in [9] and [11] but without giving a closed-form expression for $S_a(f)$.

We can therefore apply the sampling theorem, which assures the reconstruction of $u(a)$ from $4L$ samples $u(rT)$, with $T = 4/(4L)$, to find

$$u^{(4L)}(a) = \sum_{r=0}^{4L-1} u^{(4L)}(r/L) i_{4L}(a - r/L)$$

where the interpolating function is given by (36). We are thus able to capture the full $a$-dependence of the WFRT kernel by means of the interpolation formula, provided that we know the samples $u(r/L)$. However, the latter require knowledge of the summation in (48).

The solution lies in using the sampling relationship between the GS $g_n = (n)_{4L}$ and the GS $g_n = n$ of the standard chirp.
FRT. In fact, Theorem 3 holds for $N = 4L$, and this allows us to state that the CFRT kernel and the WFRT kernel take the same values at $a = r/L$. This is illustrated in Fig. 5, where $u^{(c)}(a)$ and $u^{(4L)}(a)$ are compared as functions of $a$, for $4L = 16$, when $f$ and $t$ are fixed. Applying the sampling relationship simultaneously solves the two problems of evaluating the WFRT kernel and finding the relationship between the two kinds of FRT’s, with the result

$$u^{(4L)}(a) = \sum_{r=0}^{4L-1} u^{(c)}(r/L) i_{4L}(a-r/L)$$

(50)

where $u^{(c)}(a) = \psi^{(c)}(f, t)$ is known in closed form, and the sum has a finite number of terms. Hence, (50) yields the WFRT kernel in closed form for any order $4L$.

A more explicit result is obtained by considering the marginal values of the “samples” (see Section II), namely $\psi^{(c)}(f, t) = \delta(f-t), \psi^{(c)}(f, t) = e^{-jq\pi ft}$, etc. In (50), these samples correspond to $r = 0, L, 2L, 3L$, respectively; hence

$$\psi^{(4L)}_a(f, t) = i_{4L}(a)\delta(f-t) + i_{4L}(a-1)e^{-jq\pi ft} + i_{4L}(a-2)\delta(f+t) + i_{4L}(a-3)e^{jq\pi ft}$$

$$+ \sum_{r=1}^{3} \sum_{k=0}^{L-1} \psi^{(c)}_{r/L+k}(f, t) i_{4L}(a-r/L - k).$$

(51)

In the particular case $L = 1$, the last term disappears, and we find

$$\psi^{(4)}_a(f, t) = i_a(\delta(f-t) + i_{a-1}e^{-jq\pi ft}$$

$$+ i_{a-2}\delta(f+t) + i_{a-3}e^{jq\pi ft}$$

which corresponds to a WFRT of order 4. \(^*(4)\)

Having established the kernel (34) in the form (50), we can obtain the expression of the standard WFRT of order $4L$ as

$$S^{(4L)}_a(f) = \sum_{r=0}^{4L-1} S^{(c)}_{r/L}(f) i_{4L}(a-r/L)$$

although a more detailed result can be obtained by using (51). This relationship states that to find the WFRT of a signal $s(t)$, for any fraction $a$, we must first calculate the CFRT for $4L$ fractions, namely, $S^{(c)}_0(f), S^{(c)}_1(f), \ldots, S^{(c)}_{4L-1}(f)$, and then build up $S^{(4L)}_a(f)$ with the weights $i_{4L}(a-r/L)$. Note that this sequence contains the terms $S^{(c)}_0(f) = s(f), S^{(c)}_1(f) = S(f)$, $S^{(c)}_2(f) = s(-f), S^{(c)}_3(f) = S(-f)$ for any order $L$. Thus, in the particular case $4L = 4$, the chirped samples disappear, in accordance with (52). Fig. 6 illustrates the standard WFRT’s of order 4 of the rectangular signal for different fractions $a$. Fig. 7 illustrates the standard WFRT’s of different orders $4L$ for the fraction $a = 0.3$.

**Remark 1:** The standard WFRT was obtained by Shih [10] for $4L = 4$ and recently extended by Liu et al. [11] to a general-order $4L$. These authors, however, did not appreciate the roles of the sampling theorem and of the sampling relationship.

**Remark 2:** Incidentally, we note that the sampling relationship $u^{(4L)}(r/L) = u^{(c)}(r/L)$ allows the closed-form evaluation of the subkernels (48). In fact, $U^{(4L)}_m$ are Fourier coefficients, and by (37), we find

$$U^{(4L)}_m = \frac{1}{4L} \sum_{r=0}^{4L-1} u^{(c)}(r/L)W_{4L}^m$$

(53)

where $u^{(c)}(r/L)$ is the standard chirp kernel defined by (46), and $W_{4L} = e^{j\pi r/N}$.

**Remark 3:** Again, from the sampling relationship, we see that the WFRT enjoys the same properties as the CFRT and, in particular, the relationship with the Wigner distribution (55) (see Section IV-D) when $a$ takes the discrete values $r/L$.

C. Standard WFRT as an Approximation of the Standard CFRT

The sampling relationship between the CFRT and the WFRT of order $N = 4L$ states that the WFRT is an approximation of the CFRT, which improves as $N$ increases (see Fig. 7). Indeed

$$\lim_{N \to \infty} e^{-j\pi (\sqrt{2}/2)N} = e^{-j\pi \alpha}$$

(54)

that is, as $N$ increases, the eigenvalues of the WFRT tend to the eigenvalues of the CFRT. Considering the signal $s(t)$, as expressed in the form (7), its CFRT $S^{(c)}_a(f)$, and its $N$th-order WFRT $S^{(N)}_a(f)$, as expressed as in (25), we find that the convergence of $S^{(N)}_a$ to $S^{(c)}_a$ is in the $L^2$ norm. In fact

$$\left\| S^{(N)}_a(f) - S^{(c)}_a(f) \right\|^2$$

$$= \sum_{n=0}^{\infty} \left( e^{-j\pi (\sqrt{2}/2)n} - e^{-j\pi \alpha} \right) S_n \varphi_n(f) \right)^2$$

$$\leq 2 \sum_{n=N}^{\infty} |S_n|^2$$

which is assured by the unit amplitude of the eigenvalues and by the orthonormality of the eigenfunctions. If $s(t)$ is a signal in $L^2$, as $N$ increases, the above sum converges to zero.
D. Relationship Between CFRT and Wigner Distribution

We recall that the Wigner distribution operator is defined as
\[
\mathcal{W}_a(u, v) = \int_{\mathbb{R}} s(x + t/2) s^*(x - t/2) e^{-j2\pi ft} dt \tag{55}
\]
and is not linear. The CFRT is related to the Wigner distribution by the equivalence [4]
\[
\mathcal{R}_\alpha \mathcal{W} = \mathcal{W} \mathcal{F}_\alpha \tag{56}
\]
where \( \mathcal{R}_\alpha \) is the image-rotation operator
\[
\mathcal{R}_\alpha: \quad w_a(x, y) = w \left( x \cos \frac{1}{2} \pi \alpha + y \sin \frac{1}{2} \pi \alpha, \right.
\]
\[
\left. y \cos \frac{1}{2} \pi \alpha - x \sin \frac{1}{2} \pi \alpha \right) \tag{57}
\]
Such equivalence states that the Wigner distribution of the

Fig. 6. WFRT of order \( 4L = 4 \) of the rectangular signal for several values of \( \alpha \) (real part solid, imaginary part dashed lines).

Fig. 7. Standard WFRT with weights \( 4L = 4, 8, 16, 64, \) and 256, compared with the CFRT of the signal \( s(t) = \text{rect}(t) \), for a fraction \( \alpha = 0.3 \) (real part solid, imaginary part dotted lines).
by an angle $(\pi/2)a$. Since the proof of the above equivalence is based on the specific structure of the CFRT kernel given by (46), the conclusion is that (56) holds only when $F^a$ is interpreted as the CFRT operator. However, we can take advantage of the sampling relationship linking the 4L-WFRT to the CFRT, namely, $F^a_{4L} = F^a$ for $a = k/L$ and $k \in \mathbb{Z}$ (see Section IV-B), to conclude that (56) holds also for the 4L-WFRT for the fractions $a = k/L$. Note that as the number of weights 4L increases, the identity (56) holds at more and more values of the fraction or, equivalently, for an increasing number of rotated angles of the Wigner plane.

Incidentally, we note a further property that is unique to the CFRT. Again from the kernel structure (46), the CFRT turns out to be a canonical transformation [19], i.e., one with a kernel proportional to $e^{-j(\omega_0 + \omega_1)\alpha}$, where $\omega_0$, $\omega_1$, $\alpha$ real. Once again, from the sampling relationship, we can conclude that for the fractions $a = k/L$, the 4L-WFRT is a canonical transformation.

V. NONSTANDARD FORMS OF FRT’S

In this section, we introduce new examples of FRT’s of both chirp and weighted forms to show that their expressions can be evaluated as combinations of the standard FRT’s of the previous section. We continue to assume the HG basis.

A. Nonstandard Chirp-FRT’s

We consider the specific GS examples

1) $g_n = n + 12$ shifting
2) $g_n = 5n$ scaling
3) $g_n = \begin{cases} n, & \text{for } n \neq 6 \\ n + 12 = 18, & \text{for } n = 6 \end{cases}$ perturbation
4) $g_n = \begin{cases} n, & \text{for } n \neq 8h + 2 \\ n + 12, & \text{for } n = 8h + 2 \end{cases}$ periodic perturbation
5) $g_n = n + 4n^3$ periodicity mod 4L.

All these sequences have cardinality $|G| = \infty$ so that the corresponding FRT’s are of the chirp type. In the list, we indicated the relationship of each sequence with the standard sequence $g_n = n$ so that we can use the results of Section IV.

For the sequences 1) and 2), we obtain the FRT’s

1) $S_{\alpha}(f) = S_{\alpha}^{(c)}(f)e^{2\pi j \alpha} = e^{2\pi j \alpha}$
2) $S_{\alpha}(f) = S_{\alpha}^{(c)}(f)$.

For the sequence 3), we obtain from (42) the kernel relationship $u(a) = u^{(c)}(a) + B_6e^{-j(\pi/2)a}(e^{-j(\pi/2)a} - 1)$, where $B_6 = \varphi^{(c)}(t)\varphi^{(c)}(t)$. This yields the FRT

3) $S_{\alpha}(f) = S_{\alpha}^{(c)}(f) + S_{\alpha}(f)e^{-j3\alpha} (e^{-j3\alpha} - 1)$
where $S_{\alpha}$ is the sixth coefficient of the Hermite–Gauss expansion given by (8).

For the GS 4), from (43), we get

$u(a) = u^{(c)}(a) + \sum_{n=0}^{\infty} B_n e^{-j(\pi/2)(n+4n^3)\alpha}$.

The latter series can be evaluated by a filtering operation on the function $u^{(c)}(a)$, with the result (see Appendix D) $u(a) = u^{(c)}(a) + \sum_{n=0}^{7} S_{\alpha}^{(c)}(f)W_{8}(e^{-j3\alpha} - 1)$. The corresponding FRT is then given by

4) $S_{\alpha}(f) = S_{\alpha}^{(c)}(f) + \sum_{n=0}^{7} S_{\alpha}^{(c)}(f)W_{8}(e^{-j3\alpha} - 1)$.

For the GS 5), the kernel is given by

$u(a) = \sum_{n=0}^{4L-1} B_n e^{-j(\pi/2)(n+4n^3)\alpha}$.

The closed-form summation of this uncommon sequence, where $B_n = \varphi^{(c)}(f)\varphi^{(c)}(t)$ is the product of Hermite–Gauss functions, would appear to be a very hard, even impossible task. However, considering that the GS $g_n = n + 4n^3$ is periodic mod 4L, that is $g_{4L+n} = g_n$ (mod 4L), hence the same as the standard GS $g_n = n$, we can use Theorem 4 to obtain samples of $u(a)$ in terms of samples of $u^{(c)}(a)$. Specifically

$u(r/L) = \sum_{p=0}^{4L-1} u^{(c)}(p/L)\beta(r, p)$

with

$\beta(r, p) = \frac{1}{4L}\sum_{m=0}^{4L-1} e^{2\pi j m - (m+4n^3)r/(4L)}$.

The corresponding relationship for the FRT is

5) $S_{\alpha}(f) = \sum_{p=0}^{4L-1} S_{p/L}(f)\beta(r, p)$.

We are thus able to evaluate this FRT in closed form for any rational fraction.

All these examples of nonstandard chirp FRT’s, together with the standard chirp FRT, are illustrated in Fig. 8 for the signal $s(t) = \text{rect}(t)$ and the fraction $a = Q/3$.

B. Nonstandard Weighted FRT’s

A compression of a GS $g_n$ by a (mod 4L) operation gives a new GS with finite cardinality, namely

$g_n = (g_n)_{4L}$

with $\tilde{G} \subset \{0, 1, \ldots, 4L-1\}$ and $|\tilde{G}| \leq 4L$. Hence, starting from any nonstandard sequence of the previous subsection, for any order 4L, we can obtain a corresponding nonstandard weighted FRT.
The relationship between a given weighted FRT and the corresponding chirp FRT is easily expressed by the sampling relationship and by the sampling theorem. In fact, (60) satisfies the hypothesis of Theorem 3, and hence, \( \hat{u}(r/L) = u(r/L) \) for \( r = 0, 1, \ldots, 4L - 1 \). Moreover, \( \mathcal{G} \subset \{0, 1, \ldots, 4L - 1\} \), and therefore

\[
\hat{u}(a) = \sum_{r=0}^{4L-1} u(r/L) i_{4L}(a - r/L).
\]

This result generalizes (50) to nonstandard FRT's. Now, \( u(a) \) is the kernel of a nonstandard chirp form, and \( \hat{u}(a) \) is the kernel of the corresponding nonstandard weighted form, the interpolating function being the same as in (50). For instance, if \( g_n = n + 4n^3 \) and \( 4L = 12 \), we obtain the GS of period 12:

\[
\begin{array}{ccccccccc}
 n & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\
 \tilde{g}_n & 0 & 5 & 10 & 3 & 8 & 1 & 6 & 11 & 4 & 9 & 2 & 7 \\
\end{array}
\]

Hence, this WFRT is the weighted combination of the nonstandard CFRT given by (59), which in turn is a combination of samples of the standard CFRT evaluated for 12 fractions.

Another class of weighted FRT's is obtained by using periodic GS's: those with the property

\[ g_{4L+m} = g_m, \quad h = 0, 1, 2, \ldots, m = 0, 1, \ldots, 4L - 1. \]

For these sequences, the kernel is given by [see (47) and (48)]

\[
u(a) = \sum_{m=0}^{4L-1} U_{4L}^{(m)} e^{-j\pi/2g_m a} \tag{61}
\]

where \( U_{4L}^{(m)} \) are the subkernels of the standard WFRT of order \( 4L \) given by (53). Hence

\[
u(a) = \sum_{r=0}^{4L-1} u^{(r)}(r/L)p_r(a)
\]

with

\[
p_r(a) = \frac{1}{4L} \sum_{m=0}^{4L-1} e^{-j\pi/2g_m(a-rm/L)}. \tag{62}
\]

In conclusion, every periodic GS leads to a WFRT that is directly related to the standard CFRT. As an example, we consider the GS of period \( 4L = 16 \)

\[
\begin{array}{ccccccccccccccc}
 n & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\
 \tilde{g}_n & 0 & 9 & 2 & 3 & 4 & 4 & 5 & 6 & 7 & 4 & 4 & 5 & 6 & 7 & 0 & 6 & 7 \\
\end{array}
\]

whose kernel can be calculated from (62), with \( 4L = 16 \), in terms of 16 values of \( u^{(r)}(a) \). However, we note that the range of \( \tilde{g}_n \) is the unit cell \( \mathcal{C} = \{0, 2, 3, 5, 6, 7, 9, 44\} \) of size 8 (see Fig. 2). The sampling theorem can thus be applied with \( N = 8 \) to give

\[ u(a) = \sum_{r=0}^{7} u(r/2)i_{4L}(a - r/2) \]

where the eight samples \( u(r/2) \) can be calculated from (62). Hence, we end up with two different but equivalent expressions for the same kernel.
VI. FRT DEPENDENCE ON THE EIGENFUNCTION BASIS

In this section, we assume that the GS $g_n$ is fixed. We have seen that the possible bases are governed by Theorem 1, which can be interpreted in terms of operators as

$$\hat{U} = UA$$

where

1) $\hat{U}$ is determined by the reference basis $\{\varphi_n(t)\}$;
2) $A$ is determined by the PS $\{\alpha_{mn}\}$;
3) $\hat{U}$ gives the “perturbed” basis $\{\hat{\varphi}_n(t)\}$.

Note that by (17), the operator $A$ is unitary. Since $\hat{U}$ is a valid basis, from a given FRT operator $F^\alpha = U^D U^\alpha$, we can obtain a new FRT operator

$$\hat{F}^\alpha = \hat{U}^D \hat{U}^\alpha = UAD^\alpha A^\dagger U^\alpha$$

and this can be done for every PS $\{\alpha_{mn}\} \to A$.

However, it is not clear how effectively the perturbation changes the FRT. We know some general facts, namely, that the ordinary FT operator $\mathcal{F}$ is independent of the basis, which assures that no changes occur at the fraction $a = 1$ or at any other integer fraction (see marginal conditions). Moreover, by inspecting its structure, we see that the 4-WFRT is also independent of the basis. To gain insight into the problem and to test quantitative effects, we next consider a very simple case.

A. Perturbation of Two Eigenfunctions

Starting from an orthonormal basis of eigenfunctions $\{\varphi_n(t)\}$, we modify two of them, say, $\varphi_2(t)$ and $\varphi_6(t)$, which have the same FT eigenvalue $\mu_2 = \mu_6 = -1$, according to

$$\hat{\varphi}_2(t) = a\varphi_2(t) + b\varphi_6(t), \quad \hat{\varphi}_6 = c\varphi_2(t) + d\varphi_6(t)$$

with $\hat{\varphi}_n(t) = \varphi_n(t)$, otherwise. Hence, the PS is $\alpha_{22} = a$, $\alpha_{26} = b$, $\alpha_{62} = c$, $\alpha_{66} = d$, and $\alpha_{mn} = \delta_{mn}$ for the rest. The new eigenfunctions preserve the eigenvalues, and the conditions

$$|a|^2 + |b|^2 = 1, \quad |c|^2 + |d|^2 = 1, \quad ab^* + cd^* = 0$$

[see (17)] assure that the modified set $\hat{\varphi}_n(t)$ is still orthonormal. These conditions can be put into a matrix form as

$$AA^* = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} a & b^* \\ c & d^* \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

which states that $A$ is a unitary matrix. Since the transpose $A^*$ must also be unitary, it follows that (63) are equivalent to

$$|a|^2 + |b|^2 = 1, \quad |c|^2 + |d|^2 = 1, \quad ab^* + cd^* = 0.$$  \hspace{1cm} (64)

The reference FRT and the “perturbed” FRT are given by [see (25)]

$$S_a(f) = \sum_{n=-\infty}^{\infty} S_n \hat{\varphi}_n(f) e^{-j(\pi/2)g_n a}$$
$$\hat{S}_a(f) = \sum_{n=-\infty}^{\infty} \hat{S}_n \hat{\varphi}_n(f) e^{-j(\pi/2)\hat{g}_n a}$$

where [see (8)] $\hat{S}_2 = a^* S_2 + b^* S_6$, $\hat{S}_6 = c^* S_2 + d^* S_6$, and $\hat{S}_n = S_n$ for the rest. Hence, the “perturbation” $\Delta S_a(f) = \hat{S}_a(f) - S_a(f)$ is given by

$$\Delta S_a(f) = \left[ 2 \hat{S}_2 \varphi_2(f) - S_2 \varphi_2(f) \right] e^{-j(\pi/2)g_n a}$$
$$+ \left[ \hat{S}_6 \varphi_6(f) - S_6 \varphi_6(f) \right] e^{-j(\pi/2)g_n a}$$
$$= \{ |c|^2 [2 \varphi_2(f) - S_2 \varphi_2(f)] + c d^* S_6 \varphi_6(f) \} e^{-j(\pi/2)g_n a}$$
$$+ c d^* S_2 \varphi_2(f) \} e^{-j(\pi/2)g_n a}$$

where we used (63) and (64). Moreover, considering the orthogonality condition, we find

$$\mathcal{E} = \int_{-\infty}^{+\infty} |\Delta S_a(f)|^2 df$$
$$= 4|c|^2 \left( |S_2|^2 + |S_6|^2 \right) \sin^2 \frac{\pi}{4} (g_2 - g_6)a.$$  \hspace{1cm} (65)

Note that if $g_2 = g_6$, the perturbation has no effect. In any case, since $g_2 = g_6 \mod 4$ [see (21)], then $\Delta S_a(f) = 0$ for every integer fraction $a$. In general, however, $\Delta S_a(f) \neq 0$, although since $|c|^2 + |d|^2 = 1$, this difference is bounded.

The perturbation is illustrated in Fig. 9 for the CFRT of the rectangular signal shown in Fig. 4, where $S_2 = -0.1026$, and $S_6 = 0.1549$.

B. Example of Perturbation of an Arbitrary Number of Eigenfunctions

A curious perturbation sequence is provided by the DFT matrix $[W_K^{-mn}]/\sqrt{K}$, which is orthonormal for every order $K$. Therefore, we can let

$$\alpha_{m+h, n+h} = \frac{1}{\sqrt{K}} W_K^{-mn}, \quad m, n = 0, 1, \cdots, K-1$$
$$h = 0, 1, 2, 3$$

and $\alpha_{m,n} = \delta_{mn}$, otherwise.

By doing so, we obtain $4K$ perturbed eigenfunctions

$$\hat{\varphi}_{m+n}(t) = \frac{1}{\sqrt{K}} \sum_{m=0}^{K-1} W_K^{-mn} \varphi_{m+n}(t)$$  \hspace{1cm} (67)
Fig. 9. Effect of perturbation of two eigenfunctions with $c = 1$ and $d = 0$ on the CFRT of a rectangular signal with fraction $a = 1/2$ compared with the unperturbed CFRT (dashed).

and as many perturbed coefficients

$$S_{4n+h} = \frac{1}{\sqrt{K}} \sum_{m=0}^{K-1} W_{n, m} S_{4n}$$

(68)

whereas, for $n \geq 4K$, $\varphi_n(t)$, and $S_n$ remain unchanged. According to (67), for $h$ and $t$ fixed, applying the DFT to the sequence $\{\varphi_{4n+h}(t)\}_{n=0, ..., K-1}$ gives the perturbed sequence $\{\varphi_n(t)\}_{n=0, ..., K-1}$. Similarly, in (68), the perturbed coefficients are obtained by applying the inverse DFT.

The above procedure can be applied for any order $K$. In this case, the GS is $g_n = n$, and the perturbation $\Delta S_n(f)$ has energy

$$E = 2 \sum_{n=0}^{K-1} |S_n|^2 - 2R \left\{ \sum_{h=0}^{3} \sum_{n=0}^{K-1} \sum_{m=0}^{K-1} S_{4n+h} S_{4n+h}^* \gamma_{m, n}(\alpha) \right\}$$

(69)

with

$$\gamma_{m, n}(\alpha) = \sin(K(n+m+K\alpha))e^{j\pi[(K-2n)+n-m](1-1/K)}.$$  

(70)

Note that any other orthonormal matrix can be used to obtain finite blocks of perturbations. In general, (69) becomes

$$E = 2 \sum_{n=0}^{\infty} |S_n|^2 - 2R \left\{ \sum_{h=0}^{3} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} S_{4n+h} S_{4n+h}^* \gamma_{m, n}(\alpha) \right\}$$

(71)

with

$$\gamma_{m, n}(\alpha) = \sum_{q=0}^{\infty} \alpha_{4q+h} \alpha_{4m+h} e^{j2\pi(n+m)(1-1/K)}.$$  

(72)

VII. CONCLUSIONS

We discussed the multiplicity of the FRT and the relationships between the different versions. We saw that the class of FRT's is quite vast, and perhaps only a few may be of practical interest for applications. Until now, interest had been confined mainly to the standard CFRT, particularly for applications in optics. The practical relevance of the WFRT remains to be investigated.

An important classification of FRT's was expressed in terms of the cardinality of the GS or, equivalently, in terms of bandlimitation of the kernel with respect to the domain of the fraction $a$. If the bandwidth is not finite, the FRT turns out to be of the chirp type, whereas with finite bandwidth, the FRT is of a weighted type. Moreover, the WFRT may be viewed as a filtered version (in the domain of the fraction) of the CFRT. Our aim was to establish results in closed form, and this was achieved for the whole, although vast, class of specific cases considered. This investigation has also shown that the evaluation of all the specific forms of FRT is ultimately obtained in terms of the standard CFRT.

Although, in this paper, we confined our investigations to continuous-domain signals, most considerations also apply to periodic discrete-domain signals, as suggested in the unified approach of [16]. For periodic signals in a discrete domain, the cardinality of a complete orthonormal set of eigenfunctions is always finite so that only the WFRT form can be considered. However, it can be shown in an analogous way that WFRT's with a lower number of weights are filtered versions of WFRT's of higher order.

APPENDIX A

PROOF OF THE SAMPLING THEOREM FOR COMPLEX PERIODIC SIGNALS

The sampling theorem for periodic signals is well known; historically, it was certainly the first to be considered [21], [22], and it can be regarded as a particular case of trigonometric interpolation [23, ch. X], [24, ch. VII]. Nevertheless, recent authors [25] state that a proper formulation is not easily found in the literature, even for the particular case of real signals with harmonic support of $N$ consecutive integers. Therefore, we give the proof for the general formulation of Theorem 2 that is suitable to our purposes.

We write the periodic signal in terms of its Fourier coefficients

$$u(\alpha) = \sum_{m=\infty}^{\infty} U_m e^{-j2\pi m F_0} = \sum_{m \in \mathbb{C}} U_m e^{-j2\pi m F_0}.$$  

(72)
The sampled values are then given by
\[ u(rT) = \sum_{m=-\infty}^{\infty} U_m W_N^{-mr} = \sum_{m \in \mathbb{C}} U_m W_N^{-mr} \]  
(73)
where \( F \cdot T = 1/N \) and \( W_N = e^{j2\pi/N} \). On the other hand, the DFT of the samples is given by
\[ \tilde{U}_n = \frac{1}{N} \sum_{r=0}^{N-1} u(rT) W_N^{-nr} \]
\[ = \sum_{m=-\infty}^{\infty} U_m \left\{ \frac{1}{N} \sum_{r=0}^{N-1} W_N^{(n-m)r} \right\} \]
(74)
where the sum enclosed in braces is 1 if \( m = hN + n, h \in \mathbb{Z} \) and 0 otherwise. Hence
\[ \tilde{U}_n = \sum_{h=-\infty}^{\infty} U_{hN+n} \cdot \]
(75)
Now, from the unit cell definition \[ \text{see (33)}, \text{bandlimitation assures that in the periodic repetition (75), the nonzero coefficients} \]
\( U_m \) \( \text{do not overlap and can be recovered as} \)
\( U_m = \tilde{U}_m, m \in \mathbb{C} \).
\( \text{Hence, by (74), we can express the} \)
\( U_m \) \( \text{in (72) in terms of the} \)
samples as
\[ u(a) = \sum_{m \in \mathbb{C}} \frac{1}{N} \sum_{r=0}^{N-1} u(rT) W_N^{mr} e^{-j2\pi ma} \]
and (34) follows. The proof of (37) is a consequence of \( U_m = \tilde{U}_m, m \in \mathbb{C} \), and of (74).

**APPENDIX B**

**PROOF OF THEOREM 3**

For a fixed fraction \( \alpha \), we evaluate the energy of the difference \( \hat{u}(a) - u(a) \) in the \( (f, t) \) plane as
\[ ||\hat{u}(a) - u(a)||^2 \overset{\Delta}{=} \int_{t=-\infty}^{+\infty} \int_{f=-\infty}^{+\infty} |\hat{u}(a) - u(a)|^2 df \, dt \]
where \( \hat{u}(a) \) and \( u(a) \) are expressed as in (38). Considering that \( B_n = \varphi_n(f) \varphi_n^*(t) \) and that the \( \varphi_n \) are orthonormal, we find
\[ ||\hat{u}(a) - u(a)||^2 = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} |1 - e^{-j2\pi \varphi_n - \varphi_m}|^2 \]
Now, this energy is zero for \( a = rT \) if and only if (39) holds. Note that \( ||\hat{u}(a) - u(a)||^2 = 0 \) means that \( \hat{u}(a) = u(a) \) almost everywhere in the \( (f, t) \) plane.

**APPENDIX C**

**PROOF OF THEOREM 4**

With \( u(a) \) and \( \hat{u}(a) \) given by (38), we let \( n = hN + m \). Then, considering that \( g_{hN+m} = g_m \) and \( \hat{g}_{hN+m} = \hat{g}_m \), for the samples, we find
\[ u(\ell T) = \sum_{m=0}^{N-1} U_m W_N^{-\ell m \rho} \]
\[ \hat{u}(\ell T) = \sum_{m=0}^{N-1} \hat{U}_m W_N^{-\ell m \rho} \]
(76)
where \( U_m = \sum_{n=0}^{\infty} B_{hN+n,m} \). Now, if the values \( g_m \) for \( m = 0, 1, \cdots, N - 1 \) are a permutation of \( \{0, 1, \cdots, N - 1\} \) or, more generally, fill a unit cell of size \( N \), the first of (76) can be put into a standard DFT form \( u(\ell T) = \sum_{m=0}^{N-1} U_m W_N^{-\ell m \rho} \) by letting \( U_{\ell m} = U_m \) with \( g_m = \ell (\text{mod} N) \). Then, the inverse DFT yields
\[ U_{\ell} = \frac{1}{N} \sum_{m=0}^{N-1} u(\ell T) W_N^{-\ell m \rho} \Rightarrow u_m = \frac{1}{N} \sum_{\ell=0}^{N-1} u(\ell T) W_N^{\ell g_m - \rho \ell m} \]
Using the latter in the second part of (76) completes the proof.

**APPENDIX D**

**IDENTITIES ON HERMITE–GAUSS FUNCTIONS**

In this Appendix, we prove the identity
\[ v_\alpha(n) = \frac{1}{N} \sum_{h=0}^{N-1} B_{hN+m} e^{-j(\pi/2)(hN+m)\alpha} \]
\[ = \frac{1}{N} \sum_{r=0}^{N-1} u(e^{j(\pi/2)\alpha} - 4r/N) W_N^{-mr} \]
(77)
which may be viewed as a generalization of Mehler’s expansion (45). Note that for \( \alpha = 0 \), this identity gives (53).
For the proof, we use polyphase decomposition, which is a fundamental result of the theory of multirate systems [26]. Given a z-transform \( X(z) = \sum_{n=0}^{\infty} x_n z^{-n} \), the terms \( X_m(z) = \sum_{h=0}^{\infty} x_{hn+m} z^{-nh} \), \( m = 0, 1, \cdots, N - 1 \), are called the polyphase components of \( X(z) \). The result states that \( X_m(z) \) can be obtained from \( X(z) \) by the closed-form relationship
\[ X_m(z) = \frac{1}{N} \sum_{r=0}^{N-1} X(z e^{j(\pi/2)\alpha} - 4r/N) W_N^{-mr} \]
(78)
and (77) becomes a consequence of (78).

**APPENDIX E**

**EIGENFUNCTION BASES OF THE FT**

Let \( \{\varphi_n(t)\} \) be the given basis, which we assume to be ordered so that \( \mu_n = (-j)^n \), and denote with \( E_{\mu} \) the eigenspace of the eigenvalue \( \mu \in \{1, -j, 1, j\} \); then, \( \{\varphi_{n+m} h(t) | n \in \mathbb{N}\} \) is a complete orthonormal set for \( E_{\mu} \). We claim that \( \mu \) with \( \mu = (-j)^h \) is spanned by the linear combinations
\[ x(t) = \sum_{n=0}^{\infty} A_n \varphi_{4n+h} (t) \]
and (77) becomes a consequence of (78).

Clearly, \( x \in E_{\mu} \). Conversely, if \( x \in E_{\mu} \), the expansion with respect to the basis \( \{\varphi_n(t)\} \) gives \( x(t) = \sum_{n=0}^{\infty} A_n \varphi_n(t) \) so
that its FT is given by $\mu \hat{x}(f) = X(f) = \sum A_n \mu \varphi_n(f) \chi_n.$ Since both expansions are unique, it follows that $A_n = \mu X_n$, which implies that $X_n = 0$ for $\mu_n \neq \mu$. Hence, $x(t)$ must have the form (16). Hence, (Condition 17) is thus a consequence of the orthonormality of both $\{\varphi_n(t)\}$ and $\{\chi_n\}$.

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