Oscillation of second-order nonlinear neutral delay differential equations

Guangming Wang, Tongxing Li, Shurong Sun

Abstract—In this paper, an oscillation theorem is established for the oscillation of second-order quasilinear neutral differential equation
\[ \left[ r(t)[(x(t) + p(t)x(t - \tau))']' + q(t)f(x(t - \sigma)) \right] = 0, t \geq t_0, \]
where
\[ \int_{t_0}^{\infty} dt/r^{1/\alpha}(t) < \infty, 0 \leq p(t) \leq p_0 < 1. \]
The result obtained here complements and improves some known results in literature.

Keywords—Oscillation; Neutral delay differential equation; Second order

1. Introduction

This paper is concerned with the oscillatory behavior of the second-order nonlinear neutral delay differential equation
\[ \left[ r(t)[(x(t) + p(t)x(t - \tau))']' + q(t)f(x(t - \sigma)) \right] = 0, t \geq t_0, \]
where
\[ r \in C^1([t_0, \infty), R), p, q \in C([t_0, \infty), R), f \in C(R, R). \]

Throughout this paper, we assume that the following conditions hold.

(a) \( r(t) > 0, \int_{t_0}^{\infty} dt/r^{1/\alpha}(t) < \infty, \]
\[ 0 \leq p(t) \leq p_0 < 1, q(t) > 0, \]
and \( q(t) \) is not identically zero on any ray of the form \( [t_0, \infty) \) for any \( t_0 \geq t_0; \)

(b) \( f(u)/u^\alpha \geq k > 0, \) for \( u \neq 0, \) \( k \) is a constant;

(c) \( \alpha > 0 \) is a ratio of odd positive integers,
\[ \tau \geq 0, \sigma > 0, \sigma \geq \tau. \]

Second order neutral delay differential equations have applications in problems dealing with vibrating masses attached to an elastic bar and in some variational problems. For further applications and questions concerning existence and uniqueness of solutions of neutral delay differential equations, see Hale [4].

Recently, there has been an increasing interest in obtaining sufficient conditions for the oscillation and nonoscillation of solutions for different classes of differential equations, see for example the monographs [1,8,11], especially, the oscillatory behavior of neutral functional differential equations has been the subject of intensive study in the last three decades, we refer to the articles [2, 3, 5–7, 9, 10, 12–20].

In 1985, Grammatikopoulos et al. [5] obtained that if
\[ 0 \leq p(t) \leq 1, q(t) \geq 0, \int_{t_0}^{\infty} q(s)[1 - p(s - \sigma)]ds = \infty, \]
then second order neutral differential equation is oscillatory.
\[ \left[ x(t) + p(t)x(t - \tau) \right]' + q(t)f(x(t - \sigma)) = 0 \]

Later, Grace and Lalli [3] considered the second-order nonlinear neutral delay differential equation
\[ \left[ r(t)(x(t) + p(t)x(t - \tau))' \right]' + q(t)f(x(t - \sigma)) = 0, \]
where
\[ \frac{f(x)}{x} \geq k, x \neq 0, \int_{t_0}^{\infty} \frac{dr}{r(t)} = \infty, \]
and showed that if exists a function \( \rho \in C^1([t_0, \infty), R) \) such that
\[ \int_{t_0}^{\infty} \left( \rho(s)q(s)1 - p(s - \sigma) - \left( \frac{\rho'(s)}{4k\rho(s)} \right)^2 r(s - \sigma) \right)ds = \infty, \]
then Eq. (2) oscillates.

In [7,12], the authors established some general oscillation criteria for second order neutral delay differential equation (2) when
\[ \int_{t_0}^{\infty} \frac{dr}{r(t)} = \infty, 0 \leq p(t) \leq 1. \]

Li [6] studied the second-order neutral delay differential equation
\[ \left[ x(t) + p(t)x(t - \tau) \right]' + q(t)f(x(t - \sigma)) = 0, \]
where \( f(x)/x \geq k, \) for \( x \neq 0, \) and established some new oscillation criteria for (3) under the condition \( 0 \leq p(t) \leq 1. \)

To the best of our knowledge, under the case

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\[
\int_{t_0}^{\infty} \frac{dt}{r^{1/\alpha}(t)} < \infty,
\]

it seems to have few oscillation results for Eq. (1).

For instance, Xu and Meng [15] considered the second-order neutral delay differential equation
\[
\left[ r(t) \left| (x(t) + p(t)x(t - \tau))^\alpha - x(t) + p(t)x(t - \tau) \right| \right] + q(t)f(x(\sigma(t))) = 0,
\]
the authors obtained the sufficient condition [15], which guarantees that every solution \( x(t) \) of Eq. (4) oscillates or
\[
\lim_{t \to \infty} x(t) = 0.
\]

In 2009, Ye and Xu [16] studied the second-order quasilinear neutral delay differential equation
\[
\left( r(t)\psi(x(t)) \right)^{\alpha} - \frac{1}{2} \psi(x(t)) + q(t)f(x(\sigma(t))) = 0,
\]
where \( \psi(z(t)) = x(t) + p(t)x(t) \) and \( \alpha > 0 \).

Throughout this paper, the authors assume that the following conditions hold.

\( (A_1) \) \( r, p, q \in C([t_0, \infty), R), \) \( r(t) > 0, \) \( 0 \leq p(t) \leq 1, \)
\[
q(t) \geq 0, \quad t \geq t_0;
\]

\( (A_2) \) \( \psi, f \in C^1(R, R), \) \( \psi(x) > 0 \) and \( xf(x) > 0 \) for \( x \neq 0, \) and there exist two positive constants \( k \) and \( L \) such that
\[
\frac{f(x)}{|x^{\alpha-1}x|} \geq k, \psi(x) \leq L^{-1}, x \neq 0;
\]

\( (A_3) \) \( \tau \in C([t_0, \infty), R), \) \( \tau(t) \leq t, t \geq t_0, \)
\[
\lim_{t \to \infty} \tau(t) = \infty;
\]

\( (A_4) \) \( \sigma \in C^1([t_0, \infty), R), \) \( \sigma(t) > 0, \) \( \sigma(t) \leq t, t \geq t_0, \)
\[
\lim_{t \to \infty} \sigma(t) = \infty.
\]

Denoting
\[
\varepsilon = \frac{\alpha}{\alpha + 1} \int_{t_0}^{\infty} Q(s)ds, Q(t) = q(t)(1 - p(\sigma(t)))^\alpha, \pi(t) = \int_{t_0}^{\infty} \frac{ds}{r^\alpha(s)}.
\]

Ye and Xu [16] established some new oscillation results for Eq. (5). For example,

**Theorem 1.** [16] Suppose that \( \pi(t_0) < \infty. \) If
\[
\int_{t_0}^{\infty} \frac{Q(t)\pi^\alpha(t)}{r^\alpha(s)} \left( \frac{1}{r^\alpha(s)} \frac{\sigma'(t)}{\pi(\sigma(t))} \right) dt = \infty,
\]
and
\[
\int_{t_0}^{\infty} \frac{Q(t)\pi^\alpha(t)}{r^\alpha(s)} \left( \frac{1}{r^\alpha(s)} \frac{\sigma'(t)}{\pi(\sigma(t))} \right) dt = \infty,
\]
then Eq. (5) is oscillatory.

We find that the above theorem is wrong. We shall give the following counterexample.

**Example 1.1** Consider the differential equation
\[
(e^{2t}(x(t)+\frac{1}{2}(t-2)))^\alpha + (e^{2t-3} + \frac{1}{2}e^{2t-1})x(t-3) = 0, \quad t \geq 1.
\]
Let
\[
r(t) = e^{2t}, \psi(x(t)) = 1, q(t) = e^{2t-3} + \frac{1}{2}e^{2t-1},
\]
\[
\tau(t) = t - 2, \sigma(t) = t - 3, \alpha = 1, L = 1, k = 1.
\]
Then
\[
\pi(t) = \frac{1}{2}e^{-2t}, \pi(t_0) < \infty,
\]
\[
Q(t) = \frac{1}{2}q(t) = \frac{1}{2}(e^{3} + \frac{1}{2}e^{-1})e^{2t}.
\]
It is easy to see that \( x(t) = e^{-t} \) is a positive solution of (6).

Furthermore, we have
\[
\int_{t_0}^{\infty} \frac{Q(t)\pi^\alpha(t)}{r^\alpha(s)} \left( \frac{1}{r^\alpha(s)} \frac{\sigma'(t)}{\pi(\sigma(t))} \right) dt = \infty,
\]
and
\[
\int_{t_0}^{\infty} \frac{Q(t)\pi^\alpha(t)}{r^\alpha(s)} \left( \frac{1}{r^\alpha(s)} \frac{\sigma'(t)}{\pi(\sigma(t))} \right) dt = \infty,
\]
which satisfy all conditions of Theorem 2.3 in [16]. However, \( x(t) = e^{-t} \) is a nonoscillation solution of (6).

In this paper, we obtain a new oscillation criteria for (1). The paper is organized as follows. In the next section, we will use the Riccati transformation technique to give a theorem for the oscillation of (1), under the case \( \int_{t_0}^{\infty} \frac{dt}{r^{1/\alpha}(t)} < \infty. \) We also give an example to illustrate the main results.

## 2. Main Results

In this section, we give a new oscillation theorem for (1). For the sake of convenience, we define the following notation.

\[
\delta(t) = \int_{t_0}^{t} \frac{ds}{r^\alpha(s)}.
\]

**Theorem 1.** Assume that there exists a function \( \rho \in C^1([t_0, \infty), (0, \infty)), \)\) such that
\[
\limsup_{t \to \infty} \left[ \int_{t_0}^{t} \frac{k}{(1 + p(s))^\alpha} q(s)^{\alpha}(s) - \frac{1}{(a + 1)^\alpha} \frac{1}{\rho^\alpha(s)} \right] ds = \infty,
\]
holds, and
\[
\limsup_{t \to \infty} \left[ \int_{t_0}^{t} \frac{k}{(1 + p(s))^\alpha} q(s)^{\alpha}(s) - \frac{1}{(a + 1)^\alpha} \frac{1}{\delta(s)^{\alpha}} \right] ds = \infty.
\]

Where
\[
Q(t) = q(t)(1 - p(t - \sigma))^\alpha, \quad (\rho'(t)), \quad = \max\{0, \rho'(t)\}.
\]
Then Eq. (1) is oscillatory.
Proof. Let \( x(t) \) be a nonoscillatory solution of (1). Without loss of generality, we assume that there exists \( t_1 \geq t_0 \) such that 
\[
x(t) > 0, x(t - \sigma) > 0, \quad \text{for all } t \geq t_1.
\]
Define \( z(t) = x(t) + p(t)x(t - \tau). \) From (1), we have
\[
(r(t)(z'(t))^a)' \leq 0, t \geq t_1. \tag{11}
\]
Therefore \( r(t)(z'(t))^a \) is a nonincreasing function.

Consequently, it is easy to conclude that there exist two possible cases of the sign of \( z'(t), \) that is, \( z'(t) > 0, \) or \( z'(t) < 0 \) for \( t \geq t_2 \geq t_1. \)

Case 1, \( z'(t) > 0 \) for \( t \geq t_2, \) then we define a Riccati transformation
\[
\omega(t) = \frac{r(t)(z'(t))^a}{z^a(t)}, t \geq t_2. \tag{12}
\]
Clearly, \( \omega(t) < 0. \) Noting that \( r(t)(z'(t))^a \) is nonincreasing, we have
\[
r^\frac{1}{a}(s)z'(s) \leq r^\frac{1}{a}(t)z'(t), s \geq t \geq t_2.
\]
Dividing the above inequality by \( r^\frac{1}{a}(s) \) and integrating it from \( t \) to \( l, \) we obtain
\[
z(t) \leq z(t) + r^\frac{1}{a}(t)z'(t)\int_t^l \frac{ds}{r^\frac{1}{a}(s)} l \geq t \geq t_2.
\]
Letting \( l \to \infty \) in the above inequality, we have
\[
0 \leq z(t) + r^\frac{1}{a}(t)z'(t)\delta(t), t \geq t_2.
\]
Therefore,
\[
r^\frac{1}{a}(t)z'(t)\frac{z(t)}{z(t)}\delta(t) \leq -1, t \geq t_2.
\]
From (12), we have
\[
-1 \leq \omega(t) \delta^a(t) \leq 0, t \geq t_2. \tag{13}
\]
On the other hand, noting that \( z'(t) < 0, \) from [2], we have
\[
x(t - \tau) \geq \frac{z(t)}{1 + p_0}, \quad \text{then } x(t - \sigma) \geq \frac{z(t + \tau - \sigma)}{1 + p_0}, \text{from (1),}
\]
we obtain
\[
(r(t)(z'(t))^a)' + \frac{k}{(1 + p_0)^a}q(t)z^a(t + \tau - \sigma) \leq 0. \tag{14}
\]
Differentiating (12), we have that
\[
\omega'(t) = \left(\frac{r(t)(z'(t))^a}{z^a(t)}\right)' - \alpha \frac{r(t)(z'(t))^a}{z^a(t)} \tag{15}
\]
Noting that \( \sigma \geq \tau, \) by (12), (14) and (15) we have
\[
\omega'(t) + \frac{k}{(1 + p_0)^a}q(t) + \frac{\alpha \omega^\frac{a}{a-1}(t)}{r^\frac{1}{a-1}(t)} \leq 0, \tag{16}
\]
Multiplying (16) by \( \delta^a(t) \), and integrating on \([t_2, t]\) gives
\[
\delta^a(t)\omega(t) - \delta^a(t_2)\omega(t_2) + \alpha\int_{t_2}^t \frac{\delta^{a-1}(s)\omega(s)}{r^\frac{1}{a-1}(s)}ds + \int_{t_2}^t \frac{k}{(1 + p_0)^a}q(s)\delta^a(s)ds \leq 0. \tag{17}
\]
Set
\[
p = \frac{\alpha + 1}{\alpha}, q = \alpha + 1, a = (\alpha + 1)^{\frac{a}{a-1}}, 
\]
we obtain
\[
-\alpha \delta^{a-1}(t)\omega(t) \leq \alpha \omega^\frac{a}{a-1}(t) + \left(\frac{\alpha}{\alpha + 1}\right)^{a-1} \frac{1}{\delta(t)},
\]
so we have
\[
-\alpha \delta^{a-1}(t)\omega(t) \leq \alpha \delta^{a-1}(t)\delta^\frac{a}{a-1}(t) + \left(\frac{\alpha}{\alpha + 1}\right)^{a-1} \frac{1}{\delta(t)}r^\frac{1}{a-1}(t).
\]
Thus, it follows from (17) that
\[
\delta^a(t)\omega(t) + \int_{t_2}^t \left[\frac{k}{(1 + p_0)^a}q(s)\delta^a(s) - \left(\frac{\alpha}{\alpha + 1}\right)^{a-1} \frac{1}{\delta(s)}r^\frac{1}{a-1}(s)\right]ds \leq \delta^a(t_2)\omega(t_2).
\]
By (10), we obtain
\[
\liminf_{t \to \infty} \delta^a(t)\omega(t) = -\infty,
\]
which contradicts (13). This completes the proof.

For an application of Theorem 2.1, we will give the following example.

**Example 2.1** Consider the second-order neutral delay differential equation
\[
\left(t^2(x(t) + \frac{1}{2}x(t - \tau))^\prime\right) + kf(x(t - \sigma)) = 0, t \geq 1, \quad \tag{18}
\]
where
\[
r(t) = t^2, p(t) = p_0 = \frac{1}{2}, q(t) = t, \tau(t) = t - \tau, \\
\sigma(t) = t - \sigma, \sigma \geq \tau, \alpha = 1, f(x) = x(1 + x^2)
\]
Then
σ'(t) = 1, Q(t) = \frac{1}{2}t, δ(t) = \frac{1}{t}.

Take k = 1, ρ(t) = t, we have

\limsup_{t \to \infty} \left\{ kρ(s)Q(s) - \frac{1}{(α+1)γ} r(s-σ)((ρ'(s))^{α+1})^{γ+1} \right\} ds

= \limsup_{t \to \infty} \left\{ \frac{1}{2}s - \frac{(s-σ)^2}{4s} \right\} ds = \infty,

and

\limsup_{t \to \infty} \int_{0}^{t} \left\{ \frac{k}{1+p}(s)δ'(s) - \left( \frac{α}{α+1} \right) \frac{1}{δ(s)r^2(s)} \right\} ds

= \limsup_{t \to \infty} \int_{0}^{t} \left\{ \frac{2}{5} \frac{1}{4s} \right\} ds = \infty.

Hence, by Theorem 2.1, Eq. (18) is oscillatory.

Remark 1. The result of the paper can be extended to the more general equation of the form

\left( r(t) | z'(t) |^{α-1} z'(t) \right) + q(t)f(x(t-σ)) = 0, t ≥ t₀,

where

z(t) = x(t) + p(t)x(t-τ), α > 0, \left\{ \frac{f(u)}{|u|^{α-1}} \right\} ≥ k > 0.

The statement and the formulation of the results are left to the interested reader.

Remark 2. One can easily see the results obtained in [3,5,6,7,9,4,12,13,14,15,16,17] cannot be applied in Eq. (1), so our results are new.

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