

INTEGRATOR BACKSTEPPING USING CONTRACTION THEORY: A BRIEF METHODOLOGICAL NOTE.

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Abstract: While the use of Lyapunov function candidates for integrator backstepping has been extensively studied in the literature, little research has been conducted regarding the applicability of the so-called incremental stability approaches. This note addresses the problem of the use of an incremental approach, *i.e.* contraction theory to the integrator backstepping design on the methodological aspect. After briefly recalling basic results from contraction theory, a full contraction-based integrator backstepping procedure is presented. An example is given to illustrate the method.

Keywords: Backstepping, Stability, Nonlinear Systems, Contraction Theory.

1. INTRODUCTION

In the past several years, there has been considerable interest in recursive designs for nonlinear control schemes, namely backstepping and forwarding, and a number of textbooks treating this subject have appeared in the literature (see (Krstić, *et al.*, 1995; Freeman and Kokotović, 1996; Khalil, 1996; Sepulchre, *et al.*, 1997)), together with some applications (see for example (Fossen and Grøvlén, 1998)). These well-known techniques are traditionally based on the construction of appropriate Lyapunov function candidates.

Later on, new tools for analyzing stability referred to as *incremental stability* approaches (Fornion, *et al.*, 1999; Lohmiller, 1999; Angeli, 2002) were proposed. On the contrary to Lyapunov-based analysis where trajectories are studied with respect to a specific nominal motion, incremental stability approaches are used to analyze the behavior of trajectories with respect to one another without considering any particular motion.

Among those approaches, *contraction theory*, also called *contraction analysis* is the one that may be seen as the incremental counterpart of Lyapunov stability theory ((Lohmiller, 1999; Lohmiller and Slotine, 1998)).

As pointed out in (Angeli, 2002), incremental approaches still lack recursive methods that have now been extensively studied with Lyapunov functions. As a consequence, this paper addresses the question of constructing an integrator backstepping procedure under the framework of contraction analysis. It is to be noted that the procedure depicted in this paper has been deliberately made simple in order to give a clearer idea of the differences with Lyapunov-based recursive designs. In addition to an attempt to avoid the burden of equations that would somewhat hide some meaningful aspects, the present note is intended as a first step towards a more rigorous and detailed analysis of backstepping procedures using incremental approaches.

After briefly reviewing the concepts of contraction theory that will be used throughout this paper, a very short classification of contracting systems is proposed in Section 2. Following this classification, a particular class of contracting systems is then used to study the contracting integrator backstepping design proposed in section 3. As an illustration, a simple example is treated in section 4. Finally, brief concluding remarks end the paper.

2. SOME BASICS RESULTS OF CONTRACTION ANALYSIS

The problem considered in contraction theory is to analyze the behavior of a system, possibly subject to control, for which a nonlinear model is known of the following form

$$\dot{x} = f(x, t) \quad (1)$$

where $x \in \mathbb{R}^n$ stands for the state whereas f is a nonlinear function. By this equation, one can notice that the control may easily be expressed implicitly for it is merely a function of state and time. Contracting behavior is determined upon the exact differential relation

$$\delta \dot{x} = \frac{\partial f}{\partial x}(x, t) \delta x \quad (2)$$

where δx is a virtual displacement, *i.e.* an infinitesimal displacement at fixed time.

For the sake of clarity, thereafter are reproduced the main definition and theorem of contraction taken from (Lohmiller and Slotine, 1998).

Definition 2.1. A region of the state space is called a contraction region with respect to uniformly positive definite metric

$$M(x, t) = \Theta^T(x, t) \Theta(x, t)$$

where Θ stands for a differential coordinate transformation matrix, if equivalently

$$F = (\dot{\Theta} + \Theta \frac{\partial f}{\partial x}) \Theta^{-1} \text{ or } \frac{\partial f}{\partial x}^T M + \dot{M} + M \frac{\partial f}{\partial x}$$

are uniformly negative definite.

The last expression can be regarded as an extension of the well-known Krasovskii method using a time and state dependent metric. On a historical perspective, note that results very closed from this one —however with a state but not time dependent metric— were established in the early sixties (Hartman, 1961), though with a slightly different interpretation.

Definition 2.1 leads to the following convergence result:

Theorem 2.1. Any trajectory, which starts in a ball of constant radius with respect to the metric $M(x, t)$, centered at a given trajectory and contained at all time in a contraction region, remains in that ball and converges exponentially to this trajectory.

In the following, we will only consider *global* convergence, *i.e.* the contraction region corresponds to the whole state space.

The next combination property will also be useful in section 3.

Connecting two systems, contracting under possibly different metrics, in feedback form as

$$\frac{d}{dt} \begin{pmatrix} \delta z_1 \\ \delta z_2 \end{pmatrix} = \begin{pmatrix} F_1 & G \\ -G^T & F_2 \end{pmatrix} \begin{pmatrix} \delta z_1 \\ \delta z_2 \end{pmatrix} \quad (3)$$

where $\delta z_i = \Theta_i \delta x_i$, then the resulting global system is in turn contracting. This can be generalized to any number of contracting systems (Slotine and Lohmiller, 2001). The above statement can of course be related to the well known passivity concepts and the feedback interconnection of passive systems, which is passive as well. This result also holds with a pre- and post-multiplication by a matrix as seen above (see (Sepulchre, *et al.*, 1997, p. 34)).

Contracting systems are also exponentially stable. Exponential stability is an important concept since it enables to quantify in an explicit manner the rate of decrease of the initial condition-perturbed system towards the equilibrium. Under specific conditions (constant metric), it can be very easily proven that a contracting system exhibits a Uniform Globally Exponentially Stable (UGES) property. Furthermore, a relation with another concept of incremental stability (Fromion, *et al.*, 1999) is established (see (Jouffroy and Lotin, 2002)). Following this remark, we give hereafter a short classification accounting for different kinds of contracting systems:

Definition 2.2. A system, contracting with respect to a unitary metric $M = I$ is called a *directly contracting* system.

This definition corresponds to what was done on the early developments of contraction theory.

Definition 2.3. A system, contracting with respect to a constant metric M is called a *flat contracting* system.

The term *flat* was chosen after the work of (Hartman, 1961, section 4) where the constant

metric was characterized this way. Not surprisingly, this definition includes the class of linear time-invariant systems. Also, the equivalent Lyapunov function would be obtained from the Krasovskii method.

Definition 2.4. A system, contracting with respect to a time and/or state-dependent metric $M(x, t)$ is called a *Riemann contracting* system.

In this definition, we depart quite far from Lyapunov theory because no Lyapunov function exists for a metric $M(x, t)$ in general. Indeed, the length defined by the measure of the arc joining two distant points in space would depend on the path chosen to link these two points.

3. CONTRACTING INTEGRATOR BACKSTEPPING DESIGN

For the sake of clarity, this section will follow quite the same outline as presented in (Krstić, *et al.*, 1995, p. 33-37) while keeping the same notations. Also, this should make it easier to compare with the well-known Lyapunov-based technique.

Assumption 3.1. Consider the system

$$\dot{x} = f(x, t) + B(t)u \quad (4)$$

where $x \in \mathbb{R}^n$ is the state, t is the time variable and $u \in \mathbb{R}$ is the control input. There exists a continuously differential feedback control law

$$u = \alpha(x, t) \quad (5)$$

such that the overall system

$$\dot{x} = f(x, t) + B(t)\alpha(x, t) \quad (6)$$

is directly contracting.

Note that the framework of contraction analysis allows to deal indifferently with time varying or time invariant systems, which enables to extend the procedures designed for stationary systems very easily. In terms of interpretation, the previous result may be regarded as an incremental version of feedback passivation.

Models more complex than (3) may be considered and rendered contracting. However, this particular class of systems will be shown to be adequate for a first use of the backstepping procedure.

Lemma 3.1. Let the system (4) be augmented by an integrator :

$$\dot{x} = f(x, t) + B(t)\xi \quad (7)$$

$$\dot{\xi} = u \quad (8)$$

and suppose that (7) satisfies Assumption 3.1 with ξ as its control.

If (6) is directly contracting, then there exists a feedback control law $u(x, \xi, t)$ such that the closed-loop system is directly contracting, with its jacobian as

$$F = \begin{pmatrix} \frac{\partial}{\partial x}(f + B\alpha) & B \\ -B^T & \frac{\partial u}{\partial z} - \frac{\partial \alpha}{\partial x}B \end{pmatrix} \quad (9)$$

with $z = \xi - \alpha(x, t)$.

Using the combination property of the previous section, a sketch of proof is easily found. Indeed, defining $\xi_{des} = \alpha(x, t)$ as the virtual control desired value, the deviation of ξ from ξ_{des} is written as

$$\begin{aligned} z &= \xi - \xi_{des} \\ &= \xi - \alpha(x, t) \end{aligned} \quad (10)$$

and system equation (7) can be transformed into

$$\dot{x} = f(x, t) + B(t)\alpha(x, t) + B(t)z \quad (11)$$

while (8) together with (10) yields

$$\begin{aligned} \dot{z} &= \dot{\xi} - \dot{\alpha}(x, t) \\ &= u - \frac{\partial \alpha}{\partial t} - \frac{\partial \alpha}{\partial x}\dot{x} \\ &= u(x, z, t) - \frac{\partial \alpha}{\partial t} - \frac{\partial \alpha}{\partial x}(f + B\alpha + Bz) \end{aligned} \quad (12)$$

Equations (11) and (12) yield in turn

$$\delta \dot{x} = \frac{\partial}{\partial x}(f + B\alpha) \delta x + B\delta z \quad (13)$$

and

$$\begin{aligned} \delta \dot{z} &= \frac{\partial}{\partial x} \left(u - \frac{\partial \alpha}{\partial t} - \frac{\partial \alpha}{\partial x}(f + B\alpha + Bz) \right) \delta x \\ &\quad + \left(\frac{\partial u}{\partial z} - \frac{\partial \alpha}{\partial x}B \right) \delta z \end{aligned} \quad (14)$$

with overall virtual displacement dynamics being

$$\begin{pmatrix} \delta \dot{x} \\ \delta \dot{z} \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial x}(f + B\alpha) & B \\ \frac{\partial}{\partial x}(u - \dot{\alpha}) & \frac{\partial u}{\partial z} - \frac{\partial \alpha}{\partial x}B \end{pmatrix} \begin{pmatrix} \delta x \\ \delta z \end{pmatrix} \quad (15)$$

Now if u and thus α are chosen such that

$$\frac{\partial}{\partial x}(f + B\alpha) < 0 \quad (16)$$

$$\frac{\partial u}{\partial z} - \frac{\partial \alpha}{\partial x}B < 0 \quad (17)$$

$$\frac{\partial}{\partial x} \left(u - \frac{\partial \alpha}{\partial t} - \frac{\partial \alpha}{\partial x}(f + B\alpha + Bz) \right) = -B^T \quad (18)$$

then system (7)-(8) is contracting. Obviously, condition (18) implies jacobian (9), while conditions (16) and (17) ensure that the overall system is contracting.

Explicit expression of α and u may be found, in particular when considering the special case of exact feedback linearization. However, let us stress the fact that it is one of the key features of backstepping to allow the designer to choose the most appropriate and suitable feedback functions because they mainly depend on the required performances (speed, robustness, precision, etc...). The above procedure can then be repeated recursively, thus leading to the following corollary.

Corollary 3.1. Let the system (4) satisfying Assumption 3.1 with $\alpha(x, t) = \alpha_0(x, t)$ be augmented by a chain of k integrators so that u is replaced by ξ_1 , the state of the last integrator of the chain:

$$\begin{cases} \dot{x} &= f(x, t) + B(t)\xi_1 \\ \dot{\xi}_1 &= \xi_2 \\ \vdots & \\ \dot{\xi}_{k-1} &= \xi_k \\ \dot{\xi}_k &= u \end{cases} \quad (19)$$

By repeating recursively Lemma 3.1 with $\xi_i, 1 \leq i \leq k$, the Jacobian is obtained as

$$F = \begin{pmatrix} F_0 & B & 0 & \dots & 0 \\ -B^T & F_1 & G_1 & \ddots & \vdots \\ 0 & -G_1^T & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & F_{k-1} & G_{k-1} \\ 0 & \dots & 0 & -G_{k-1}^T & F_k \end{pmatrix} \quad (20)$$

and implies that (19) is contracting if

$$F_i < 0, \forall i \in \{0; k\} \quad (21)$$

like the terms B and $-B^T$, G_i and $-G_i^T$ represent the crossed-terms obtained in the next steps of the procedure. They are noted differently from B to stress the fact that unlike B , they were not initially present in the system structure but were introduced by the control u .

This last corollary may be of interest because firstly, on the contrary to Lyapunov analysis, it makes appear explicitly the feedback interconnection structure, instead of reducing the stability behavior into a single scalar function. Second, it does not require the energy-like form (storage function, ...) that the passivity paradigm implies. Finally, since contraction behavior is somewhat independent of the attractor, there is conceptually no significant difference between stabilization and

tracking, enabling thus a more generalized point of view.

On another aspect, note that the extension to flat contracting systems (see Definition 2.3) through several constant transformation matrices Θ_i is fairly simple.

4. AN EXAMPLE

The next example is intended as an illustration of the above procedure. As for the Lyapunov-based procedure, the distinction between the stabilization function construction and the stability behavior analysis is quite apparent.

Consider the system equation

$$\begin{cases} \dot{x} &= -x^3 + \xi_1 \\ \dot{\xi}_1 &= \xi_2 \\ \dot{\xi}_2 &= u \end{cases} \quad (22)$$

Setting the desired value ξ_{1des} as

$$\xi_{1des} = -x \quad (23)$$

will render the \dot{x} -subsystem contracting in x , and expressing deviation z as

$$\begin{aligned} z_1 &= \xi_1 - \xi_{1des} \\ &= \xi_1 + x \end{aligned} \quad (24)$$

we obtain

$$\dot{x} = -x^3 - x + \xi_1 \quad (25)$$

and

$$\begin{aligned} \dot{z}_1 &= \dot{\xi}_1 - \dot{\xi}_{1des} \\ &= \xi_2 - \frac{\partial \xi_{1des}}{\partial x} \dot{x} \\ &= \xi_2 - x^3 - x + z_1 \end{aligned} \quad (26)$$

Thus, ξ_{2des} is chosen as

$$\xi_{2des} = -2z_1 + x^3 \quad (27)$$

and the z_1 -dynamics are

$$\dot{z}_1 = -x - z_1 + z_2 \quad (28)$$

which shows that this subsystem is directly contracting in z_1 .

The next step consists in computing \dot{z}_2 :

$$\begin{aligned} \dot{z}_2 &= \dot{\xi}_2 - \dot{\xi}_{2des} \\ &= u - \frac{\partial \xi_{2des}}{\partial x} \dot{x} - \frac{\partial \xi_{2des}}{\partial \xi_1} \dot{z}_1 \\ &= u - 3x^2 \dot{x} + 2\dot{z}_1 \end{aligned} \quad (29)$$

and finally, control is obtained as

$$u = 3x^2\dot{x} - 2\dot{z}_1 - z_2 - z_1 \quad (30)$$

Time differentiation of x and z_1 is not needed as both are known functions which time derivative can be computed analytically. Hence, the control can be implemented as

$$u = 3x^2(-x^3 - x + z_1) - 2(-x - z_1 + z_2) - z_2 - z_1 \quad (31)$$

Looking at the interconnections, one can check with the overall virtual displacement dynamics

$$\begin{pmatrix} \delta\dot{x} \\ \delta\dot{z}_1 \\ \delta\dot{z}_2 \end{pmatrix} = \begin{pmatrix} -3x^2 - 1 & 1 & 0 \\ -1 & -1 & 1 \\ 0 & -1 & -1 \end{pmatrix} \begin{pmatrix} \delta x \\ \delta z_1 \\ \delta z_2 \end{pmatrix} \quad (32)$$

that the closed-loop system can finally be concluded as being contracting.

5. CONCLUDING REMARKS

This paper has addressed the question of the construction of integrator backstepping techniques based on incremental stability approaches. One of them —contraction theory— was used to “mimic” Lyapunov-based procedures, enabling thus to compare the applicability of the two methods. Qualitative remarks were made along the procedure description, and a simple example was provided for purpose of illustration.

As mentioned in the introduction, there is still a need for a more comprehensive study. Current research includes further formalization as well as the use of contraction theory to ensure robust design in a quantitative way. Also, other useful designs as for example integrator forwarding (see (Sepulchre, *et al.*, 1996)) are to be investigated.

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