Refined Enumerations of Totally Symmetric
Self-Complementary Plane Partitions
and Lattice Path Combinatorics

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Abstract
This article is a short explanation of some of the results obtained in my papers “On refined
enumerations of totally symmetric self-complementary plane partitions I, II”. We give Pfaffian
expressions for some of the conjectures in the paper “Self-complementary totally symmetric
plane partitions” (J. Combin. Theory Ser. A 42, 277–292) by Mills, Robbins and Rumsey,
using the lattice path method.

1 Introduction
In the paper [8] Mills, Robbins and Rumsey presented several conjectures on the enumeration of
the totally symmetric self-complementary plane partitions. The aim of this article is to obtain
Pfaffian expressions for the refined enumeration and doubly refined enumeration of the totally
symmetric self-complementary plane partitions (see Theorem 1.4). In [4, 5], we obtain more Pfaffian
or determinant expressions, and certain constant term identities for the conjectures.

A plane partition is an array \( \pi = (\pi_{ij})_{i,j \geq 1} \) of nonnegative integers such that \( \pi \) has finite
support (i.e. finitely many nonzero entries) and is weakly decreasing in rows and columns. If
\( \sum_{i,j \geq 1} \pi_{ij} = n \), then we write \( |\pi| = n \) and say that \( \pi \) is a plane partition of \( n \), or \( \pi \) has weight
\( n \). A part of a plane partition \( \pi = (\pi_{ij})_{i,j \geq 1} \) is a positive entry \( \pi_{ij} > 0 \). The shape of \( \pi \) is the
ordinary partition \( \lambda \) for which \( \pi \) has \( \lambda_i \) nonzero parts in the \( i \)th row. Consider the elements of \( \mathbb{P}^3 \),
regarded as the lattice points of \( \mathbb{R}^3 \) in the positive orthant. The Ferrers graph \( F(\pi) \) of \( \pi \) is the set
of all lattice points \( (x, y, z) \in \mathbb{P}^3 \) such that \( z \leq \pi_{ij} \). A subset \( F \) of \( \mathbb{P}^3 \) is a Ferrers graph if and only
if it satisfies
\[
x_1 \leq x_2, \ y_1 \leq y_2, \ z_1 \leq z_2 \text{ and } (x_2, y_2, z_2) \in F \Rightarrow (x_1, y_1, z_1) \in F.
\]

Hereafter we identify a plane partition and its Ferrers graph, and write \( \pi \) for \( F(\pi) \). The symmetric
group \( S_3 \) is acting on \( \mathbb{P}^3 \) as permutations of the coordinate axes. A plane partition is said to be
totally symmetric if its Ferrers graph is mapped to itself under all 6 permutations in \( S_3 \).

A plane partition \( \pi \subseteq X_{r,s,t} := [r] \times [s] \times [t] \) is \((r, s, t)\)-self-complementary if we have, for all
\( p \in X_{r,s,t}, \ p \in \pi \) if and only if \( \sigma_{r,s,t}(p) \not\in \pi \). Let \( T_n \) denote the set of all plane partitions which is
contained in the cube \( X_{2n}, (2n, 2n, 2n)\)-self-complementary and totally symmetric.

In [8] Mills, Robbins and Rumsey have introduced a class \( B_n \) of triangular shifted plane partitions
\[
\begin{array}{cccc}
b_{11} & b_{12} & \ldots & b_{1,n-1} \\
b_{22} & b_{2,n-1} \\
\vdots & & & \\
b_{n-1,n-1}
\end{array}
\]
Let $U_{\text{max}}$ be the shifted shape of $U_c$. For example, $C_2$ consists of the following seven elements:

\[
\begin{array}{cccccccc}
3 & 3 & 3 & 3 & 3 & 3 & 3 & 2 \\
3 & 2 & 1 & 2 & 1 & 2 & & \\
& & & & & & & 1 \\
\end{array}
\]

They have established a bijection between $T_n$ and $B_n$.

Let $\mu$ be a strict partition. A shifted plane partition $\tau$ of shifted shape $\mu$ is an arbitrary filling of the cells of $\mu$ with nonnegative integers such that each entry is weakly decreasing in rows and columns. In this article we allow parts to be zero for shifted plane partitions of a fixed shifted shape $\mu$. Here we consider a more general set $\mathcal{P}_{n,m}$ of shifted plane partitions which appeared in [7, Theorem 1].

**Definition 1.1.** Let $m$ and $n \geq 1$ be nonnegative integers. Let $\mathcal{B}_{n,m}$ denote the set of shifted plane partitions $b = (b_{ij})_{1 \leq i \leq j}$ subject to the constraints that

\[(B1)\text{ the shifted shape of } b \text{ is } (n + m - 1, n + m - 2, \ldots, 2, 1);\]

\[(B2)\text{ max}\{n - i, 0\} \leq b_{ij} \leq n \text{ for } 1 \leq i \leq j \leq n + m - 1.\]

The main object we study in this article is the following set $\mathcal{P}_{n,m}$, which is bijective with the set $\mathcal{B}_{n,m}$ defined above.

**Definition 1.2.** Let $m$ and $n \geq 1$ be nonnegative integers. Let $\mathcal{P}_{n,m}$ denote the set of plane partitions $c = (c_{ij})_{1 \leq i,j}$ subject to the constraints that

\[(C1)\text{ }c \text{ has at most } n \text{ columns;}\]

\[(C2)\text{ }c \text{ is column-strict and each part in the } j\text{th column does not exceed } n + m - j.\]

If a part in the $j\text{th}$ column of $c$ is equal to $n + m - j$ (that can happen only in the first row, i.e. $c_{1j} = n + m - j$), we call the part a saturated part.

The important fact is that we construct a bijection between $\mathcal{B}_{n,m}$ and $\mathcal{P}_{n,m}$ in [4]. By this bijection, the statistics on $\mathcal{B}_{n,m}$ defined by Mills, Robbins and Rumsey in [8] correspond to the following statistics $\mathcal{B}_r(c)$ on $\mathcal{P}_{n,m}$.

**Definition 1.3.** For $c \in \mathcal{P}_{n,m}$, let $\mathcal{B}_r(c)$ be the number of parts equal to $r$ plus the number of saturated parts less than $r$, i.e.

\[
\mathcal{B}_r(c) = \sharp\{(i,j) : c_{ij} = r\} + \sharp\{1 \leq k < r : c_{1,n+m-k} = k\}. \tag{1.1}
\]

Especially $\mathcal{B}_1(c)$ is the number of 1’s in $c$ and $\mathcal{B}_{n+m}(c)$ is the number of saturated parts in $c$. It is also easy to see that $\mathcal{B}_{n+m-1}(c) = \mathcal{B}_{n+m}(c)$ since, if a part of $c \in \mathcal{P}_{n,m}$ is equal to $n + m - 1$, then it is saturated.

Let $\bar{S}_n = (\bar{s}_{ij})_{1 \leq i,j \leq n}$ be the skew-symmetric matrix of size $n$ whose $(i,j)$th entry $\bar{s}_{ij}$ is $(-1)^{j-i-1}$ for $1 \leq i < j \leq n$. Let $B_{n,m}^N(t,u) = (b_{ij}^{(m)}(t,u))_{0 \leq i \leq n-1, 0 \leq j \leq n+N-1}$ be the $n \times (n+N)$ matrix whose $(i,j)$th entry is

\[
b_{ij}^{(m)}(t,u) = \begin{cases} 
\delta_{i,j} & \text{if } i+j = 0, \\
\binom{i+m-1}{j-i} + \binom{i+m-1}{j-i-1}tu & \text{if } i+j = 1, \\
\binom{i+m-2}{j-i} + \binom{i+m-2}{j-i-1}(t+u) + \binom{i+m-2}{j-i-2}tu & \text{otherwise}.
\end{cases} \tag{1.2}
\]

For example,

\[
B_{3,0}^N(t,u) = \begin{pmatrix} 1 & 0 & 0 & 0 \\
0 & 1 & tu & 0 \\
0 & 0 & 1 & t + u \\
& & & tu \end{pmatrix}.
\]

We define the $n \times (n+N)$ matrices $B_{n,m}^N(t) = B_{n,m}^N(t,1)$ and $B_{n,m}^N(t,1)$. The results of this article is the following theorem.
Let $n \geq 1$ be non-negative integers, and let $N$ be an even integer such that $N \geq n + m - 1$.

(i) If $r$ is a positive integer such that $2 \leq r \leq n + m$, then the generating function for all plane partitions $c \in \mathcal{P}_{n,m}$ with the weight $t^{U_1(c)}t^{U_r(c)}$ is

$$
\sum_{c \in \mathcal{P}_{n,m}} t^{U_1(c)}t^{U_r(c)} = \text{Prop}(\frac{O_n}{-B^{N}_{n,m}(t,u)}J_nS_{n+N}^{N}(t,u)) \quad (1.3)
$$

(ii) If $r$ is a positive integer such that $1 \leq r \leq n + m$, then the generating function for all plane partitions $c \in \mathcal{P}_{n,m}$ with the weight $t^{U_r(c)}$ is given by

$$
\sum_{c \in \mathcal{P}_{n,m}} t^{U_r(c)} = \text{Prop}(\frac{O_n}{-B^{N}_{n,m}(t)}J_nS_{n+N}^{N}(t)) \quad (1.4)
$$

Now we assign weight

$$
l^{U_r(c)}x^c = \prod_{k=1}^{m+n} t^{U_k(c)} \prod_{c \geq 1} x_i^{n \text{'s in } c}
$$

to each $c \in \mathcal{P}_{n,m}$. We prove Theorem 1.4 from the minor summation formula [6] and the following theorem, which can be proved with the lattice path method.

**Theorem 1.5.** Let $m$ and $n \geq 1$ be non-negative integers, and put $N = n + m$. Let $\lambda$ be a partition with $\ell(\lambda) \leq n$. Then the generating function of all plane partitions $c \in \mathcal{P}_{n,m}$ of shape $\lambda'$ with the weight $l^{U_r(c)}x^c$ is given by

$$
\sum_{c \in \mathcal{P}_{n,m}, \mu(c) = \lambda'} t^{U_r(c)}x^c = \det \left( e_{\lambda'}^{(N-i)}(t_{1}, x_{1}, \ldots, t_{N-i-1}, x_{N-i-1}, T_{N-i}x_{N-i}) \right)_{1 \leq i, j \leq n},
$$

where $T_i = \prod_{k=i}^{N} t_k$.

In fact, we give a lattice path realization of each $c \in \mathcal{P}_{n,m}$. Let $V = \{(x, y) \in \mathbb{N}^2 : 0 \leq y \leq x\}$ be the vertex set, and direct an edge from $u$ to $v$ whenever $v - u = (1,-1)$ or $(0,-1)$.

(i) We assign the weight

$$
\begin{cases}
\prod_{k=j}^{N} t_k \cdot x_j & \text{if } j = i, \\
 t_j x_j & \text{if } j < i,
\end{cases}
$$

to the horizontal edge from $u = (i, j)$ to $v = (i, j - 1)$.

(ii) We assign the weight 1 to the vertical edge from $u = (i, j)$ to $v = (i, j - 1)$.

Let $u_j = (N-j, N-j)$ and $v_j = (\lambda_j + N-j, 0)$ for $j = 1, \ldots, n$, and let $u = (u_1, \ldots, u_n)$ and $v = (v_1, \ldots, v_n)$. We claim that the $c \in \mathcal{P}_{n,m}$ of shape $\lambda'$ can be identified as $n$-tuples of nonintersecting $D$-paths in $\mathcal{P}(u,v)$. For example, the plane partition

$$
\begin{array}{cccccccc}
8 & 8 & 7 & 5 & 5 & 3 & 3 \\
7 & 7 & 6 & 3 & 3 & 2 \\
5 & 5 & 5 & 2 & 2 \\
3 & 2 & 2 & 1 & 1 \\
2 & 1 & 1 \\
1 & & & & & & & \\
\end{array}
$$

corresponds to the lattice paths illustrated in Figure 1.
Figure 1: Lattice Paths \((n = 7, m = 3, \lambda' = (65^242^21)), T_i = \prod_{k=i}^{n+m} t_k.\)

References


