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Learning Trajectories in Mathematics Education

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Many successful recent approaches to developing innovative mathematics curricula and to conducting research on learning and teaching mathematics have used the construct of "learning trajectories" as a foundation. However, the developers and authors have interpreted and applied this idea in different ways, leading to a need for discussions of these variations and a search for clarifications and shared meanings. Further, the construct of learning trajectories is less than a decade old, but palpably has many roots in previous theories of learning, teaching, and curriculum. The purpose of this special issue is to present several research perspectives on learning trajectories with the intention of encouraging the broader community to reflect on, better define, adopt, adapt, or challenge the concept. This brief article introduces learning trajectories from our perspective. The other articles provide elaboration, examples, and discussion of the construct. They purposefully are intended to be illustrative, exploratory, and provocative with regard to the learning trajectories construct; they are not a set of verification studies.

BACKGROUND

Overviews of recent research-based mathematics curricula reveal several shared characteristics. These include the following: creating and maintaining connections between research and curriculum development as integrated, interactive, processes; using a broad range of scientific methodologies; maintaining close connec-

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tions between tasks and children's mathematical thinking; and using some version of "learning trajectories" (Clements, 2002).

As one example, a team from a realistic mathematics education (RME) group could begin to design curriculum with an anticipatory thought experiment. They formulate a hypothetical learning trajectory that involves conjectures about both a possible learning route that aims at significant mathematical ideas and a specific means that can be used to support and organize learning along this route. The trajectory is conceived of through a thought experiment in which the historical development of mathematics is used as a heuristic. More recently, children's informal solution strategies have also been considered. The original design is a set of instructional tasks with guidelines suggesting an order for the tasks and the types of thinking and learning in which the students can engage as they participate in the instructional tasks. This original design is often not worked out in detail because tasks are revised extensively during classroom testing. That is, the tasks that are actually used in the classroom are determined on a day-to-day basis considering what was learned from enacting the preceding tasks in the classroom. In this second phase, the educational experiment, this preliminary design is refined in a series of intense cyclic processes of deliberations on instructional tasks, the teacher's role, and the wider classroom culture, as these are constituted in the classroom (Gravemeijer, 1999). In the third phase, a "best-case" instructional sequence is constructed. The goal is to develop and describe a more general description of the hypothetical learning trajectories that emerged in specific classrooms that underlies the instructional sequence and to justify it with both theoretical deliberations and empirical data (Gravemeijer, 1994a, 1994b, 1999). This overarching goal is that the local instructional theory will provide a framework that teachers can use to construe hypothetical learning trajectories that fit their own classroom contexts. See Gravemeijer's article in this special issue for the most recent theoretical description.

Learning trajectories can have significance beyond curriculum development. There is evidence that superior teachers use a related conceptual structure. For example, in one study of a reform-based curriculum, the few teachers that had worthwhile, in-depth discussions saw themselves not as moving through a curriculum, but as helping students move through a progression or range of solution methods (Fuson, Carroll, & Drueck, 2000); that is, simultaneously using and modifying a type of hypothetical learning trajectory.

THE LEARNING TRAJECTORY CONSTRUCT

In his seminal work, Simon (1995) stated that a "hypothetical learning trajectory" included "the learning goal, the learning activities, and the thinking and learning in which the students might engage" (p. 133). The name "hypothetical learning trajectory" reflects its roots in a particular constructivist perspective. That is, although the name emphasizes learning over teaching, Simon's description is clearly intended to characterize an essential aspect of pedagogical thinking (i.e., determine the goal, create tasks connected to children's thinking and learning, etc.).

The nascency and complex nature of learning trajectories has led to a variety of interpretations and applications. For example, in contrast to the traditional RME approach and to Simon's (1995) approach, some only emphasize the developmental progressions of learning (what Simon calls hypothetical learning processes) during the creation of a particular curricular or pedagogical context. We believe that, although studying either psychological developmental progressions or instructional sequences separately can be valid research goals and studies of each can and should inform mathematics education, the power and uniqueness of the learning trajectories construct stems from the inextricable interconnection between these two aspects. For our purposes, then, we conceptualize learning trajectories as descriptions of children's thinking and learning in a specific mathematical domain and a related, conjectured route through a set of instructional tasks designed to engender those mental processes or actions hypothesized to move children through a developmental progression of levels of thinking, created with the intent of supporting children's achievement of specific goals in that mathematical domain (cf. Clements, 2002; Gravemeijer, 1999; Simon, 1995).

Consider the developmental progression aspect and the instructional task aspect in turn. First (from our perspective), one specifies learning models that reflect natural developmental progressions (at least for a given age range of students in a particular culture) identified in theoretically and empirically grounded models of children's thinking, learning, and development (Carpenter & Moser, 1984; Griffin & Case, 1997). That is, researchers build a cognitive model of students' learning that is sufficiently explicit to describe the processes involved in the construction of the goal mathematics across several qualitatively distinct structural levels of increasing sophistication, complexity, abstraction, power, and generality. This constructivist aspect distinguishes the learning trajectory approach from previous instructional design models that, for example, used reductionist techniques to break a goal competence into subskills, based on an adult's perspective. This is illustrated by Fuson's (1997) curriculum, which was based on a model of children's solving of word problems, including moving through increasingly difficult types of word problems based on the model. The theory is that learning consistent with such natural developmental progressions is more effective, efficient, and generative for the student than learning that does not follow these paths.[1] Research indicates that providing knowledge of children's thinking and learning in the targeted

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[1]However, such progressions are not isolated from cultural influences, especially intentional instruction. For example, we postulate that progress through nongenetic levels of thinking (e.g., in geometry) is determined more by social influences, and specifically instruction, than by age-linked development (Clements, Battista, & Sarama, 2001).

subject-matter domain can substantially affect curriculum design by focusing it on teaching and learning (Tamir, 1988; Walker, 1992). This leads our discussion to the second aspect.

The second aspect of learning trajectories is an instructional sequence. In our work, these are composed of key tasks designed to promote learning at a particular conceptual level or benchmark in the development progression. Extant research is used to identify tasks as effective in promoting the learning of students at each level by encouraging children to construct the concepts and skills that characterize the succeeding level. That is, we hypothesize the specific mental constructions (i.e., mental actions-on-objects) and patterns of thinking that constitute children's thinking at each level. We design tasks that include external objects and actions that mirror the hypothesized mathematical activity of students as closely as possible; for example, objects can be shapes or sticks and actions can be creating, copying, uniting, disembedding, and hiding both individual units and composite units. Tasks require children to apply, externally and mentally, the actions and objects of the goal level of thinking (see Clements & Battista, 2000, for further description and examples of mathematical objects/concepts and mathematical actions/processes that operate on them). These tasks are, of course, sequenced corresponding to the order of the developmental progressions to complete the hypothesized learning trajectory. The main theoretical claim is that such tasks will constitute a particularly efficacious educational program. However, there is no implication that the task sequence is the only, or the best, path for learning and teaching, only that it is hypothesized to be one fecund route. Further, societally determined values and goals are substantive components of any curriculum (Confrey, 1996; Hiebert, 1999; National Research Council, 2002; Tyler, 1949); research cannot ignore or determine these components (cf. Lester & Wiliam, 2002).

Thus, a complete hypothetical learning trajectory includes all three aspects: the learning goal, developmental progressions of thinking and learning, and sequence of instructional tasks. The synergism between the latter two aspects has already been described. Less obvious is that their integration can produce novel results, even within the local theoretical fields of psychology and pedagogy. The enactment of an effective, complete learning trajectory can actually alter developmental progressions or expectations previously established by psychological studies because it opens up new paths for learning and development. This, of course, reflects the traditional, if oversimplified, debate between Vygotsky (1934/1986) and Piaget and Szeminska (1952) regarding the priority of development over learning. We believe that learning trajectory research, along with other research corpi, supports the Vygotskian position that, at least in some domains and some ways, learning and teaching tasks can change the course of development. Such an enactment based on the fine-grain cognitive analysis of the developmental progression and the similarly detailed analysis of the instruction tasks provides a more elaborated theoretical base for curriculum and instruction than is often available and can also open instructional approaches or avenues not previously considered. Note that these simultaneous detailed analyses distinguish the hypothetical learning trajectory construct from other lines of inquiry, even such closely related work as the van Hiele's simultaneous dissertations (van Hiele, 1959/1985; van Hiele-Geldof, 1984).

Creation and use of learning trajectories always implies conceptual analysis: "The actual process of thinking remains invisible and so do the concepts it uses and the raw material of which they are composed" (von Glaserfeld, 1995, p. 77). Further, an overarching research goal in the field of learning trajectories is to generate knowledge of learning and teaching. Therefore, scientific processes (e.g., documenting decisions, rationales, and conditions; hypothesizing mechanisms; predicting events; and checking those predictions) must be carefully followed and recorded. Further, authors of models of teaching and learning should consider a wide variety of individual, social, and contextual aspects (e.g., Bauersfeld, 1980; Cobb, 2001; McClain & Cobb, 2001; Schofield, 1995; Secada, 1992). That is, a concerted effort should be made to view the curriculum, and the teaching and learning process, through multiple conceptual lenses (Schoenfeld, 2002), examining both underlying assumptions and data from as many alternate, often incompatible, perspectives as possible (Lester & Wiliam, 2002).

The construct of hypothetical learning trajectories is a cognitive tool grounded in constructivism. It has been adapted for use with a social perspective (e.g., Cobb, 2001), reconceptualizing the learning trajectory construct as a "sequence (or set) of (taken-as-shared) classroom mathematics practices that emerges through interaction (especially through classroom discourse—with the proactive involvement of the teacher)" (Yackel, personal communication, July 30, 2003). Yackel does not use learning trajectories to describe the learning of individual students; this interpretation differs from ours, which includes analyses at both the group and individual levels. However, it does reveal another application, and thus unique contribution, of the hypothetical learning trajectory construct.

In this description, we have taken the perspective of the researcher or researchers/curriculum developer who often writes for a general audience. It is important to note that learning trajectories could and should be reconceptualized or created by small groups or individual teachers, so they are based on more intimate knowledge of the particular students involved—their extant knowledge, learning preferences, and engagement in certain task types or contexts (Simon and Tzur, this issue). Indeed, a priori learning trajectories are always hypothetical in that the actual learning and teaching, and the teacher's recognition of these, cannot be completely known in advance (Simon, 1995). The teacher must construct new models of children's mathematics as they interact with children around the instructional tasks and thus alter their own knowledge of children and future instructional strategies and paths. Thus, the realized learning trajectory, the taken-as-shared practices and understandings, are emergent.

ARTICLES IN THIS SPECIAL ISSUE

This special issue contains articles by several leaders in inventing, elaborating, or contesting the construct of hypothetical learning trajectories. Simon (1995) introduced the construct hypothetical learning trajectory as a way to describe the pedagogical thinking involved in teaching mathematics for understanding. In their article, Simon and Tzur (this issue) elaborate this formulation within their constructivist framework by describing (a) a mechanism, reflection on task-effect relation, that is an elaboration of Piaget's (2001) reflective abstraction and (b) the ways such a mechanism can structure the use of each of several components of the hypothetical learning trajectory. Their mechanism, which explicates the role of mathematical tasks in mathematics concept development, addresses the learning paradox (Bereiter, 1985; Pascual-Leone, 1976).

Gravemeijer (this issue) describes what his interpretation of hypothetical learning trajectories offers to the reform of mathematics education. He contrasts that interpretation with classical instructional design theories that, in his opinion, do not fit mathematics education that tries to capitalize on the inventions of the students. This contrast is especially useful in that it clarifies some unique characteristics of hypothetical learning trajectories. Further, Gravemeijer illustrates his interpretation with a local instruction theory on addition and subtraction, showing how classroom teachers can use such theories to construe hypothetical learning trajectories fitting their classroom contexts.

Steffe (this issue) emphasizes that, "through the construction of learning trajectories that are co-produced by children, it is possible to construct learning trajectories of children that include an account of one's own ways and means of acting and operating as a teacher." He abstracts learning trajectories from acts of teaching children and argues that this is of critical importance, illustrating the process with research on children's construction of partitioning schemes and commensurate fractions.

Clements, Wilson, and Sarama (this issue) describe the genesis of a hypothetical learning trajectory involving young children's composition of geometric shape and present a multimethod evaluation of the developmental progression underlying it (however, instructional sequences were always a component of the research). Clements and Sarama's complete approach is to consider research on both aspects—learning and teaching for the initial hypothetical learning trajectory—and then design, iteratively test, and recognize that individual differences, variations, alternate routes, and further refinements are not only a goal, but a continual requirement (Clements, 2002; Clements & Battista, 2000; Sarama & Clements, in press). They are presently conducting research on the complete hypothetical learning trajectory.

Battista (this issue) argues that, to be scientific, assessment must be linked to research on student learning and cognition. He provides an illustration in the framework of his work on cognition-based assessment, an assessment system to detail the cognitive underpinnings of the progress students make in developing understanding of specific mathematics topics. This, then, also emphasizes the developmental progression aspect only, although the ideas were formed in the context of curriculum development and evaluation (similar to the work of Clements, Wilson, and Sarama, this issue).

Lesh and Yoon (this issue) present a challenge to an overly narrow view of learning trajectories as inflexible, narrow channels of learning (a view we do not believe is held by authors of the other articles, but a possible misinterpretation of the construct) in emphasizing a learning terrain that the authors visualize as an inverted genetic inheritance tree, in which great-grandchildren can trace their evolution from multiple lineages. From our view, Lesh and Yoon illustrate important issues, but we believe the differences can be at least in part an issue of grain size. That is, students who create various models can be following different but equally important learning trajectories (in the small); teachers would benefit from assessing these. More important, especially viewing learning trajectories as occurring in a social setting, each model-eliciting activity can (and probably should) be placed in larger cycles within a broad hypothetical learning trajectory that describes the teachers' and researchers' mathematical goals.

Baroody, Cibulskis, Lai, and Li (this issue) discuss all these articles in their reaction. Baroody and his colleagues provide a useful comparison to previous approaches, comments and critiques of all the other articles, and suggestions for extending the various approaches.

FINAL WORDS

We believe that the notion of hypothetical learning trajectories is a unique and substantive contribution to the field. The construct differs from other models in that it involves self-reflexive constructivism and includes the simultaneous consideration of mathematics goals, models of children's thinking, teachers' and researchers' models of children's thinking, sequences of instructional tasks, and the interaction of these at a detailed level of analyses of processes.

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