Global chaos synchronisation of identical chaotic systems via novel sliding mode control method and its application to Zhu system

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Abstract: Synchronisation of chaotic systems is an important research problem in chaos theory. In this research work, a novel sliding mode control method is proposed for the global chaos synchronisation of identical chaotic systems. The general result derived using novel sliding mode control method is established using Lyapunov stability theory. As an application of the general result, the problem of global chaos synchronisation of identical Zhu chaotic systems (2010) is studied and a new sliding mode controller is derived. Numerical simulations have been shown to illustrate the phase portraits of Zhu chaotic system and the sliding mode controller design for the global chaos synchronisation of identical Zhu chaotic systems.

Keywords: chaos; chaotic systems; sliding mode control; SMC; Lyapunov stability theory; chaos synchronisation; Zhu system.


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## 1 Introduction

Chaotic systems are dissipative nonlinear dynamical systems having at least one positive Lyapunov exponent. The Lyapunov exponent of a chaotic system is a measure of the divergence of points which are initially very close and this can be used to quantify chaotic systems (Leonov, 2008). Each nonlinear dynamical system has a spectrum of Lyapunov exponents, which are equal in number to the dimension of the state space. The largest Lyapunov exponent of a nonlinear dynamical system is called the maximal Lyapunov exponent (MLE).

An important paradigm of a 3-D chaotic system was discovered by Lorenz (1963) while he was studying a 3-D weather model. Subsequently, many chaotic systems were found in the literature such as Rössler (1976) system, ACT system (Arneodo et al., 1980), Sprott (1994) systems, Chen (1999) system, Lü system (Lü and Chen, 2002), Liu system (Liu et al., 2004), Cai system (Cai and Tan, 2007), Tigan system (Tigan and Opris, 2008), Zhu system (Zhu et al., 2010), Pan system (Pan et al., 2010), Sundarapandian system (Sundarapandian, 2013a, 2013b, 2013c), Li-Wu system (Li et al., 2013), etc.

Chaos theory has applications like chemical reactions (Li and Zhu, 2004; Villegas et al., 2012), lasers (Antonelli and Mecozzi, 2009; Hu et al., 2011), vibrations (Liu et al., 2008; Luo and Lv, 2009), turbines (Asgari et al., 2013), combustion engines (Wendeker et al., 2003), power systems (Yu et al., 2003), robotics (Volos et al., 2012), etc. In communications, chaos has applications like cryptosystems (Usama et al., 2010; Rhouma and Belghith, 2010), secure communications (Suzuki and Imai, 2006; Chen and Min, 2008; Fallahi and Leung, 2010), etc.

Next, we describe the complete synchronisation of chaotic systems (A) and (B) as follows. If the chaotic system (A) is called the master or drive system and the controlled chaotic system (B) is called the slave or response system, then the complete synchronisation aims to control the slave system (B) so that the states of the slave system (B) track the states of the master system (A) asymptotically.

Various control schemes have been developed to investigate the global chaos synchronisation problem such as PC method (Pecora and Carroll, 1990), OGY method (Ott et al., 1990), active control method (Ho and Hung, 2002; Chen, 2005; Sundarapandian, 2011a; Sarasu and Sundarapandian, 2011; Zhang et al., 2012), adaptive control method (Liao and Tsai, 2000; Sundarapandian and Karthikeyan, 2011; Sundarapandian, 2011b, 2013d; Sundarapandian and Pehlivan, 2012), backstepping method (Yu and Zhang, 2004; Park, 2006; Sundarapandian, 2013c), time-delay feedback method (Guo and Zhong, 2009), sliding mode control (SMC) (Zarrabi et al., 2012; Shang and Wang, 2013; Zen et al., 2013; Wu et al., 2014; Kaur and Janardhanan, 2014), etc.

In this paper, we use a novel sliding control method for deriving a general result for the global chaos synchronisation of identical chaotic systems using SMC theory. The sliding control method is an effective control tool having advantages of low sensitivity to parameter variations in the plant and disturbances affecting the plant.

In the SMC design, the control dynamics has two sequential modes, viz. the reaching mode and the sliding mode. In accordance with this general design, the sliding controller design consists of two parts: a hyperplane design and controller design. First, a hyperplane is designed using the pole-placement approach in the modern control theory and then a controller is designed using the sliding condition. The stability of the overall system is guaranteed by the sliding condition and by a stable hyperplane.

The rest of the paper is organised as follows. In Section 2, we discuss the problem statement for the complete synchronisation problem of identical chaotic systems. In Section 3, we derive a general result for the global chaos synchronisation of identical chaotic systems using novel sliding control method. In Section 4, we describe the Zhu chaotic system (Zhu et al., 2010) and its properties. Phase portraits of the Zhu system are described using MATLAB. In Section 5, we describe the sliding mode controller design for the identical Zhu chaotic systems using novel sliding control method and its numerical simulations using MATLAB. Finally, conclusions are given in Section 6.

## 2 Problem statement

This section gives a problem statement for the global chaos synchronisation of identical chaotic systems.

As the master or drive system, we take the chaotic system given by

$$\dot{x} = Ax + f(x)$$  \hspace{1cm} (1)

In equation (1), $x \in \mathbb{R}^n$ denotes the state of the system, $A \in \mathbb{R}^{n \times n}$ denotes the matrix of system parameters and $f(x) \in \mathbb{R}^n$ contains the nonlinear parts of the system.

As the slave or response system, we take the controlled chaotic system given by

$$\dot{y} = Ay + f(y) + u$$  \hspace{1cm} (2)

In equation (2), $y \in \mathbb{R}^n$ denotes the state of the system and $u$ is the control.

For the complete synchronisation of systems (1) and (2), the synchronisation error is defined as

$$e = y - x$$  \hspace{1cm} (3)
A simple calculation yields the error dynamics as

\[ \dot{e} = Ae + \psi(x, y) + u, \]  

where

\[ \psi(x, y) = f(y) - f(x). \]  

Thus, the complete synchronisation problem can be defined as follows: find a controller \( u(x, y) \) so as to render the synchronisation error \( e(t) \) to be globally asymptotically stable for all values of \( e(0) \in \mathbb{R}^n \), i.e.,

\[ \lim_{t \to \infty} \|e(t)\| = 0 \quad \text{for all } e(0) \in \mathbb{R}^n. \]  

3 A novel SMC method for the global chaos synchronisation of chaotic systems

First, we set the design by setting the control as

\[ u(t) = -\psi(x, y) + Bv(t) \]  

In equation (7), \( B \in \mathbb{R}^n \) is chosen such that \((A, B)\) is completely controllable.

By substituting (7) into (4), we get the closed-loop error dynamics

\[ \dot{e} = Ae + Bv \]  

The system (8) is a linear time-variant control system with single input \( v \).

Hence, we have converted the origin problem of complete chaos synchronisation of the identical chaotic systems (1) and (2) into an equivalent control problem of globally stabilising the error system (8) by a suitable choice of the feedback control (7).

We start the sliding controller design by defining the sliding variable as

\[ s(e) = Ce = c_1 e_1 + c_2 e_2 + \cdots + c_n e_n, \]  

where \( C \in \mathbb{R}^{1 \times n} \) is a constant vector to be determined.

The sliding manifold \( S \) is defined as the hyperplane \( S = \{ e \in \mathbb{R}^n : s(e) = Ce = 0 \} \).

We shall assume that a sliding motion occurs on the hyperplane \( S \).

In sliding mode, the following equations must be satisfied:

\[ s = 0 \quad \text{and} \quad \dot{s} = CAe + CBv = 0 \]  

We assume that \( CB \neq 0 \)

The sliding motion is influenced by the equivalent control derived from (11) as

\[ v_{eq}(t) = -(CB)^{-1}CAe(t) \]  

By substituting (13) into (8), we obtain the equivalent error dynamics in the sliding phase as

\[ \dot{e} = Ae - (CB)^{-1}CAe = Ee, \]  

where

\[ E = \left[ I - B(CB)^{-1}C \right] A. \]  

We note that \( E \) is independent of the control and has at most \((n - 1)\) non-zero eigenvalues, depending on the chosen switching surface, while the associated eigenvectors belong to \( \ker(C) \).

Since \((A, B)\) is controllable, we can use sliding control theory (Utkin, 1977) to choose \( B \) and \( C \) so that \( E \) has any desired \((n - 1)\) stable eigenvalues.

This shows that the dynamics in the sliding mode is globally asymptotically stable.

Finally, for the sliding controller design, we apply a novel sliding control law, viz.

\[ s = -ks - qs^2 \text{sgn}(s) \]  

In equation (16), \( \text{sgn}(\cdot) \) denotes the sign function and the SMC constants \( k > 0, q > 0 \) are found in such a way that the sliding condition is satisfied and that the sliding motion will occur.

By combining equations (11), (13) and (16), we finally obtain the SMC \( v(t) \) as

\[ v(t) = -(CB)^{-1} \left[ C(kI + A)e + qs^2 \text{sgn}(s) \right] \]  

Next, we establish the main result of this section.

Theorem 1: The sliding mode controller law defined by (7) achieves global and asymptotic synchronisation of the identical chaotic systems (1) and (2) for all initial conditions \( x(0), y(0) \in \mathbb{R}^n \), where \( v \) is defined by the novel sliding control law (16), \( C \in \mathbb{R}^{1 \times n} \) is such that \((A, B)\) is controllable, \( C \in \mathbb{R}^{1 \times n} \) is such that \( CB \neq 0 \) and that the matrix \( E \) defined by (15) has \((n - 1)\) stable eigenvalues.

Proof: Upon substitution of the control laws (7) and (17) into the error dynamics (4), we get the closed-loop error dynamics as

\[ \dot{e} = Ae - B(CB)^{-1} \left[ C(kI + A)e + qs^2 \text{sgn}(s) \right] \]  

We shall show that the error system (18) is globally asymptotically stable by considering the quadratic Lyapunov function

\[ V(e) = \frac{1}{2} s^2(e) \]  

\[ s = 0 \quad \text{and} \quad \dot{s} = CAe + CBv = 0 \]  

We assume that \( CB \neq 0 \)
The sliding mode motion is characterised by the equations
\[ s(e) = 0 \quad \text{and} \quad \dot{s}(e) = 0 \]  
(20)

By the choice of \( E \), the dynamics in the sliding mode is globally asymptotically stable.

When \( s(e) \neq 0, \dot{V}(e) > 0. \)

Also, when \( s(e) \neq 0 \), differentiating \( V \) along the error dynamics (18) or the equivalent dynamics (16), we get
\[ \dot{V}(e) = s \dot{s} = -ks^2 - qs^3 \text{sgn}(s) < 0. \]

Hence, by Lyapunov stability theory (Leonov, 2008), the error dynamics (18) is globally asymptotically stable for all \( e(0) \in \mathbb{R}^n \). This completes the proof.

4 Properties of the Zhu chaotic system

This section describes the qualitative properties of the Zhu chaotic system (Zhu et al., 2010), which is a new 3-D chaotic system.

The Zhu system is described by the dynamics
\[
\begin{align*}
\dot{x}_1 &= -x_1 - ax_2 + x_2 x_3 \\
\dot{x}_2 &= bx_3 - x_1 x_3 \\
\dot{x}_3 &= -c x_3 + x_1 x_2
\end{align*}
\]  
(22)

where \( x_1, x_2, x_3 \) are the state variables and \( a, b, c \) are positive parameters.

The Zhu system (22) is chaotic when the parameter values are taken as
\[ a = 1.5, \quad b = 2.5, \quad c = 4.9 \]
(23)

The 3-D phase portrait of the Zhu system (22) is depicted in Figure 1, when the parameter values are chosen as in (23) and the initial conditions are taken as follows:
\[ x_1(0) = 1.2, \quad x_2(0) = 0.8, \quad x_3(0) = 1.4 \]
(24)

The 2-D projections of the phase-portrait of the Zhu system (22) are depicted in Figures 2 to 4.

The system (22) has Lyapunov exponents in the chaotic case as
\[ L_1 = 0.6747, \quad L_2 = 0, \quad L_3 = -4.0738 \]
(25)

The MLE of the Zhu system is \( L_1 = 0.6747 \).

The Lyapunov dimension of the Zhu system is found as
\[ D_L = 2 + \frac{L_2 + L_3}{|L_3|} = 2.1656 \]

The dynamics of the Lyapunov exponents is shown in Figure 5.
5 Sliding mode controller design for the global chaos synchronisation of Zhu systems

This section describes the sliding mode controller design for the global chaos synchronisation of Zhu (2010) chaotic systems based on the novel method described by Theorem 1 in Section 3.

As the master system, we consider the Zhu chaotic system given by

\[
\begin{align*}
\dot{x}_1 &= -x_1 - ax_2 + x_2 x_3 \\
\dot{x}_2 &= bx_2 - x_1 x_3 \\
\dot{x}_3 &= -cx_3 + x_1 x_2
\end{align*}
\]  

(26)

where \(x_1, x_2, x_3\) are the state variables and \(a, b, c\) are positive parameters.

As the slave system, we consider the controlled Zhu chaotic system given by

\[
\begin{align*}
\dot{y}_1 &= -y_1 - ay_2 + y_2 y_3 + u_1 \\
\dot{y}_2 &= by_2 - y_1 y_3 + u_2 \\
\dot{y}_3 &= -cy_3 + y_1 y_2 + u_3
\end{align*}
\]  

(27)

where \(y_1, y_2, y_3\) are the state variables and \(u_1, u_2, u_3\) are the controls.

The complete synchronisation error is defined by

\[
\begin{align*}
e_1 &= y_1 - x_1 \\
e_2 &= y_2 - x_2 \\
e_3 &= y_3 - x_3
\end{align*}
\]

(28)

\[P(\omega, \theta, \gamma) = 1+\]

Then the error dynamics is obtained as

\[
\begin{align*}
\dot{e}_1 &= -e_1 - a e_2 + y_2 y_3 - x_2 x_3 + u_1 \\
\dot{e}_2 &= b e_2 - y_1 y_3 + x_1 x_3 + u_2 \\
\dot{e}_3 &= -c e_3 + y_1 y_2 - x_1 x_2 + u_3
\end{align*}
\]

(29)

In matrix form, we can write the error dynamics (28) as

\[
\dot{e} = Ae + \psi(x, y) + u,
\]

(30)

where

\[
A = \begin{bmatrix}
-1 & -a & 0 \\
0 & b & 0 \\
0 & 0 & -c
\end{bmatrix},
\]

(31)

\[
\psi(x, y) = \begin{bmatrix}
y_2 y_3 - x_2 x_3 \\
y_1 y_3 + x_1 x_3 \\
y_1 y_2 - x_1 x_2
\end{bmatrix},
\]

(32)

and

\[
u = \begin{bmatrix}
u_1 \\
u_2 \\
u_3
\end{bmatrix}
\]

(33)

We follow the procedure given in Section 3 for the construction of the novel sliding controller to achieve complete synchronisation of the identical Zhu systems (26) and (27).

First, we set \(u\) as

\[
u(t) = -\psi(x, y) + Bv(t),
\]

(34)

where \(B\) is selected such that \((A, B)\) is completely controllable.
A simple choice of $B$ is

$$B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$  \hfill (35)

The controllability matrix $Z = [B \ AB \ A^2B]$ is calculated as follows.

$$Z = \begin{bmatrix} 1 & -2.5 & -1.25 \\ 1 & 2.5 & 6.25 \\ 1 & -4.9 & 24.01 \end{bmatrix}.$$  \hfill (36)

The controllability matrix $Z$ has full rank.

Thus, it follows that $(A, B)$ is completely controllable.

The Zhu system displays a strange attractor when the parameter values are taken as

$$a = 1.5, \ b = 2.5, \ c = 4.9$$  \hfill (37)

The sliding mode variable is selected as

$$s = Ce = \begin{bmatrix} 1 \\ -2.1 \\ 1 \end{bmatrix} e$$  \hfill (38)

which renders the sliding motion globally asymptotically stable.

Next, we take the sliding mode gains as

$$k = 5 \text{ and } q = 0.2.$$  \hfill (39)

From equation (17) of Section 3, we obtain the novel sliding control $v$ as

$$v = 40e_1 - 172.5e_2 + e_3 + 2s^2 \text{ sgn}(s)$$  \hfill (40)

As an application of Theorem 1 to the identical Zhu chaotic systems, we obtain the following main result of this section.

**Theorem 2:** The identical Zhu chaotic systems (26) and (27) are globally and asymptotically synchronised for all initial conditions $x(0), \ y(0) \in \mathbb{R}^3$ with the sliding controller $u$ defined by (34), where $\psi(x, y)$ is defined by (32), $B$ is defined by (35), and $v$ is defined by (40).

For numerical simulations, we use MATLAB software for solving systems of differential equations using the classical fourth order Runge-Kutta method with step size $h = 10^{-8}$.

The parameters of the Zhu chaotic systems (25) and (26) are taken as in the chaotic case, viz.,

$$a = 1.5, \ b = 2.5, \ c = 4.9$$

As an initial condition for the Zhu system (25), we take

$$x_1(0) = 2.7, \ x_2(0) = 1.5, \ x_3(0) = 2.4$$

As an initial condition for the Zhu system (26), we take

$$y_1(0) = -5.6, \ y_2(0) = 4.3, \ y_3(0) = 8.7$$

Figures 6 to 8 show the complete synchronisation of the Zhu chaotic systems (25) and (26).

Figure 9 shows the time-history of the synchronisation errors $e_1, e_2, e_3$.

Figures 6 to 8 show that the states $(x_1, y_1), (x_2, y_2)$ and $(x_3, y_3)$ are synchronised in two seconds, while Figure 9 shows that the chaos synchronisation errors $e_1, e_2, e_3$ decay to zero in two seconds.
Conclusions

In this research work, a novel SMC result has been proposed for the complete synchronisation of a pair of identical chaotic systems. Lyapunov stability theory has been used to prove this main result of the work. Next, as an application of the main result, a sliding controller has been designed for achieving complete chaos synchronisation of identical Zhu chaotic systems (Zhu et al., 2010). Numerical simulations using MATLAB have been provided to illustrate all the Zhu chaotic systems and the novel sliding mode controller design for the global chaos synchronisation of Zhu chaotic systems. As we have presented a general procedure using novel SMC for the complete synchronisation of identical chaotic systems, this procedure can be applied for other chaotic systems as well and the efficiency of this procedure in chaos applications may be studied.

References


