A combinatorial property on angular orders of plane point sets

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Abstract

We study the following combinatorial property of point sets in the plane: For a set \( S \) of \( n \) points in general position and a point \( p \in S \) consider the points of \( S - p \) in their angular order around \( p \). This gives a star-shaped polygon (or a polygonal path) with \( p \) in its kernel. Define \( c(p) \) as the number of convex angles in this star-shaped polygon around \( p \), and \( c(S) \) as the sum of all \( c(p) \), for \( p \in S \). We show that for every point set \( S \), \( c(S) \) is always at least \( \frac{1}{\sqrt{2}} n^2 - O(n) \). This bound is shown to be almost tight. Consequently, every set of \( n \) points admits a star-shaped polygonization with at least \( \sqrt{n^2} - O(1) \) convex angles.

Keywords:
Combinatorial problems
Point set
Convex angle
Star-shaped polygon

1. Introduction

Let \( S \) be a set of \( n \) points in the plane in general position, that is, no three points lie on a common line. When connecting points of \( S \) with straight-line segments, one can form triangles, simple polygons or polygonizations (simple polygons using all points of \( S \)). These elementary constructions lead to difficult combinatorial problems on point sets, like to determine, among all sets \( S \) of \( n \) points, the minimum number of empty triangles [3], the minimum size of the largest convex polygon in \( S \) (the Erdős–Szekeres theorem) [5], or the maximum possible number of polygonizations [6]. Describing properties of point sets sheds light on this kind of problems. In this note we study the following combinatorial property of point sets: For a set \( S \) of \( n \) points in general position in the plane and a point \( p \in S \) consider the points of \( S - p \) in their angular order around \( p \). This gives a star-shaped polygon with \( p \) in its kernel. Define \( c(p) \) as the number of convex angles in this star-shaped polygon around \( p \). If \( p \) is an interior point of \( S \), then we obtain a star-shaped polygon with \( p \) in its kernel. If \( p \) is an extreme point of \( S \) then the ordered set \( S - p \) gives a polygonal path. An angle \( \angle q_{i-1}, q_i, q_{i+1} \) at \( q_i \) defined by three consecutive vertices of \( P \), or \( P' \) respectively, is convex with respect to \( p \) if the four points \( p, q_{i-1}, q_i, q_{i+1} \) form a convex quadrilateral. We also say that \( q_i \) is a convex vertex. See Fig. 1.

Fig. 1. The star-shaped polygon of \( S - p \). \( q_i \) is a convex vertex, and \( q_{i-1} \) is reflex.
In a polygonal path $P'$ we do not consider the angle at the first and at the last vertex. Let $c_S(p)$ denote the number of convex angles of the polygon $P$, respectively the polygonal path $P'$, obtained by the angular order of $S - p$ around $p$. We have $0 \leq c_S(p) \leq n - 1$, because for $p$ an extreme point of $S$ that only sees one reflex chain, clearly $c_S(p) = 0$. On the other hand if $p$ lies inside a convex polygon with vertex set $S - p$ then all $n - 1$ consecutive angles around $p$ are convex. We define $c(S) = \sum_{p \in S} c_S(p)$ and $c(n) = \min_S c(S)$, where the minimum is taken over all point sets $S$ of $n$ points in general position in the plane. We show that $c(n) \geq \frac{1}{\sqrt{2}}n^2 - O(n)$. In other words, for every point set $S$ of $n$ points in general position the sum of convex angles around all points $c(S) \geq \frac{1}{\sqrt{2}}n^2 - O(n)$. This bound is almost tight, since there exist point sets $S$ such that $c(S) \leq \frac{n^2}{\sqrt{2}} - O(n)$.

This result also implies that every set of $n$ points in general position in the plane admits a star-shaped polygonization with at least $\frac{n^2}{2} - O(1)$ convex angles.

The definition of $c(S)$ relates to the efficient algorithm of Dobkin et al. [4] to count the number of empty triangles in a point set $S$: For $p \in S$, consider the star-shaped polygon of $S - p$ around $p$. Then each interior diagonal of this polygon gives rise to an empty triangle of $S$ incident to $p$. For counting the number of empty triangles of $S$ it is sufficient to construct this star-shaped polygon for each point $p \in S$ and count the number of interior diagonals. Then every empty triangle is encountered exactly three times.

The definition of $c(S)$ also is related to the reflexivity [1,2] of a point set $S$, that is, the smallest number of reflex vertices in a polygonization of $S$. Here, we focus on star-shaped polygonizations.

We finally remark that star-shaped polygonizations also have important applications in computational geometry, see for example [7], due to their simplicity and the facility of triangulating them. In this sense, finding the simplest polygonization of a point set is a useful issue. Star-shaped polygonizations with the maximum number of convex angles might be among the simplest ones.

2. Proofs

**Lemma 2.1.** Let a point set $S$ of $n$ points in general position in the plane be given, let $S' \subset S$ and $p \in S'$. Then $c_{S'}(p) \geq k$ implies $c_S(p) \geq \frac{k^2}{2}$.

**Proof.** Let $Q = \{q_1, \ldots, q_k\}$ denote the ordered subset of convex vertices of the angular order of $S' - p$ around $p$. If $p$ is an extreme point of $S'$, then also include the first and last vertex $q_0$ and $q_{k+1}$ of $Q$ in $Q$. For three consecutive vertices $q_{i-1}, q_i, q_{i+1}$ of $Q$ let $S_{q_i} \subset S - p$ be the ordered subset of $S - p$ from $q_{i-1}$ to $q_{i+1}$. We show that $S_{q_i}$ contains at least one convex vertex around $p$ in $S$, different from $q_{i-1}$ and $q_{i+1}$. Assume the angle at $q_i$ in $S_{q_i}$ around $p$ is not convex, and let $s_1, s_2 \in S'$ be the previous and next point to $q_i$ in the angular order of $S' - p$ around $p$. Two rays from $p$ through $s_1$ and $q_i$ define a cone $s_1, p, q_i$ with apex $p$. Also consider the cone $q_i, p, s_2$ with apex $p$. Then at least one of the two cones $s_1, p, q_i$ and $q_i, p, s_2$ contains a point of $S$ in its interior such that it lies outside the triangle $s_1, p, q_i$, or outside the triangle $q_i, p, s_2$, respectively. Assume $S$ contains a point in the cone $s_1, p, q_i$ outside the triangle $s_1, p, q_i$. Among all these points, consider the point $z$ with maximal distance to the line $s_1q_i$. Consider $z_2$, the line through $z$ parallel to the line through $s_1$ and $q_i$. All points of $S_{q_i}$ inside the cone $s_1, p, q_i$ lie on that side of $z_2$ that contains $p$. Hence, the angle at $z$ is convex in $S_{q_i}$. It follows that $S_{q_i}$ contains at least one convex angle, different from $q_i-1$ and $q_{i+1}$. For $p$ an extreme point of $S$, we can divide the vertices of the polygonal path $P'$ of $S - p$ into $\left\lceil \frac{k}{2} \right\rceil$ groups $S_{q_{2i-1}},$ for $i = 1, \ldots, \left\lfloor \frac{k}{2} \right\rfloor$. Each group contains at least one convex angle. For $p$ an interior point of $S$, the following cases arise:

1. If $p$ lies outside the convex hull of $Q$ then again we can consider the polygonal path as in the previous case.

2. If $p$ lies inside the convex hull of $Q$:
   (a) If $k$ is even, we can divide the points of $S$ into $\frac{k}{2}$ groups $S_{q_{2i-1}}$, each of them containing a convex angle.
   (b) If $k$ is odd, then consider the cone $q_k, p, q_1$ with apex $p$:
      i. If it contains a convex angle of $S - p$ around $p$ in its interior, then we divide the remaining points into $\frac{k+1}{2}$ groups $S_{q_{2i-1}}$, each of them containing a convex angle.
      ii. If the cone $q_k, p, q_1$ does not contain a convex angle in its interior, then there is a convex angle in the cone $q_k, p, q_2$, or possibly the convex angle is at $q_1$, and we can divide the remaining points into $\frac{k-1}{2}$ groups.

In each case, this gives at least $\left\lceil \frac{k}{2} \right\rceil$ convex angles. □

**Theorem 2.2.** $\frac{1}{\sqrt{2}}n^2 - O(n) \leq c(n) \leq 2n^2 - O(n)$.

**Proof.** Let $S$ be any set of $n$ points in general position in the plane. If for each point $p$ of $S$, $c_S(p) \geq \frac{1}{\sqrt{2}}\sqrt{n}$, then clearly $c(S) = \sum_{p \in S} c_S(p) \geq \frac{1}{\sqrt{2}}n^2$. Thus, assume that $S$ contains a point $p$ with $c_S(p) < \frac{1}{\sqrt{2}}\sqrt{n}$. Let $Q$ denote the ordered subset of convex vertices from the points of $S - p$ in their angular order around $p$. $Q$ partitions $S - p - Q$ into $k \leq \frac{1}{\sqrt{2}}\sqrt{n}$ chains of reflex vertices. We also include the two points of $Q$ that bound a reflex chain to that chain. It might be the case that a chain only consists of two vertices of $Q$. Note that $k = \lvert Q \rvert + 1$ if $p$ is an extreme point of $S$, and $k = \lvert Q \rvert$ otherwise. Denote these chains as $R_1, \ldots, R_k$. We have $\sum_{j=1}^k \lvert R_j \rvert = n - \lvert Q \rvert - 1$. Each set $R_j$ with $\lvert R_j \rvert > 2$ forms an empty convex polygon. Hence, for each point $q$ of $R_j, j = 1, \ldots, k$, we have $c_{R_j}(q) = \lvert R_j \rvert - 3$, or $c_{R_j}(q) = 0$ if $\lvert R_j \rvert = 2$. By Lemma 2.1, each point of $R_j$ sees at least $\left\lceil \frac{\lvert R_j \rvert - 3}{2} \right\rceil$ convex angles in $S$. Summing up at all points of $S - p$ we get at least $\sum_{j=1}^k \lvert R_j \rvert \left\lceil \frac{\lvert R_j \rvert - 3}{2} \right\rceil \geq \sum_{j=1}^k \left\lceil \frac{\lvert R_j \rvert - 3}{2} \right\rceil \geq \frac{2n^2}{\sqrt{2}} - \frac{3n\sqrt{n}}{2}$ convex angles in total. Elementary calculations show that this sum is
minimized if all $|R_j|$ are equal, given that $\sum_{j=1}^{k} |R_j|$ is a fixed number. We thus can assume $|R_j| = \frac{\sqrt{n}}{k} + O(1)$ for all $j$.

We get

$$c(S) \geq \sum_{j=1}^{k} \frac{|R_j|^2}{2} - \sum_{j=1}^{k} 3|R_j| \geq k \left( \frac{n^2}{2k^2} - \frac{n}{k}O(1) + O(1) \right) - \frac{3}{2}(n+k-1)$$

$$\geq \frac{n^2}{2k} - O(n)$$

$$\geq \frac{1}{\sqrt{2}}n^{\frac{3}{2}} - O(n).$$

A set $S$ of $n$ points with $c(S) \leq 2n^{\frac{3}{2}}$ is shown in Figs. 2 and 3, where also star-shaped polygons and a polygonal path are drawn. Here $n = m^2$ for $m \in \mathbb{N}$. The same construction can be adapted for other values of $n$ as well. There are $\sqrt{n}$ points that form a regular polygon. For each edge of the polygon, a chain of $\sqrt{n} - 1$ points is placed inside the polygon very close to the midpoint of the edge in such a way that these points together with the two endpoints of the corresponding edge of the regular polygon form a convex polygon.

To count $c(S)$ we distinguish whether a point $p$ is an extreme point of $S$ (1), see Fig. 2; $p$ is an interior point of $S$ that is neither the first nor the last point of a chain (2), see Fig. 3(left); or $p$ is an interior point of $S$ that is the first or the last point of a chain (3), see Fig. 3(right). For each point $p$ of type (1) we count $c_S(p) = 3\sqrt{n} - 5$. Since there are $\sqrt{n}$ points of type (1), this gives $3n - 5\sqrt{n}$ convex angles. For each point $p$ of type (2) we count $c_S(p) = 2\sqrt{n} - 2$. Also for each point $p$ of type (3) we count $c_S(p) = 2\sqrt{n} - 2$. $S$ has $\sqrt{n}(\sqrt{n} - 1)$ interior points, this gives $2n^2 - 4n + 2\sqrt{n} - 2\sqrt{n}$ convex angles. We obtain $c(S) = \sum_{p \in S} c_S(p) = 2n^2 - 4n + 2\sqrt{n} - 2\sqrt{n} - 3\sqrt{n}$. □

**Corollary 2.3.** Every set $S$ of $n$ points in general position in the plane admits a star-shaped polygonization of $S$ with at least $\sqrt{\frac{n}{2}} - O(1)$ convex angles.

**Proof.** By Theorem 2.2, for at least one point $p \in S$, $c_S(p)$ is at least $\sqrt{\frac{n}{2}} - O(1)$. If $p$ is an interior point, replace an edge $uv$ (where preferably $u$ or $v$ is a reflex vertex) of the star-shaped polygon of $S - p$ by the edges $up$ and $vp$. If $p$ is an extreme point, connect $p$ to the first and last vertex of the polygonal path of $S - p$. In each case, a star-shaped polygonization is obtained. □

**Acknowledgement**

The authors would like to thank an anonymous referee for his valuable comments and suggestions.

**References**


