Graph Drawing Beyond Planarity: Some Results and Open Problems

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Outline

• Graph Drawing (GD) beyond planarity
• Combinatorial relationships
• Optimization trade-offs and algorithms
• Open problems
GD beyond planarity
Relational Data Sets
Graph Drawing

\[ G = (V, E) \]
\[ V = \{1, 2, 3, 4, 5, 6\} \]
\[ E = \{(1,3) \, (1,6) \, (2,3) \, (2,5) \, (2,4) \, (2,6) \, (3,5) \, (4,5) \, (4,6)\} \]

The drawing must be readable.
Readability and Crossings

edge crossings significantly affect the readability (see, e.g., Sugiyama et al., Warshall, North et al., Batini et al., mid 80s) - confirmed by cognitive experimental studies (Purchase et al., 2000-2002)

rich body of graph drawing techniques assume the input is a planar (planarized) graph and avoid edge crossings as much as possible
The planarization handicap

for dense enough or constrained enough drawings, many edge crossing are unavoidable

FlyCircuit Database, NTHU
Mutzel’s intuition about crossings

34 crossings:
minimum “skewness”
(number of edges whose deletion makes it planar)

24 crossings:
minimum number of crossings
Experiments of Eades, Hong, Huang

Observations from eye tracking

- **No crossings**: eye movements were smooth and fast.
- **Large crossing angle**: eye movements were smooth, but a little slower.
- **Small crossing angle**: eye movements were very slow and no longer smooth (back-and-forth movements at crossing points).
Example

[Didimo, L., Romeo, “A Graph Drawing Application to Web Site Traffic Analysis”, JGAA 2011]
Beyond planarity

The visual complexity not only depends on the number of crossings but also on the type of crossings.

Challenge: compute drawings where some "bad" crossing configurations are forbidden (minimized).
Drawings with forbidden crossing configurations

- RAC
- SKEWNESS-h (h=1)
- h-PLANAR (h=3)
- h-QUASI-PLANAR (h=3)
Drawings with forbidden crossing configurations

- Strong 1-visibility drawing
- Weak 1-visibility drawing
Most explored research directions

**Turán-type:** find upper bounds on the edge density

**Recognition:** how hard is it to test whether a graph admits a drawing with a forbidden configuration?

**Fáry-type:** given a drawing (with jordan arcs), is there a straight-line drawing that preserves the given topology?
New research directions

- Study the **combinatorial relationships** between different families of nearly planar graphs.

- Study **trade-offs** between crossing complexity and other aesthetic criteria.
Combinatorial relationships between nearly planar graphs
1-planarity, quasi-planarity and 1-visibility

[Evans et al., 2014]
RAC and 1-planarity

[ Eades, L., 2013 ]
Theorem

A maximally dense RAC graph is 1-planar. Also, for every integer $i$ such that $i \geq 0$ there exists a 1-planar graph with $n = 8 + 4i$ vertices and $4n - 10$ edges that is not a RAC graph. Finally, for every integer $n > 85$, there exists a RAC graph with $n$ vertices that is not 1-planar. [Eades, L., 2013]
Some details about the proof
Preliminaries: edge coloring

- red edges do not cross
- each green edge crosses with a blue edge
  - red-blue (embedded planar) graph = red + blue edges
  - red-green (embedded planar) graph = red + green edges
Preliminaries: $G_{rb}$ and $G_{rg}$ in a maximally dense RAC graph

Each internal face of $G_{rb}$ ($G_{rg}$) has at least two red edges [Didimo, Eades, L., 2011]
Preliminaries: $G_{rb}$ and $G_{rg}$ in a maximal RAC graph

Notation:

- $m_r, m_b, m_g =$ number of red, blue, and green edges
- $f_{rb} =$ number of faces of the red-blue graph $G_{rb}$

Assumption:

- $m_g \leq m_b$
Maximally dense RAC graphs are 1-planar

Approach:

• suppose we can show that $G_{rb}$ and $G_{rg}$ are both maximal planar graphs; then:
Maximally dense RAC graphs are 1-planar

Approach:

• suppose we can show that $G_{rb}$ and $G_{rg}$ are both maximal planar graphs; then:

![Diagram](attachment:image.png)
$G_{rb}$ and $G_{rg}$ are maximal planar graphs (1)

- the following is proven first:

**Claim 1**: the external face of $G_{rb}$ and $G_{rg}$ is a 3-cycle

- then, we consider the internal faces of $G_{rb}$ that share at least one edge with the external face (fence faces)

  there are at least 1 and at most 3 fence faces
$G_{rb}$ and $G_{rg}$ are maximal planar graphs (2)

• ...and prove the following

**Claim 2:** If $G$ is maximal, $G_{rb}$ has three fence faces and each fence face is a 3-cycle

• obs: at least two fence faces consist of red edges

\[ \alpha + \beta + \gamma \geq 360^\circ \]
\[ \alpha < 90^\circ \]
\[ \Rightarrow \beta \geq 90^\circ \text{ and } \gamma \geq 90^\circ \]
$G_{rb}$ and $G_{rb}$ are maximal planar graphs (3)

since: (1) each internal face of $G_{rb}$ has at least 2 red edges; (2) the external face of $G_{rb}$ is a red 3-cycle; (3) at least two fence faces are red 3-cycles \( \Rightarrow \quad 2m_r \geq 2(f_{rb} - 3) + 3 + 3 + 3 \)

since $m_r$ and $f_{rb}$ are integers, we obtain

By Euler’s formula for planar graphs \( \Rightarrow \quad m_r + m_b \leq n + f_{rb} - 2 \)

\( m_b \leq n - 4 \)
$G_{rb}$ and $G_{rb}$ are maximal planar graphs (4)

G is a maximally dense RAC graph $\Rightarrow m_b + m_r + m_g = 4n - 10$

$m_b \leq n - 4$

$m_r + m_g \geq 3n-6$

since by assumption $m_g \leq m_b$ and since both $G_{rg}$ and $G_{rb}$ are planar $\Rightarrow G_{rg}$ and $G_{rb}$ are both maximal planar graphs

Therefore a maximal RAC graph is 1-planar
RAC Graphs that are not 1-planar

There exists a graph $G$ with less than $4n-10$ such that $G$ is a RAC graph but is not 1-planar.
RAC Graphs that are not 1-planar

There exists a graph $G$ with less than $4n - 10$ such that $G$ is a RAC graph but is not 1-planar.

$\geq 4$
Not all 1-planar graphs with $4n-10$ edges are maximal RAC

$G_0$ has $n=8$ vertices and $4n-10=22$ edges; for $i \geq 0$, $G_i$ has $n=8+4i$ vertices and $4n-10$ edges
Not all 1-planar graphs with $4n - 10$ edges are maximal RAC.

$G_o$ has $n = 8$ vertices and $4n - 10 = 22$ edges; for $i \geq 0$, $G_i$ has $n = 8 + 4i$ vertices and $4n - 10$ edges; they are 1-planar graphs.

We show that $G_i$ cannot be realized as a RAC graph (by induction on $i$).
$G_o$ is not RAC realizable

every vertex has degree 5 or 6.

for every 3-cycle there is a $K_4$

for every $K_4$, there is a 4-cycle through the other vertices
$G_o$ is not RAC realizable

if $G_o$ were RAC realizable, the external face of the realization would be a 3-cycle
$G_o$ is not RAC realizable

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\( G_o \) is not RAC realizable

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...summarizing....
Area Requirement Beyond Planarity

- RAC
- skewness-h
- h-planar
RAC straight-line drawings of planar graphs may require quadratic area

(Angelini et al., JGAA 2011)
Area req. of h-planar drawings

**h-planar (constant h)** straight-line drawings (and RAC straight-line drawings) of planar graphs may require **quadratic area** [Di Giacomo et al., 2012]
Area req. of skewness-$h$ drawings

skewness-$h$ (constant $h$) straight-line drawings of planar graphs may require quadratic area

$O(n^2)$ area, if planar
linear area upper bound

4-quasi-planar

h-quasi-planar drawings
Bounded treewidth

\[ G \text{ has treewidth } \leq k \iff G \text{ is a partial } k\text{-tree} \]
G has treewidth $\leq k \iff G$ is a partial $k$-tree
Bounded treewidth

G has treewidth \( \leq k \iff G \) is a partial \( k \)-tree
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G has treewidth $\leq k \iff G$ is a partial $k$-tree
Bounded treewidth

G has treewidth $\leq k \iff G$ is a partial $k$-tree
every n-vertex graph with bounded treewidth admits an $h$-quasi planar straight-line drawing in linear area such that the value of $h$ does not depend on $n$

[Di Giacomo, Didimo, L., Montecchiani, 2013]
Applying the result

every $h$-colorable graph has a linear area s.l. drawing
[Wood, CGTA, 2005]

Di Giacomo et al., 2013
Ingredients

study the relationship between $(c, t)$-track layouts and h-quasi planar straight-line drawings

new technique to compute a $(2, t)$-track layout of a partial k-tree
Open problems
Inclusion properties and RAC graphs

Characterize those 1-planar graphs that have a RAC drawing.

Recognizing those graphs that have a RAC drawing is NP-hard. Does this problem remain NP-hard for those graphs with $n$ vertices and $4n-10$ edges?
Area-crossing complexity trade-offs

Do partial $k$-trees admit a $O(1)$-quasi planar straight line drawing in linear area and constant aspect ratio?

For, example, do outerplanar graphs admit a $3$-quasi planar straight line drawing in linear area and constant aspect ratio?

Do all planar graphs have a sub-quadratic area $h$-quasi planar straight-line drawing with constant $h$?
## Other problem categories

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<th>Turan-type</th>
<th>Recognition</th>
<th>Fary-type</th>
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<tbody>
<tr>
<td><strong>RAC</strong></td>
<td>O(n)</td>
<td>NP-hard (linear-time for 2-layer)</td>
<td>-</td>
</tr>
<tr>
<td><strong>1-planar</strong></td>
<td>O(n)</td>
<td>NP-hard (linear time for given rot. syst.)</td>
<td>character. test, drawing</td>
</tr>
<tr>
<td><strong>3-quasi-planar</strong></td>
<td>O(n)</td>
<td>??</td>
<td>??</td>
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<td><strong>skewness-1</strong></td>
<td>O(n)</td>
<td>polynomial</td>
<td>character. test, drawing</td>
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