Fuzzy Variable Structure Control

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Abstract—A new methodology is presented to improve the design and tuning of a fuzzy logic controller (FLC) using variable structure control (VSC) theory. A VSC-type rule base is constructed and the fundamentals of FLC explored quantitatively by VSC theory. A very concise mathematical expression for the FLC is presented, in which the Lyapunov stability criterion can be applied to guide the design and tuning. A fuzzy two-term control based on VSC theory. The attention is focused on the points listed below.

I. INTRODUCTION

Fuzzy logic control (FLC) was first introduced [15] and applied in the 1970’s in an attempt to design controllers for systems that are structurally difficult to model. Since then, fuzzy logic control has become an active and fruitful research area with many practical applications to industrial processes being reported as well as studies of the theory itself [13], [14]. However, presently there is no systematic procedure for the design of a FLC. The fundamentals of FLC are still missing and the stability is not guaranteed. Applying mature classical control theory to FLC seems necessary [14].

It is well known that variable structure control (VSC) can provide very robust performance [1], [2]. The similarities between VSC and FLC have recently been discussed [3], [4], however, due to insufficient mathematical tools, this comparison has only been qualitative. On the other hand, a quantitative method for analysing FLC has been provided [8], and applied on fuzzy two-term control [9], [10]. However, their mathematical derivation and the expression are complex and difficult to follow. It is also well known that the performance of FLC is not satisfactory on higher order systems [11], [12]. Many papers have also appeared on learning and tuning of FLC [7], [17]–[22], however, a systematic way is still missing because the relationship between scaling gains and performance is not very clear.

Based on the above points, a fuzzy variable structure control (FZ-VSC) is introduced for designing and tuning of FLC based on VSC theory. The attention is focused on the points listed below.

A. Mathematical Design and Analysis of Fuzzy Two-Term Control Based on VSC Theory

A VSC-type control rule base is introduced. The rule base is decomposed into inference cells (IC) on which a very concise mathematical model of FZ-VSC can be obtained based on VSC theory.

B. Stability Design Based on Lyapunov Theory

Based on the simple analytical model of FZ-VSC, a Lyapunov function can be easily found for helping the design and tuning. A hierarchical method can be used to form a higher order FZ-VSC by a two-dimensional (2-D) rule base instead of a usual higher dimensional one.

C. A More Systematic Procedure for Gain Tuning

A more systematic and simple gain tuning can be obtained with the help of the VSC theory.

Finally, a simple simulation on a higher order system demonstrates the effectiveness of FZ-VSC.

II. VARIABLE STRUCTURE CONTROL

Suppose a $n$th-order system is described as

$$x^{(n)} = f(x,t) + U$$  \hfill (1)

where $x$ is the state vector, and $U$ is the control variable. A switching function $S$ is defined as [1]

$$S = \left( \frac{d}{dt} + \lambda \right)^{(n-1)} e$$  \hfill (2)

where $e = xd - x_d$ is the desired signal, and $\lambda$ is the constant. A switching surface defined by $S = 0$ represents the desired dynamics, which is insensitive to the parameters of the system. These trajectories describe a new type of motion called sliding mode. After a proper switching function $S$ is obtained, the problem in designing VSC becomes one of designing a control $U$ such that any state $e$ outside the switching surface $(S = 0)$ is driven to reach the surface in finite time.

A sufficient condition for this behavior is to choose a Lyapunov function $V$ and force the control to meet the Lyapunov stability criterion shown in (3)

$$V = \frac{1}{2} S^2$$

$$\frac{d}{dt} V = \frac{1}{2} S^2 \leq -\eta |S|, \quad \eta > 0.$$  \hfill (3)

Therefore, the basic switching law of VSC is of the form [1], [2]

$$U = u_e + K \text{sgn}(S)$$  \hfill (4)

where $u_e$ is the estimated equivalent control to compensate the estimated undesirable dynamics, $\text{sgn}(\cdot)$ is a sign function and $K$ is a constant. Due to switching delays and negligible time constant, it is rare in real systems that ideal sliding mode occurs. To avoid rapidly changing values of the control, a thin boundary layer (BL) is introduced around the switching surface to smooth out the control discontinuity. Then the switching law (4) can be modified as

$$U = u_e + K \text{sat}(S/\phi) = \begin{cases} u_e + K \text{sgn}(S) & |S| \geq \phi \\ u_e + K S/\phi & |S| \leq \phi \end{cases}$$  \hfill (5)

where $\phi$ is the thickness of the boundary layer. The modified switching law and the boundary layer for a second-order system ($n = 2$) is illustrated in Fig. 1.

III. FUZZY VARIABLE STRUCTURE CONTROL

The most commonly used FLC is the fuzzy two-term control shown in Fig. 2, which includes PI-type and PD-type FLC. The fuzzy two-term control can also be considered and designed as a second-order fuzzy variable structure control [3], [4].
A. VSC-Type Rule Base for a Fuzzy Two-Term Control

The rule base for FZ-VSC should generate a switching function $S(E, R)$ and establish the connection between $S$ and control $u$. Based on VSC theory, a VSC-type rule base with seven labels can be generated as in Table I, where NL $\cdots$ PL are labels of negative large, $\cdots$, positive large. As inputs of fuzzy two-term control are $E = K_c e$ and $R = K_d r$ ($r = \epsilon$), this one-dimensional (1-D) rule base should be decoupled into a 2-D one. If $K_r = \lambda K_d$, the switching function $S(E, R)$ can be defined as

$$S(E, R) = K_d (\lambda \epsilon + r) = E + R. \ \ (6)$$

After fuzzification, $E$ and $R$ become fuzzy variables expressed in labels. The operation (6) becomes addition of labels, which leads to a 2-D rule base shown in Fig. 3. This VSC-type rule base is actually similar to the one derived from the phase plane technique [5].

B. Analytical Methods for Fuzzy Control

Assumption: The membership functions (MF’s) used for input/output variables are chosen as triangular shapes as shown in Fig. 4. All input variables and the output variable have $N$ MF’s. All MF’s of input variables $E$ and $R$ have an equal spread $2A$. All MF’s of the output variable $u$ has an equal spread $2B$.

1) Decomposing Rule Base into Inference Cells: The rule base can be divided into many square blocks with output rules on the four corners as shown in Fig. 4. As all the fuzzy inference operations can be calculated on these blocks, they are called inference cells (IC) in this paper.

To maintain generality, the IC$(i, j)$ is chosen for the analysis. Two MF’s from $E$ ($i$th and $i + 1$th) and $R$ ($j$th and $j + 1$th) are folded together to form the IC. Two lines from MF’s divide the IC into four regions (IC1–IC4) as shown in Fig. 5. One of the diagonal lines is called the $S$-line on which all the points have the same distance to the $(S = 0)$. The output from the inference are shown on the relevant corner and indexed by the sequence number of input MF’s.

The absolute position of IC$(i, j)$ in the rule base is from $[iA, jA]$ to $[(i + 1)A, (j + 1)A]$. Its relative position in the IC plane is from

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**TABLE I**

<table>
<thead>
<tr>
<th>S</th>
<th>NL</th>
<th>NM</th>
<th>NS</th>
<th>ZR</th>
<th>PS</th>
<th>PM</th>
<th>PL</th>
</tr>
</thead>
<tbody>
<tr>
<td>$u$</td>
<td>NL</td>
<td>NM</td>
<td>NS</td>
<td>ZR</td>
<td>PS</td>
<td>PM</td>
<td>PL</td>
</tr>
</tbody>
</table>

---
As there always exists $[0, 0]$ to $[A, A]$. The input data $(E, R)$ to the rule base can always be mapped into the relative input data $(e^*, r^*)$ in the IC$(i, j)$ by the following mapping formula:

$$
E = iA + e^*, \quad (i = \cdots, -1, 0, 1, \cdots)
$$

$$
R = jA + r^*, \quad (j = \cdots, -1, 0, 1, \cdots). \quad (7)
$$

All of the fuzzy operations, including “fuzzification,” “inference,” and “defuzzification,” can be done in the IC.

2) Fuzzy Inference Operation: The inference method used in this paper is Mamdani’s max-min method.

a) Fuzzification: From (7), inputs $(E, R)$ have their mapped data, $(e^*, r^*)$ in IC$(i, j)$. Because of the MF overlapping in Fig. 4, the grade for $e^*$ are $\mu_i$ and $\mu_i + 1$; the grade for $r^*$ are $\mu_j$ and $\mu_j + 1$. As there always exists $\mu_i + \mu_i + 1 = 1, \mu_j + \mu_j + 1 = 1$, therefore, it is easy to obtain all the fuzzified values of the input, $(e^*, r^*)$

$$
\mu_i = 1 - \frac{e^*}{A}, \quad \mu_{i+1} = \frac{e^*}{A},
$$

$$
\mu_j = 1 - \frac{r^*}{A}, \quad \mu_{j+1} = \frac{r^*}{A}. \quad (8)
$$

b) Inference Operation:

i) The inference rule “if $E$ is $E_i$ and $R$ is $R_j$, output is $u_{k-1}$” follows the linear operation indicated by the index formula

$$
k = i + j + 1. \quad (9)
$$

The output rule strength can be obtained from the output MF’s in Fig. 4

$$
u_k = kB. \quad (10)
$$

ii) The minimum operation is used to get the output grade as shown in (11). Their positions on the IC$(i, j)$ are shown in Fig 5

$$
\mu_1 = \min(\mu_i, \mu_j) \quad \text{for output } u_{k-1}
$$

$$
\mu_2 = \min(\mu_i, \mu_{j+1}) \quad \text{for output } u_k
$$

$$
\mu_3 = \min(\mu_{i+1}, \mu_j) \quad \text{for output } u_k
$$

$$
\mu_3 = \min(\mu_{i+1}, \mu_{j+1}) \quad \text{for output } u_{k+1}. \quad (11)
$$

iii) The maximum operation is used to obtain the maximum effect of two outputs $u_k$ on the S-line

$$
\mu_2 = \max(\mu_21, \mu_22). \quad (12)
$$

![Fig. 5. Functional composition of the inference cell IC$(i, j)$.]

**Table II**

<table>
<thead>
<tr>
<th>Region</th>
<th>$\mu_1$</th>
<th>$\mu_2$</th>
<th>$\mu_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>IC1</td>
<td>$\mu_1$</td>
<td>$\mu_2$</td>
<td>$\mu_3$</td>
</tr>
<tr>
<td>IC2</td>
<td>$\mu_1$</td>
<td>$\mu_2$</td>
<td>$\mu_3$</td>
</tr>
<tr>
<td>IC3</td>
<td>$\mu_1$</td>
<td>$\mu_2$</td>
<td>$\mu_3$</td>
</tr>
<tr>
<td>IC4</td>
<td>$\mu_1$</td>
<td>$\mu_2$</td>
<td>$\mu_3$</td>
</tr>
</tbody>
</table>

C. Theoretical Analysis and Design of VSC-Type Fuzzy Two-Term Control

1) Mathematical Description of VSC-Type Fuzzy Two-Term Control: Different inputs, $(e^*, r^*)$ may fall in the different regions (IC1–IC4), which causes different results from the max-min operation. All the possible results are summarized in Table II.

Following (8)–(13) and in Table II, the crisp output result $u_1$ for the region IC$_i$ can be derived as follows (see the Appendix for the detailed derivation):

$$
u_1 = \frac{B}{A} \gamma S + kB(1 - \gamma_1) \quad (14)
$$

or

$$
u_1 = kB + \frac{B}{A} \gamma(S - kA), \quad (l = 1, 2, 3, 4) \quad (15)
$$

where

$$
\gamma_1 = (1 + \mu_i)^{-1} = (2 - e^*/A)^{-1}
$$

$$
\gamma_2 = (1 + \mu_j)^{-1} = (2 - r^*/A)^{-1}
$$

$$
\gamma_3 = (1 + \mu_{i+1})^{-1} = (1 + e^*/A)^{-1}
$$

$$
\gamma_4 = (1 + \mu_{j+1})^{-1} = (1 + r^*/A)^{-1}
$$

$S = E + R = (k - 1)A + e^* + r^*$

$k = i + j + 1$

$\gamma_l$ ($l = 1, 2, 3, 4$) is a nonlinear parameter in the region IC$_i$.

2) The Analysis of FZ-VSC:

a) A nonlinear VSC at each layer: From (15), it is easy to see that FZ-VSC is a nonlinear control because it has a nonlinear gain parameter $\gamma$ and a nonlinear relay term $kB$.

It is easy to prove that the constant $\gamma$ line is the square parallel to the IC border as shown in Fig. 6. The parameter $\gamma$ is monotonically increasing from $\gamma_{\min} = 2/3$ at the center of IC to $\gamma_{\max} = 1$ at the IC border.

The control $u$ is monotonically increasing from

$$
u_{\min} = (k - 1)B \quad \text{at } S = (k - 1)A
$$

through

$$
u_{\max} = kB \quad \text{at } S = kA
$$
which is used to compensate the undesirable effects.

Just like conventional VSC [1], [2], the functionally the continuous approximation of the function \( g \) can be considered a conventional VSC with a nonlinear gain at each layer, which is explained graphically in Fig. 7. The control outside the boundary layer can be imagined as the switching surface at each layer, which is a real one when \( k = 0 \) and a pseudo one when \( k \neq 0 \).

Equation (17) can be written as

\[
 u = kB + B_{\gamma \text{sat}} \left( \frac{S^*}{A} \right)
\]

(17)

where

\[
 \text{sat}(S^*/A) = \left\{ \begin{array}{ll}
 \text{sgn}(S^*) & |S^*| < A \\
 S^*/A & |S^*| \geq A
\end{array} \right.
\]

(16)

Just like conventional VSC [1], [2], the function \( \text{sat}(S^*/A) \) is actually the continuous approximation of the function \( \text{sgn}(S^*) \) obtained by introducing a boundary layer. Thus, the control shown by (17) can be considered a conventional VSC with a nonlinear gain at each layer, which is explained graphically in Fig. 7. The \( S \)-line \( S = kB \) can be imagined as the switching surface at each layer, which is a real one when \( k = 0 \) and a pseudo one when \( k \neq 0 \).

**A general expression for fuzzy two-term control:** Equation (17) is basically for a PD-type control. For the PI-type control in Fig. 2, we have the control at \( t_k \) interval

\[
 U^{P}(t_k) = u(t_{k-1}) + K_u u(t_k)
\]

\[
 u(t_{k-1}) = K u \sum_{i=0}^{k-1} u(t_i)
\]

\[
 u(t) = k(t)B + B_{\gamma} g(t) \text{sat} \left( \frac{S^*(t)}{A} \right)
\]

Thus, both PD-type and PI-type control can be expressed in a general form

\[
 U = u + K_u B \gamma \text{sat} \left( \frac{S^*}{A} \right)
\]

(18)

where the estimated equivalent control is

\[
 u_e = K_u k B, \text{ for } U^{P}
\]

and

\[
 u_e = u(t_{k-1}) + K_u k B, \text{ for } U^{P^{+}}
\]

which is used to compensate the undesirable effects.

c) The relationship between \( k \) and the nonlinearity of FZ-VSC: When \( k = 0 \), fuzzy two-term control becomes a nonlinear two-term control as shown in (19). When \( k \neq 0 \), the nonlinear relay exists and the control is more nonlinear

\[
 u = \gamma S B /A = \gamma (E + R) B /A.
\]

(19)

d) The relationship between gamma and the nonlinearity of FZ-VSC: It is the overlapping of MF’s that causes the nonlinearities of the fuzzy control. This is because a less overlapping of two MF’s will generate a more specific, i.e., more linear control; while a more overlapping of MF’s will generate a less specific, i.e., more fuzzy control. As shown in Fig. 6, the larger \( \gamma \) means the less overlapping of MF’s and the smaller \( \gamma \) means the more overlapping of MF’s. The control (14) becomes a linear two-term control when \( \gamma = \gamma_{\text{min}} = 1 \), and reaches the highest nonlinearity when \( \gamma = \gamma_{\text{max}} = 2/3 \). In other words, FZ-VSC has the self-tuning property because of the nonlinearly varying parameter \( \gamma \).

e) The relationship between number of MF’s \( N \) and the nonlinearity of FZ-VSC: The more MF’s, i.e., the larger \( N \), will have more IC’s. The more IC’s will have more “linear region”. Therefore, the control becomes more linear when \( N \) increases. It becomes a linear control \( u = BS/A = S \) when \( N \rightarrow \infty \), and a pure nonlinear two-term control when \( N = 2 \).

**D. Higher Order Fuzzy Variable Structure System**

For a \( n \)-th order system

\[
 x^{(n)} = f(x, t) + U
\]

(20)

a \( (n-1) \)-th-order switching function is needed [1], [2]. A higher dimensional rule base can be used to build this higher order FZ-VSC [3]. However, a higher dimensional rule base is very difficult to build and implement. Therefore, a hierarchical method is proposed to form a higher order switching function by a normal 2-D rule base. A switching line for a \( n \)-th order system is

\[
 \sigma_{n-1} = \left( \lambda + \frac{d}{dt} \right) e^{n-1} + \sum_{k=0}^{n-1} C_{n-1}^k \lambda^{n-k-1} e^k
\]

(21)

where

\[
 C_{n-1}^k = \frac{(n-1)!}{(n-k-1)!k!}
\]

This switching line can be formed by a hierarchical method as shown below

\[
 \begin{align*}
 \sigma_1 &= \lambda \epsilon + \dot{\epsilon} \\
 \sigma_2 &= \lambda \sigma_1 + \dot{\sigma}_1 \\
 &\vdots \\
 \sigma_{n-1} &= \lambda \sigma_{n-2} + \dot{\sigma}_{n-2}.
\end{align*}
\]

(22)

Then the \( (n-1) \)-th order switching function of FLC \( S_{n-1} = K_\sigma \sigma_{n-1} \) can be implemented by a normal 2-D rule base shown in Fig. 8. Being similar to (18), the control output \( U \) can be expressed as

\[
 U = u + K_u B \gamma \text{sat} \left( \frac{S_{n-1}}{A} \right)
\]

(23)

with

\[
 S_{n-1} = S_{n-1} - kB \text{ and } S_{n-1} = K_\delta \sigma_{n-1}.
\]
E. Stability Design

Consider a nth-order system shown in (20). Suppose the upper bound for the undesirable dynamics exists

$$\sum_{i=1}^{n-1} C_{i-1}^{n-1} \lambda^{(n-i)} e^{(i)} + x_d - f \leq F$$

$$(i = 1, 2, \ldots, n - 1).$$

(24)

The derivation of (21) becomes

$$\dot{\sigma}_{n-1} = \sum_{i=0}^{n-1} C_{i-1}^{n-1} \lambda^{(n-i)} e^{(i+1)}$$

$$= \sum_{i=1}^{n-1} C_{i-1}^{n-1} \lambda^{(n-i)} e^{(i)} + e^{(n)}$$

$$= \sum_{i=1}^{n-1} C_{i-1}^{n-1} \lambda^{(n-i)} e^{(i)} + x_d - f - U.$$  

(25)

As $S_{n-1} = K_d \sigma_{n-1}$, placing (24) into (25), we have

$$\dot{S}_{n-1} = K_d \dot{\sigma}_{n-1} \leq K_d (F - U).$$

(26)

Suppose $u_c$ can compensate the most of undesirable effects as shown in (27)

$$|F + u_c| \leq \Delta F.$$  

(27)

By using the ideal function $\text{sgn}(S_{n-1}^*)$ instead of the approximation $\text{sat}(S_{n-1}/A)$ in (23) [1],[2], we have

$$\dot{S}_{n-1} \leq K_d \{\Delta F - K_u B^T \text{sgn}(S_{n-1}^*)\}.$$  

(28)

A Lyapunov function at kth layer can be chosen as

$$V(S_{n-1}^*, t) = \frac{1}{2K_d} S_{n-1}^{*2}.$$  

(29)

As $\dot{S}_{n-1}^* = \dot{\sigma}_{n-1}^*$ at the kth layer, we have

$$\dot{V} = \frac{1}{K_d} \dot{S}_{n-1}^* S_{n-1} \leq \Delta F S_{n-1}^* - K_u B_\gamma S_{n-1}^*.$$  

(30)

If the control at every layer can be designed so that

$$B_\gamma K_u = \Delta F + \eta, \quad \text{with } \eta > 0$$

(31)

then the stability is guaranteed by

$$\dot{V} \leq -\eta |S_{n-1}^*|^2.$$  

(32)

The explanation of (31) is when the state is at the pseudo switching surface on the kth layer, $u = kB$ is supposed to be enough to compensate for the undesirable effect. If the state is moving above it, the control is increased, and vice versa. Therefore, it can be imagined that FZ-VSC tries to keep the stability at each layer. As the undesirable effect reduces, it will moves gradually to the lower layer, and finally converges along the switching surface at the layer where $k = 0$.

a) Hybrid design to enhance the stability: It can be seen from (24) that the term $\sum_{i=1}^{n-1} C_{i-1}^{n-1} \lambda^{(n-i)} e^{(i)}$ may have a negative affect on the stability. A hybrid conventional/fuzzy control can be designed to cancel this term

$$U_{\text{hybrid}} = U + \sum_{i=1}^{n-1} C_{i-1}^{n-1} \lambda^{(n-i)} e^{(i)}.$$  

(33)

Through this modification, the upper bound on the undesirable effects will be reduced and the stability will be improved. For a second-order plant, a second-order FZ-VSC is needed, then the compensation term in (32) reduces to $\lambda \dot{e}$. As the modification increases the derivative effect, the hybrid conventional/fuzzy PI-type control in (33) is actually one kind of fuzzy PID control [6], which may improve the performance considerably

$$U_{PID} = U^{*} + K_u \lambda \dot{e}.$$  

(34)

F. Gain Tuning

As the parameter $\gamma$, $B$ and $k$ are determined by the structure of the MF’s, only three I/O scaling gains $K_d$, $\gamma$ and $K_u$ can be tuned
by the user. The structure of scaling gains is the same for a higher order FZ-VSC.

1) According to the VSC theory [1], [2], the ratio \( \lambda \) is usually chosen for the satisfactory dynamics of the switching surface. Good dynamics of the switching surface is essential to the stability and robustness of the entire system.

2) From the stability design (27)–(31), we can see that the gain \( K_a \) has little influence to the stability. However, too much input saturation may cause instability. Then \( K_a \) should be chosen to have a reasonable \( K_a \) so that both inputs have the best resolution and less saturation.

3) The output gain \( K_u \) significantly affects the stability. Generally speaking, smaller \( K_u \) produces a smaller control signal which will reduce chattering (one feature of VSC) greatly. As there is the accumulation in PI-type control, the output gain \( K_u \) may be much smaller than that in PD-type control in order to keep the stability criterion (31) valid. Therefore the chattering in PI-type control is usually much smaller than PD-type. Too large \( K_u \) may also cause instability, in this sense, \( K_u \) behaves similarly to the proportional gain of the conventional two-term control.

IV. SIMULATION EXAMPLE

The conventional fuzzy two-term control is usually good for low-order systems, but may not be satisfactory for higher order systems which requires a higher dimensional switching surface. Therefore, a higher order FZ-VSC may be helpful.

\[
5 = s^3 + 4.5s^2 + 5.5s + 10.5
\]  

A third-order linear model shown in (34) is chosen for the simulation. The proper switching surface should be a 2-D one according to the VSC theory [1], [2]. A third-order FZ-VSC is built in Fig. 9, and compared with conventional fuzzy two-term control in the simulation.

The quantitative criteria for measuring the performance are chosen as IAE and ITAE. Smaller numbers imply better performance

\[
IAE = \int |e| \, dt, \quad ITAE = \int t |e| \, dt.
\]  

A 2-D rule base in Fig. 3 is used for both of them with A=B. Inputs to the FZ-VSC are normalized to \([-1, 1]\), thus \( A = B = 1/3 \). Both controllers have three gains \( (K_d, \lambda, K_a) \) to be tuned. For simplicity, all initial gains start at one in order to have a good input resolution and no saturation. The tuning steps for this FZ-VSC are listed as follows:

Step 1: To increase \( \lambda \) to get a faster transient.
Step 2: To adjust \( K_a \) to obtain a faster and stable performance.
Step 3: If Steps 1 and 2 fail to achieve a satisfactory result, it may be helpful to adjust \( K_d \) or \( \lambda \) and \( K_a \) together without too much change in \( K_a \).

For convenience, the conventional two-term fuzzy control is labeled by FZ-PD₁/FZ-PI₁, and the third-order FZ-VSC is labeled by FZ-PD₃/FZ-PI₃. After a few tries, the performances are shown in Tables III and IV, and Figs. 10 and 11. FZ-PD₁, FZ-PI₁ and FZ-PD₃ are tuned by the first two steps only, but FZ-PD₃ needs all three steps.

1) It will be hard for conventional two-term fuzzy control to achieve better results on higher order system because of its 1-D switching line.
2) With VSC theory, fuzzy control can be easily expanded to a higher order one and achieve better results because of its higher dimensional switching surface.

3) Gain tuning for FZ-VSC becomes more systematic and easier as the effect of each gain is more clear according to the VSC theory.

V. CONCLUSIONS

The rule base can be mapped into small inference cells in order to make the mathematical analysis simpler. The VSC theory can be applied to the design of an FLC so that a very concise mathematical model of FLC can be derived. FLC designed by VSC theory can be considered as a piecewise VSC with nonlinear gain which provides the self-tuning property—the mystery of FLC. Lyapunov stability criterion can also be applied to guide the design and tuning. Input gains have less effect on the system stability, as one of them can be determined by the switching surface while the other only affects the control resolution. The output gain has a significant effect on the system stability. The gain tuning becomes simpler and more systematic. With the hierarchical method, FZ-VSC can easily be designed for higher order systems to achieve better results.

APPENDIX

MATHEMATICAL DERIVATION OF FZ-VSC

For region IC₁:

From Table II and (8), we have

\[ \sum_{j=1}^{3} \mu_{ij} = \mu_i + \mu_j + \mu_{j+1} = \mu_i + 1 = 2 - \frac{e^*}{A} = \gamma_1^{-1}. \]  
(A1)

From Table II, (8) and (11), we obtain

\[ \sum_{j=1}^{3} \mu_{i}u_{k+1} = \mu_i u_{k+1} + \mu_j u_k + \mu_{j+1} u_{k+1} = B[k - 1]A + e^* + r^* + B(1 - e^*/A) \]

as \( S = E + R = (k - 1)A + e^* + r^* \)

\[ \sum_{j=1}^{3} \mu_{i}u_{k+1} = \frac{B}{A}S + kB(\gamma_1^{-1} - 1). \]  
(A2)

From (11), (A1), and (A2), the crisp output of FZ-VSC in region IC₁ can be obtained

\[ u_1 = \frac{\sum_{j=1}^{3} \mu_{i}u_{k+1} - 2}{\sum_{j=1}^{3} \mu_{i}} = \frac{B}{A}S + kB(\gamma_1^{-1} - 1) \]

\[ = \frac{B}{A}\gamma_1 S + kB(1 - \gamma_1) \]

\[ = kB + \frac{B}{A}\gamma_1(S - kA). \]  
(A3)

For other regions:

By using the same method and procedure on other regions (IC₂–IC₅), we can easily obtain all the possible crisp outputs of FZ-VSC as shown in (14) and (15).

REFERENCES