COMPLEX HYPERBOLIC STRUCTURES ON DISC BUNDLES OVER SURFACES I. GENERAL SETTINGS. A SERIES OF EXAMPLES

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Abstract. We study oriented disc bundles M over a closed orientable surface Σ that arise from certain discrete subgroups in $\mathrm{PU}(2,1)$ generated by reflections in ultraparallel complex geodesics in the complex hyperbolic plane $\mathbb{H}^2_{\mathbb{C}}$. The results obtained allow us to construct the first examples of

- Disc bundles M over Σ that satisfy the equality $2(\chi+e)=3\tau$,
- Disc bundles M over Σ that satisfy the inequality $\frac{1}{2}\chi<e$,
- Disc bundles M over Σ that admit both real hyperbolic and complex hyperbolic structures,
- Discrete and faithful representations $\varrho:\pi_1\Sigma\to\mathrm{PU}(2,1)$ with fractional Toledo invariant, and
- Nonhomeomorphic disc bundles M over the same Σ and with the same τ,

where χ stands for the Euler characteristic $\chi(\Sigma)$ of Σ, e, for the Euler number $e(M)$ of M, and τ, for the Toledo invariant of M. To get a satisfactory explanation of the equality $2(\chi+e)=3\tau$, we conjecture that there exists a holomorphic section in all our examples.

Constructing examples is based on a new version of Poincaré's Polyhedron Theorem where requirements concerning the tessellation have a form which is as local as possible. This version can be easily adapted to be applied in subtle situations lacking the concept of convexity.

A more important feature of the examples is that, in [Ana], we will distinguish some examples having the same e, χ, and τ with new discrete invariants, which will lead to a detailed knowledge about the corresponding Teichmüller space.

In [AGu], we apply the introduced methods to construct a trivial bundle carrying complex hyperbolic structure.

CONTENTS

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1. Introduction

Dealing with geometry and topology of 4-manifolds, it is natural to study whether a topological 4-manifold M admits a classic geometric structure: real hyperbolic, complex hyperbolic, quaternionic, etc. Henceforth, M is an oriented disc bundle over a closed orientable surface Σ, χ denotes the Euler characteristic $\chi(\Sigma)$ of Σ, and e, the Euler number $e(M)$ of M.

Gromov, Lawson, Thurston [GLT] and Kuiper [Kui] (see also Kapovich [Kap] and Luo Feng [Luo]) found various sufficient conditions in terms of χ and e that provide the existence of a real hyperbolic structure on the bundle M. Our article studies the case of complex hyperbolic geometry. In this case, there is one more discrete invariant — the Toledo invariant of a representation $\varrho:\pi_1\Sigma\to\mathrm{PU}(2,1)$ defined by M — related to the complex (= Riemannian) structure on M. It is shown in [Tol] that the Toledo invariant τ takes values in $\frac{2}{3}\mathbb{Z}$ and satisfies the inequality $|\tau|\le|\chi|$. There are not so many known disc bundles M carrying a complex hyperbolic structure. For a trivial one, $\mathbb{R}$-*Fuchsian*, M is homeomorphic to the tangent bundle $\mathrm{T}\,\Sigma$ of Σ which results in $e=\chi$ and $\tau=0$. For the other trivial one, $\mathbb{C}$-*Fuchsian*, that can be characterized by $\chi=\tau$ [Tol], M is homeomorphic to the square root of $\mathrm{T}\,\Sigma$, and $e=\frac{1}{2}\chi$. The first nontrivial examples of complex hyperbolic disc bundles were constructed by Goldman, Kapovich, and Leeb [GKL]. Their examples satisfy the relations $e=\chi+|\tau/2|$ and $\chi\le e\le\frac{1}{2}\chi$. Thus, $\mathbb{R}$-*Fuchsian* and $\mathbb{C}$-*Fuchsian* bundles provide the extreme values for e in all known examples.

The main purpose of the present article is to study discrete subgroups H in $\mathrm{PU}(2,1)$ generated by the reflections $R_1,\dots,R_n$ in ultraparallel complex geodesics $M_1,\dots,M_n$ in $\mathbb{H}^2_{\mathbb{C}}$ with the defining relations $R_n\dots R_1=1$, $R_i^2=1$ such that a suitable torsion-free subgroup of index 2 or 4 in H produces a bundle M in question. A fundamental domain for H is bounded by a cycle of bisectors B_i's such that the neighbouring bisectors have a common slice and M_i is the middle slice of B_i. The results obtained allow us to construct a large variety of new manifolds. It is worthwhile mentioning that our construction has some features of the construction described in [GLT]. Also, it has some common features with Kuiper's construction [Kui] since both mimic well-known plane examples. However, the complex hyperbolic situation is more subtle: to check that bisectors intersect properly is not an easy task, bisectors have nonconstant angle along their common slice, to prove the fact that the fundamental domain is fibred and to calculate the Euler number of M requires additional efforts.

At the end of this article, we apply our methods to construct some series of explicit examples of complex hyperbolic disc bundles. Clearly, we simultaneously obtain a family of compact 3-manifolds (circle bundles over closed orientable surfaces) admitting a spherical CR-structure. All the examples of the series satisfy the inequality $\frac{1}{2}\chi<e$ and the equality $2(\chi+e)=3\tau$ (with negative χ,e,τ). The inequality was never valid for previously known examples, whereas the equality was valid only in the $\mathbb{C}$-Fuchsian case. It easily follows from the adjunction formula that the equality is a necessary condition for the existence of a holomorphic section of the bundle M. It is intriguing to conjecture that there exists a holomorphic section in *all our examples.* Such a section would produce a holomorphic disc D in $\mathbb{B}^4\subset\mathbb{C}^2$ whose boundary $\partial D\subset\mathbb{S}^3$ is a fractal curve, ∂D is the limit set of the corresponding group. Also, in this case, M is not Stein. The only known non-Stein complex hyperbolic disc bundle is $\mathbb{C}$-Fuchsian. The fact that the $\mathbb{R}$-Fuchsian bundles are Stein manifolds is proven in [BSh].

Combining our results with some known facts, we arrive at the following.

As was shown in [Kui], the inequality $|e|\le\frac{1}{3}|\chi|$ is sufficient for the existence of a real hyperbolic structure on a bundle. Since some of our examples satisfy this inequality, we obtain the first disc bundles admitting both structures: real hyperbolic and complex hyperbolic. Passing on to the corresponding circle bundles, we see that there exist circle bundles over closed orientable surfaces admitting simultaneously conformally flat and spherical CR-structures. Compare our examples with those constructed by Schwartz in [Sch1] and [Sch2].

In a preliminary version [GKL1] of [GKL], it was conjectured that τ is always an even integer for any discrete and faithful representation $\pi_1\Sigma\to\mathrm{PU}(2,1)$. For many of our examples, τ is not integer, implying, in particular, that the corresponding representation $\pi_1\Sigma \to \mathrm{PU}(2,1)$ cannot be lifted to $\mathrm{SU}(2,1)$.

In [GKL], for any Σ with $\chi(\Sigma) < 0$ and for any even integer τ subject to the Toledo necessary condition $|\tau| \le |\chi|$, a complex hyperbolic disc bundle M over Σ was constructed with Toledo invariant τ. Therefore, each of our examples with τ integer provides a couple of nonhomeomorphic complex hyperbolic disc bundles over the same Σ and with the same τ. This implies that there exist discrete and faithful representations $\varrho : \pi_1\Sigma \to \mathrm{PU}(2,1)$ lying in the same connected component of the space of representations but in different connected components of the Teichmüller space, the space of discrete, faithful, and type-preserving representations.

In order to prove that H is discrete, we need a new version of Poincaré's Polyhedron Theorem. In the case of nonconstant curvature, the 3-faces of a fundamental polyhedron, bisectors in our case, are not totally geodesic, and the angle between them along a common 2-face is not usually constant. A typical condition in known versions of Poincaré's Polyhedron Theorem is that the faces of adjacent polyhedra intersect properly. Since we cannot explore the concept of convexity, this condition is rather difficult to verify in the complex hyperbolic case. Our basic strategy to overcome the difficulties is to find requirements concerning the tessellation in a form which is as local as possible. Although Poincaré's Polyhedron Theorem in the presented form looks very particular, it is easy to adapt the proof to a suitable generalized version (see [AGr]).

Whenever reasonable, we work without coordinates. In fact, we rewrite a number of well-known facts and formulae using this approach (some of them can be found in Goldman [Gol] or in Sandler [San]), so, to a certain extent, our exposition is self-contained. Nevertheless, direct references would not simplify the exposition, since, in most cases, we need something developed in the proof. Some proofs contain straightforward and boring verifications. In such cases, we point out what the reader can omit without loss of anything useful.

Sections 2, 3, and 4 establish our conventions regarding complex hyperbolic geometry and provide technical tools. We believe that other classic geometries such as real hyperbolic, quaternionic, etc., can be treated in a similar way [AGr]. Those sections also contain some crucial facts and concepts: the meridional displacement, Proposition 3.9, Theorem 4.2, and Corollary 4.3. We should warn the reader that our definitions — such as those of geodesics, bisectors, complex geodesics, etc. — are frequently more general than the common ones. This is mostly due to the use of geometry of positive points in subsequent articles.

Section 5 deals with the properties of bisectors that we need in the proof of Theorem 6.2.7. From the lemmas of this section and from Lemma 6.2.4, one can extract those properties of 3-faces that allow to compose an abstract version of Poincaré's Polyhedron Theorem provable in the same way.

The geometrical core of the article begins in Section 6. We exhibit the general construction of fundamental polyhedra and prove our version of Poincaré's Polyhedron Theorem (Theorem 6.2.7). Exploring the concept of transversality of bisectors and planning further applications, we find a criterion (Criterion 6.3.2) which reduces the problem of intersection of bisectors to the question of transversality. In order to obtain a disc bundle, we require that fundamental polyhedra can be cut into transversal triangles of bisectors and prove that each transversal triangle can be properly fibred into discs (Theorem 6.5.1). Having in mind the examples that we are going to construct in Section 7, we establish Criterion 6.5.3 which decides whether a triangle of bisectors is transversal. The transversal triangles turn out to be important geometrical objects. They can serve as building blocks for constructing fundamental polyhedra, and not only in the way used in this article. Also, they are naturally equipped with an isometry of its vertices (complex geodesics). Lemmas 6.5.4 and 6.5.5 used in the proof of Theorem 6.5.1 are also applied in Section 7: with the help of the isometry mentioned above, they permit us to control the 'fractional Euler number' of triangles.

Section 7 is an application of the previous sections: we construct a series of explicit examples. Each of them is given by a torsion-free subgroup of finite index in the discrete group F_n generated by two rotations U and V, being $U^n=V^n=(V^{-1}U)^2=1$ the defining relations ($V^{-1}U$ is the reflection in a complex geodesic). The fundamental domain for F_n is a quadrangle of bisectors glued from two transversal triangles. Applying the numerical criteria of the previous sections, we reduce the verification of discreteness to some explicit inequalities (see Theorem 7.1.2). Finally, in Subsection 7.2, we list some of the most interesting examples.

The introduced methods are unexpectedly powerful. For instance, in [Ana], they will be applied to a detailed study of the Teichmüller space of the complex hyperbolic manifolds in question, that is, the moduli space of faithful and discrete representations included in $\operatorname{Hom}(\pi_1\Sigma,\operatorname{PU}(2,1))/\operatorname{PU}(2,1)$. Some of the examples found in the present work having the same χ, e, and τ will be distinguished by a series of new discrete invariants of the complex hyperbolic structure on the bundle, and will give rise to open locus on the corresponding connected components of the Teichmüller space. The meridional displacement will turn out to be a particular aspect of a new rich structure related to a classic geometry.

Also, these method allow to construct a trivial complex hyperbolic bundle [AGu].

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2. Preliminaries

2.1. Tangent Bundle, Hermitian Structure, Levi-Civita Connection, Curvature. Let V be a $\mathbb{C}$-vector space, $\dim_{\mathbb{C}} V=3$, equipped with a hermitian form of signature $++-$. The form identifies $V^*\simeq\overline{V}$, where $\overline{V}$ stands for the $\mathbb{C}$-vector space defined[1] by $c\cdot\overline{v}\rightleftharpoons\overline{\overline{c}\cdot v}$ for $\overline{v}\in\overline{V}$, $v\in V$, $c\in\mathbb{C}$. Thus, we can equip $\operatorname{Lin}_{\mathbb{C}}(V,V)\simeq V^*\otimes_{\mathbb{C}}V\simeq\overline{V}\otimes_{\mathbb{C}}V$ with the hermitian form defined as $\langle\overline{v}_1\otimes v_2,\overline{v}_3\otimes v_4\rangle\rightleftharpoons-\langle v_3,v_1\rangle\cdot\langle v_2,v_4\rangle$. For a nonisotropic $p\in V$, we have the orthogonal decomposition $V=\mathbb{C}p\dot{+}p^\perp$, $v=v^p+{}^pv$, where $v^p\rightleftharpoons\dfrac{\langle v,p\rangle}{\langle p,p\rangle}p\in\mathbb{C}p$ and ${}^pv\rightleftharpoons v-\dfrac{\langle v,p\rangle}{\langle p,p\rangle}p\in p^\perp$. Clearly, $v^{c\cdot p}=v^p$ and ${}^{c\cdot p}v={}^pv$ for $0\neq c\in\mathbb{C}$. For a nonisotropic $p\in V$ and for $v\in V$, denote $v_p\rightleftharpoons\langle-,p\rangle\otimes{}^pv\in V^*\otimes_{\mathbb{C}}p^\perp$. Obviously, $v_p(p^\perp)=0$.

Depending on the context, we will use elements of V to denote points in $\mathbb{CP}V$. We regard any $\varphi\in\operatorname{Lin}_{\mathbb{C}}(V,V)$ as a tangent vector $t_\varphi\in\mathrm{T}_p\,\mathbb{CP}V$ by defining $t_\varphi f\rightleftharpoons\dfrac{d}{d\varepsilon}\Big|_{\varepsilon=0}\hat f\big(p+\varepsilon\varphi(p)\big)$ for a local smooth function f on $\mathbb{CP}V$ and its lift $\hat f$ to V. In this well-known way, we identify $\mathrm{T}_p\,\mathbb{CP}V\simeq\operatorname{Lin}_{\mathbb{C}}(\mathbb{C}p,V/\mathbb{C}p)$ [Man]. Hence, for a nonisotropic p, $\mathrm{T}_p\,\mathbb{CP}V\simeq\langle-,p\rangle\otimes p^\perp$ is equipped with a hermitian form defined by $\langle t_1,t_2\rangle=-\langle p,p\rangle\langle v_1,v_2\rangle$ for $t_1=\langle-,p\rangle\otimes v_1\in\mathrm{T}_p\,\mathbb{CP}V\ni t_2=\langle-,p\rangle\otimes v_2$, where $v_1,v_2\in p^\perp$.

We will denote $\mathrm{B}\,V\rightleftharpoons\{p\in\mathbb{CP}V\mid\langle p,p\rangle<0\}$, $\partial\,\mathrm{B}\,V\rightleftharpoons\{p\in\mathbb{CP}V\mid\langle p,p\rangle=0\}$, and $\overline{\mathrm{B}}V\rightleftharpoons\mathrm{B}\,V\cup\partial\,\mathrm{B}\,V$. As we will shortly show, the hermitian metric over the complex hyperbolic space $\mathbb{H}^2_{\mathbb{C}}\rightleftharpoons\mathrm{B}\,V$ coincides, up to the scale factor of 4, with the one introduced in [Gol].

For $p_1,p_2\in\mathrm{B}\,V$, let us calculate the length of the curve (which turns out to be a geodesic) $c(s)\rightleftharpoons(1-s)p_1+sp_2$, $s\in[0,1]$, assuming $\langle p_1,p_1\rangle=\langle p_2,p_2\rangle=-1$ and $\langle p_1,p_2\rangle=-a\leq0$ (which implies $a\geq1$). It is easy to verify that $c'(s)\rightleftharpoons\big\langle-,c(s)\big\rangle\otimes\dfrac{{}^{c(s)}(p_2-p_1)}{\big\langle c(s),c(s)\big\rangle}$ is the tangent vector to the curve. Then $\big\langle c'(s),c'(s)\big\rangle=\dfrac{a^2-1}{\big(1+2(a-1)s(1-s)\big)^2}$ and the length is $\ell(c)=\displaystyle\int_0^1\frac{\sqrt{a^2-1}\;ds}{1+2(a-1)s(1-s)}=\ln(a+\sqrt{a^2-1})$. Using projective coordinates $p_1=[z_1,1]$ and $p_2=[z_2,1]$ on the projective line containing p_1

[1] Here and in what follows, the symbol $\rightleftharpoons$ stands for 'equals by definition.'

and p_2, we have $2\operatorname{dist}(z_1,z_2)=\ln\dfrac{|1-z_1\overline{z}_2|+|z_1-z_2|}{|1-z_1\overline{z}_2|-|z_1-z_2|}$ which is the usual distance formula for the disc model of the hyperbolic plane. Being the distance a monotonic function of the *tance*

$$\operatorname{ta}(p_1,p_2)\rightleftharpoons\frac{\langle p_1,p_2\rangle\langle p_2,p_1\rangle}{\langle p_1,p_1\rangle\langle p_2,p_2\rangle}=a^2,$$

it is frequently convenient to use tance[2] instead of distance.

Let $X(p)=\langle -,p\rangle\otimes x(p)$, where $x(p)\in p^\perp$, be a smooth tangent vector field defined out of $\partial\,\mathrm{B}\,V$ (normally, dealing with vector fields or functions, we will locally lift their argument, 'living' in $\mathbb{CP}V$, to that in V) and let $v\in V$. It is easy to verify that

$$\nabla_{v_p}X\rightleftharpoons\Big(\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}x\big(p+\varepsilon\langle p,p\rangle\,{}^p v\big)\Big)_p$$

defines a connection such that $v_p\langle X,Y\rangle=\langle\nabla_{v_p}X,Y\rangle+\langle X,\nabla_{v_p}Y\rangle$.

2.1.1. Lemma. *Let $p\in V$ be nonisotropic and let $x,y\in V$. Then*

$$\nabla_{x_p}y_*=-\langle y,p\rangle x_p,\quad x_p\langle y,*\rangle=\langle y,{}^p x\rangle\langle p,p\rangle,\quad x_p\langle *,y\rangle=\langle {}^p x,y\rangle\langle p,p\rangle,\quad x_p\langle *,*\rangle=0.$$

Proof is straightforward. As examples, we prove the first and the last. For the first, we have $y_*=\langle -,*\rangle\otimes\Big(y-\dfrac{\langle y,*\rangle}{\langle *,*\rangle}*\Big)$, therefore,

$$\nabla_{x_p}y_*=\bigg(\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\Big(y-\frac{\langle y,\,p+\varepsilon\langle p,p\rangle\,{}^p x\rangle}{\langle p+\varepsilon\langle p,p\rangle\,{}^p x,p+\varepsilon\langle p,p\rangle\,{}^p x\rangle}\big(p+\varepsilon\langle p,p\rangle\,{}^p x\big)\Big)\bigg)_p=$$

$$=-\bigg(\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\Big(\frac{\langle y,p\rangle+\varepsilon\langle p,p\rangle\langle y,{}^p x\rangle}{\langle p,p\rangle+\varepsilon^2\langle p,p\rangle^2\langle {}^p x,{}^p x\rangle}\big(p+\varepsilon\langle p,p\rangle\,{}^p x\big)\Big)\bigg)_p=$$

$$=-\bigg(\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\Big(\frac{\langle y,p\rangle+\varepsilon\langle p,p\rangle\langle y,{}^p x\rangle}{\langle p,p\rangle}\big(p+\varepsilon\langle p,p\rangle\,{}^p x\big)\Big)\bigg)_p=$$

$$=-\bigg(\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\Big(\Big(\frac{\langle y,p\rangle}{\langle p,p\rangle}+\varepsilon\langle y,{}^p x\rangle\Big)\big(p+\varepsilon\langle p,p\rangle\,{}^p x\big)\Big)\bigg)_p=-\big(\langle y,{}^p x\rangle p+\langle y,p\rangle\,{}^p x\big)_p=$$

$$=-\langle -,p\rangle\otimes{}^p\big(\langle y,{}^p x\rangle p+\langle y,p\rangle\,{}^p x\big)=-\langle -,p\rangle\otimes\big(\langle y,p\rangle\,{}^p x\big)=-\langle y,p\rangle x_p.$$

For the last,

$$x_p\langle *,*\rangle=\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\big\langle p+\varepsilon\langle p,p\rangle\,{}^p x,\ p+\varepsilon\langle p,p\rangle\,{}^p x\big\rangle=\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\big(\langle p,p\rangle+\varepsilon^2\langle p,p\rangle^2\langle {}^p x,{}^p x\rangle\big)=0\ \blacksquare$$

Let $x,y\in V$, $x\neq 0$. The vector field

$$\operatorname{Tn}(x,y)(*)\rightleftharpoons\frac{y_*}{\langle x,*\rangle}$$

is defined out of $\mathbb{CP}x^\perp\cup\partial\,\mathrm{B}\,V$. Clearly, $\operatorname{Tn}(p,-y)(p)=y_p$ if $\langle p,p\rangle=-1$.

2.1.2. Lemma. *For a nonisotropic $p\in V$, $x,y\in p^\perp$, and any nonisotropic $q\notin p^\perp$,*

$$\nabla_{\operatorname{Tn}(p,x)}\operatorname{Tn}(p,y)(q)=-\frac{\langle y,q\rangle x_q}{\langle p,q\rangle^2}-\frac{\langle p,{}^q x\rangle\langle q,q\rangle y_q}{\langle q,p\rangle\langle p,q\rangle^2}$$

(in particular, $\nabla_{\operatorname{Tn}(p,x)}\operatorname{Tn}(p,y)(p)=\nabla_{\operatorname{Tn}(p,y)}\operatorname{Tn}(p,x)(p)=0$) and $\big[\operatorname{Tn}(p,x),\operatorname{Tn}(p,y)\big](p)=0$.

Proof is routine. By the first two equalities of Lemma 2.1.1,

[2] If one of p_1,p_2 is isotropic, we define conventionally $\operatorname{ta}(p_1,p_2)$ as being ∞ or 1 when $\langle p_1,p_2\rangle\neq 0$ or $\langle p_1,p_2\rangle=0$, respectively.

$$\nabla_{\mathrm{Tn}(p,x)}\mathrm{Tn}(p,y)(q)=\nabla_{\mathrm{Tn}(p,x)(q)}\Big(\frac{y_*}{\langle p,*\rangle}\Big)=\frac{\nabla_{\mathrm{Tn}(p,x)(q)}y_*}{\langle p,q\rangle}-\frac{\mathrm{Tn}(p,x)(q)\big(\langle p,*\rangle\big)y_q}{\langle p,q\rangle^2}=$$

$$=-\frac{\langle y,q\rangle\,\mathrm{Tn}(p,x)(q)}{\langle p,q\rangle}-\frac{\Big\langle p,\dfrac{{}^qx}{\langle p,q\rangle}\Big\rangle\langle q,q\rangle y_q}{\langle p,q\rangle^2}=-\frac{\langle y,q\rangle x_q}{\langle p,q\rangle^2}-\frac{\langle p,{}^qx\rangle\langle q,q\rangle y_q}{\langle q,p\rangle\langle p,q\rangle^2}.$$

Let f be the lift to V of an (analytic) function defined in some neighbourhood of p (thus, $f(cq)=f(q)$ for any $0\neq c\in\mathbb{C}$). By definition,

$$\mathrm{Tn}(p,y)f(q)=\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}f\Big(q+\varepsilon\langle q,q\rangle\frac{{}^qy}{\langle p,q\rangle}\Big).$$

In the sequel, we will repeatedly apply identities of a kind as follows:

$$\frac{d}{d\delta}\Big|_{\delta=0}\bigg(\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}f\Big(\ldots+\frac{\varepsilon}{1+c\varepsilon\delta}\,q+\ldots\Big)\bigg)=\frac{d}{d\delta}\Big|_{\delta=0}\bigg(\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}f(\ldots+\varepsilon q+\ldots)\bigg).$$

In this way,

$$\mathrm{Tn}(p,x)\big(\mathrm{Tn}(p,y)f\big)(p)=\frac{d}{d\delta}\Big|_{\delta=0}\bigg(\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}f\Big(p+\delta x+\varepsilon\langle p+\delta x,p+\delta x\rangle\frac{{}^{p+\delta x}y}{\langle p,p+\delta x\rangle}\Big)\bigg)=$$

$$=\frac{d}{d\delta}\Big|_{\delta=0}\Bigg(\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}f\bigg(p+\delta x+\varepsilon\Big(1+\delta^2\frac{\langle x,x\rangle}{\langle p,p\rangle}\Big)\,{}^{p+\delta x}y\bigg)\Bigg)=$$

$$=\frac{d}{d\delta}\Big|_{\delta=0}\bigg(\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}f\big(p+\delta x+\varepsilon\,{}^{p-\delta x}y\big)\bigg)=\frac{d}{d\delta}\Big|_{\delta=0}\Bigg(\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}f\bigg(p+\delta x+\varepsilon\Big(y-\frac{\langle y,\delta x\rangle}{\langle p,p\rangle}(p+\delta x)\Big)\bigg)\Bigg)=$$

$$=\frac{d}{d\delta}\Big|_{\delta=0}\bigg(\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}f\Big(p+\delta x+\varepsilon y-\varepsilon\delta\frac{\langle y,x\rangle}{\langle p,p\rangle}p\Big)\bigg)=\frac{d}{d\delta}\Big|_{\delta=0}\Bigg(\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}f\bigg(p\Big(1-\varepsilon\delta\frac{\langle y,x\rangle}{\langle p,p\rangle}\Big)+\delta x+\varepsilon y\bigg)\Bigg).$$

From $f(cp)=f(p)$, we derive

$$\mathrm{Tn}(p,x)\big(\mathrm{Tn}(p,y)f\big)(p)=\frac{d}{d\delta}\Big|_{\delta=0}\bigg(\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}f\Big(p+\frac{\delta}{1-\varepsilon\delta\langle y,x\rangle/\langle p,p\rangle}\,x+\frac{\varepsilon}{1-\varepsilon\delta\langle y,x\rangle/\langle p,p\rangle}\,y\Big)\bigg)=$$

$$=\frac{d}{d\delta}\Big|_{\delta=0}\Big(\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}f(p+\delta x+\varepsilon y)\Big).$$

Therefore, $\mathrm{Tn}(p,x)\big(\mathrm{Tn}(p,y)f\big)(p)=\mathrm{Tn}(p,y)\big(\mathrm{Tn}(p,x)f\big)(p)$ ■

2.1.3. Corollary. *∇ is a Levi-Civita connection* ■

2.1.4. Proposition (compare with [Gol, p. 54])**.** *For a nonisotropic $p\in V$ and $x_1,x_2,y\in p^\perp$, the curvature tensor*

$$R(x_{1p},x_{2p})y_p=\langle p,p\rangle\Big(\langle y,x_1\rangle x_2-\langle y,x_2\rangle x_1+\big(\langle x_2,x_1\rangle-\langle x_1,x_2\rangle\big)y\Big)_p.$$

Furthermore, assuming $p\in\mathrm{B}\,V$, $\langle x_1,x_1\rangle=\langle x_2,x_2\rangle=1$, $\langle x_1,x_2\rangle=a+ib$, $a,b\in\mathbb{R}$, $a^2+b^2\le 1$, the sectional curvature $S(x_{1p},x_{2p})=-\Big(1+\dfrac{3b^2}{1-a^2}\Big)$. It varies in $[-4,-1]$, being[3] *(-4) exactly if $x_2\in\mathbb{C}x_1$, and (-1), exactly if $\langle x_1,x_2\rangle\in\mathbb{R}$.*

Proof is immediate. As above, we consider the fields $X_i\rightleftharpoons\mathrm{Tn}(p,x_i)$, $i=1,2$, and $Y\rightleftharpoons\mathrm{Tn}(p,y)$. By Lemma 2.1.2, $\nabla_{X_2}Y(q)=-\dfrac{\langle y,q\rangle x_{2q}}{\langle p,q\rangle^2}-\dfrac{\langle p,{}^qx_2\rangle\langle q,q\rangle y_q}{\langle q,p\rangle\langle p,q\rangle^2}$. Hence,

$$\nabla_{X_1}\nabla_{X_2}Y(p)=\frac{1}{\langle p,p\rangle}\nabla_{x_{1p}}\Big(-\frac{\langle y,*\rangle x_{2*}}{\langle p,*\rangle^2}-\frac{\langle p,{}^*x_2\rangle\langle *,*\rangle y_*}{\langle *,p\rangle\langle p,*\rangle^2}\Big)=$$

[3] with respect to the well-known totally geodesic surfaces: projective line and real plane [Gol].

$$=-\frac{x_{1p}\big(\langle y,*\rangle\big)}{\langle p,p\rangle^3}\,x_{2p}-\frac{x_{1p}\big(\langle p,{}^*x_2\rangle\big)}{\langle p,p\rangle^3}\,y_p=-\frac{\langle y,x_1\rangle}{\langle p,p\rangle^2}\,x_{2p}-\frac{x_{1p}\big(\langle p,{}^*x_2\rangle\big)}{\langle p,p\rangle^3}\,y_p.$$

Since

$$x_{1p}\big(\langle p,{}^*x_2\rangle\big)=x_{1p}\Big(\Big\langle p\,,\,-\frac{\langle x_2,*\rangle *}{\langle *,*\rangle}\Big\rangle\Big)=-x_{1p}\Big(\frac{\langle *,x_2\rangle\langle p,*\rangle}{\langle *,*\rangle}\Big)=-x_{1p}\big(\langle *,x_2\rangle\big)=-\langle x_1,x_2\rangle\langle p,p\rangle,$$

we obtain

$$\nabla_{X_1}\nabla_{X_2}Y(p)=\frac{1}{\langle p,p\rangle^2}\big(\langle x_1,x_2\rangle y_p-\langle y,x_1\rangle x_{2p}\big).$$

Analogously,

$$\nabla_{X_2}\nabla_{X_1}Y(p)=\frac{1}{\langle p,p\rangle^2}\big(\langle x_2,x_1\rangle y_p-\langle y,x_2\rangle x_{1p}\big).$$

By Lemma 2.1.2,

$$R\Big(\frac{x_{1p}}{\langle p,p\rangle},\frac{x_{2p}}{\langle p,p\rangle}\Big)\frac{y_p}{\langle p,p\rangle}=R(X_1,X_2)Y(p)=\big(\nabla_{X_2}\nabla_{X_1}Y-\nabla_{X_1}\nabla_{X_2}Y+\nabla_{[X_1,X_2]}Y\big)(p)=$$

$$=\frac{1}{\langle p,p\rangle^2}\,\Big(\langle y,x_1\rangle x_2-\langle y,x_2\rangle x_1+\big(\langle x_2,x_1\rangle-\langle x_1,x_2\rangle\big)y\Big)_p,$$

implying the desired fact. Now, $S(x_{1p},x_{2p})=\dfrac{\big(R(x_{1p},x_{2p})x_{1p},x_{2p}\big)}{(x_{1p},x_{1p})(x_{2p},x_{2p})-(x_{1p},x_{2p})^2}=-\Big(1+\dfrac{3b^2}{1-a^2}\Big)$ ■

2.2. Kähler Potential

2.2.1. Lemma. *Let $u\in V$. We define a 1-form $P_u(x_p)\rightleftharpoons-\frac{1}{2}\operatorname{Im}\Big(\frac{\langle p,p\rangle\langle {}^px,u\rangle}{\langle p,u\rangle}\Big)$ for $p\notin\mathbb{C}\mathbb{P}u^\perp\cup\partial\,\mathrm{B}\,V$. Let $u_1,u_2\in V$, $\langle u_1,u_2\rangle\ne0$. For $p\notin\mathbb{C}\mathbb{P}u_1^\perp\cup\mathbb{C}\mathbb{P}u_2^\perp$, we define*[4] *$f_{u_1,u_2}(p)\rightleftharpoons\frac{1}{2}\operatorname{Arg}\Big(\frac{\langle u_1,p\rangle\langle p,u_2\rangle}{\langle u_1,u_2\rangle}\Big)$. Then the form $dP_u=\omega$ is a Kähler form and $P_{u_1}-P_{u_2}=df_{u_1,u_2}$.*

Proof is a routine use of Lemmas 2.1.1 and 2.1.2 and involves the identity of Maurer-Cartan. Let $p\in V$ be nonisotropic and let $x,y\in p^\perp$. For $X\rightleftharpoons\mathrm{Tn}(p,x)$ and $Y\rightleftharpoons\mathrm{Tn}(p,y)$, we have

$$X\big(P_u(Y)\big)(p)=-\frac{1}{2}\operatorname{Im}\bigg(\frac{1}{\langle p,p\rangle}\,x_p\Big(\frac{\langle *,*\rangle\langle {}^*y,u\rangle}{\langle p,*\rangle\langle *,u\rangle}\Big)\bigg)=-\frac{1}{2}\operatorname{Im}\bigg(x_p\Big(\frac{\langle {}^*y,u\rangle}{\langle p,*\rangle\langle *,u\rangle}\Big)\bigg)=$$

$$=-\frac{1}{2}\operatorname{Im}\Big(\frac{1}{\langle p,p\rangle\langle p,u\rangle}\,x_p\big(\langle {}^*y,u\rangle\big)-\frac{\langle y,u\rangle}{\langle p,p\rangle^2\langle p,u\rangle^2}\,x_p\big(\langle p,*\rangle\langle *,u\rangle\big)\Big).$$

From $x_p\big(\langle {}^*y,u\rangle\big)=-x_p\Big(\frac{\langle y,*\rangle\langle *,u\rangle}{\langle *,*\rangle}\Big)=-x_p\big(\langle y,*\rangle\big)\frac{\langle p,u\rangle}{\langle p,p\rangle}=-\langle y,x\rangle\langle p,u\rangle$ and from $x_p\big(\langle p,*\rangle\langle *,u\rangle\big)=\langle p,p\rangle^2\langle x,u\rangle$, we obtain $X\big(P_u(Y)\big)(p)=\frac{1}{2}\operatorname{Im}\Big(\frac{\langle y,x\rangle}{\langle p,p\rangle}+\frac{\langle y,u\rangle\langle x,u\rangle}{\langle p,u\rangle^2}\Big)$. By Lemma 2.1.2,

$$dP_u\Big(\frac{x_p}{\langle p,p\rangle},\frac{y_p}{\langle p,p\rangle}\Big)=dP_u\big(X,Y\big)(p)=X\big(P_u(Y)\big)(p)-Y\big(P_u(X)\big)(p)-P_u\big([X,Y]\big)(p)=$$

$$=\frac{1}{2}\operatorname{Im}\Big(\frac{\langle y,x\rangle}{\langle p,p\rangle}+\frac{\langle y,u\rangle\langle x,u\rangle}{\langle p,u\rangle^2}\Big)-\frac{1}{2}\operatorname{Im}\Big(\frac{\langle x,y\rangle}{\langle p,p\rangle}+\frac{\langle x,u\rangle\langle y,u\rangle}{\langle p,u\rangle^2}\Big)=\frac{1}{2}\operatorname{Im}\Big(\frac{\langle y,x\rangle}{\langle p,p\rangle}-\frac{\langle x,y\rangle}{\langle p,p\rangle}\Big)$$

implying that $dP_u(x_p,y_p)=\frac{1}{2}\operatorname{Im}\big(\langle x_p,y_p\rangle-\langle y_p,x_p\rangle\big)=\operatorname{Im}\langle x_p,y_p\rangle$.

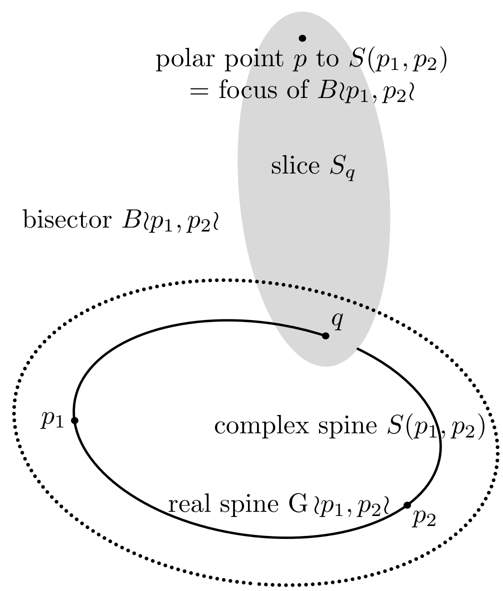
For $x\in p^\perp$,

[4] In what follows, we understand by Arg a function taking values in $[0,2\pi]$. However, in this Lemma, it is better to read Arg as a multi-valued and, hence, smooth function defined in $\mathbb{C}\setminus\{0\}$.

$$df_{u_1,u_2}(x_p)=\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}f_{u_1,u_2}\big(p+\varepsilon\langle p,p\rangle x\big)=\frac12\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\operatorname{Arg}\Big(\big\langle u_1,p+\varepsilon\langle p,p\rangle x\big\rangle\big\langle p+\varepsilon\langle p,p\rangle x,u_2\big\rangle\Big)=$$
$$=\frac12\operatorname{Im}\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\Big(\ln\big\langle u_1,p+\varepsilon\langle p,p\rangle x\big\rangle+\ln\big\langle p+\varepsilon\langle p,p\rangle x,u_2\big\rangle\Big)=$$
$$=\frac12\operatorname{Im}\Big(\frac{\langle p,p\rangle\langle u_1,x\rangle}{\langle u_1,p\rangle}+\frac{\langle p,p\rangle\langle x,u_2\rangle}{\langle p,u_2\rangle}\Big)=\big(P_{u_1}-P_{u_2}\big)(x_p)\ \blacksquare$$

2.3. Bisectors. For any projective line $S\subset\mathbb{CP}V$, there is a unique $p\in\mathbb{CP}V$ such that $S=\mathbb{CP}p^\perp$. We call p the *polar point to* S. We will denote by $S(p_1,p_2)\risingdotseq\mathbb{CP}(\mathbb{C}p_1+\mathbb{C}p_2)$ the projective line passing through two different points $p_1,p_2\in\mathbb{CP}V$.

Let $S\subset V$ be an $\mathbb{R}$-vector subspace, $\dim_\mathbb{R}S=2$, such that the hermitian form is real and nondegenerated (of signature $++$ or $+-$) over S. We call $\mathrm{G}\,S\risingdotseq\mathbb{CP}S\risingdotseq\mathbb{RP}^1S$ *extended geodesic* (or simply *geodesic*). For $p_1,p_2\in\mathbb{CP}V$, $0\neq\operatorname{ta}(p_1,p_2)\neq1$, there exists a unique extended geodesic $\mathrm{G}\wr p_1,p_2\wr$ containing p_1,p_2. It is easy to verify that $\mathrm{G}\wr p_1,p_2\wr$ is given in $S(p_1,p_2)$ by the equation $b(x,p_1,p_2)\risingdotseq\dfrac{\langle p_1,x\rangle\langle x,p_2\rangle}{\langle p_1,p_2\rangle}-\dfrac{\langle p_2,x\rangle\langle x,p_1\rangle}{\langle p_2,p_1\rangle}=0$. In $\mathbb{CP}V$, this equation describes[5] the (*extended*) *bisector* $B\wr p_1,p_2\wr$ whose *complex spine* is $S(p_1,p_2)$, whose *focus* $p\notin\partial\,\mathrm{B}\,V$ is the polar point to $S(p_1,p_2)$, and whose *real spine* is $\mathrm{G}\wr p_1,p_2\wr$. Let $q\in\mathrm{G}\wr p_1,p_2\wr$. The projective line $S_q\risingdotseq S(p,q)$ that connects q with the focus p is a *slice* of the bisector. Obviously, the polar point to any slice belongs to the real spine. These definitions differ slightly from common ones: for instance, what we call a bisector is an extor in [Gol].



Out of $\partial\,\mathrm{B}\,V$, the bisector is given by zeroes of the function $f(x)\risingdotseq\dfrac{b(x,p_1,p_2)}{\langle x,x\rangle}$. This implies that v_q is tangent to $B(p_1,p_2)$, where $q\in B\wr p_1,p_2\wr\setminus\partial\,\mathrm{B}\,V$ and $v\in q^\perp$, if and only if $v_q(f)=0$. By Lemma 2.1.1, this is equivalent to $t(v,q,p_1,p_2)\risingdotseq\dfrac{\langle p_1,v\rangle\langle q,p_2\rangle+\langle p_1,q\rangle\langle v,p_2\rangle}{\langle p_1,p_2\rangle}-\dfrac{\langle p_2,v\rangle\langle q,p_1\rangle+\langle p_2,q\rangle\langle v,p_1\rangle}{\langle p_2,p_1\rangle}=0$.

2.3.1. Lemma. *Let* $p_1,p_2\in\mathbb{CP}V$, $0\neq\operatorname{ta}(p_1,p_2)\neq1$, *and let* $q\in B\wr p_1,p_2\wr\setminus\partial\,\mathrm{B}\,V$. *Then* $n(q,p_1,p_2)\risingdotseq\langle-,q\rangle\otimes i\cdot\Big(\dfrac{\langle q,p_2\rangle}{\langle p_1,p_2\rangle}p_1-\dfrac{\langle q,p_1\rangle}{\langle p_2,p_1\rangle}p_2\Big)\neq0$ (*unless* $\langle q,p_1\rangle=\langle q,p_2\rangle=0$) *is a vector normal*[6] *to* $B\wr p_1,p_2\wr$ *at* q.

Proof. The equation $t(v,q,p_1,p_2)=0$ is equivalent to $\operatorname{Re}\big\langle n(q,p_1,p_2),v_q\big\rangle=0$ $\blacksquare$

2.3.2. Remark. Every geodesic G of signature $+-$ possesses exactly two isotropic points v_1,v_2 called *vertices.* For a given $g\in\mathrm{G}\cap\mathrm{B}\,V$, we can choose representatives $v_1,v_2,g\in V$ such that $\begin{pmatrix}0&-\frac12\\-\frac12&0\end{pmatrix}$ is the Gram matrix for v_1,v_2 and such that $g=v_1+v_2$. The formula $g(\alpha)=\alpha^{-1}v_1+\alpha v_2$, $\alpha>0$, lists

[5] Also, it is possible to define a bisector as the hypersurface equidistant (equitant) from two different points.

[6] with the orientation of $B\wr p_1,p_2\wr$ taken into account: the region given by $\operatorname{Im}\dfrac{\langle p_1,x\rangle\langle x,p_2\rangle}{\langle p_1,p_2\rangle}\geq0$ is on the side of the indicated normal vector.

all negative points in G and $\big\langle g(\alpha),g(\alpha)\big\rangle=-1$. For arbitrary $p_1,p_2,g\in\mathrm G\cap\mathrm B\,V$, we can assume that $g=v_1+v_2$, $p_i=\alpha_i^{-1}v_1+\alpha_iv_2$, $\alpha_i>0$, $i=1,2$. Then $\dfrac{\langle p_1,g\rangle\langle g,p_2\rangle}{\langle p_1,p_2\rangle}=-\dfrac{(\alpha_1^{-1}+\alpha_1)(\alpha_2^{-1}+\alpha_2)}{2(\alpha_1^{-1}\alpha_2+\alpha_1\alpha_2^{-1})}<0$.

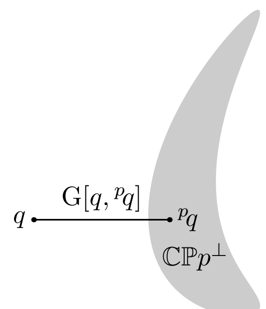
It follows that, for any $x\in B\wr p_1,p_2\wr\cap\mathrm B\,V$, we have $\dfrac{\langle p_1,x\rangle\langle x,p_2\rangle}{\langle p_1,p_2\rangle}<0$ since we can write $x=g+cp$ for some $g\in\mathrm G\wr p_1,p_2\wr\cap\mathrm B\,V$ and $c\in\mathbb C$, where p stands for the focus of $B\wr p_1,p_2\wr$.

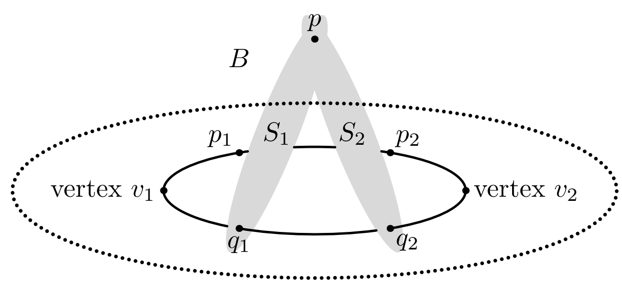
Similar considerations are applicable to the positive part of G.

Let $p_1,p_2\in\overline{\mathrm B}V$. We will denote by $\mathrm G[p_1,p_2]$ (or by $\mathrm G[p_1,p_2\rangle$) a definite geodesic segment connecting p_1 and p_2 (usually the one included in $\mathrm B\,V$). A similar notation, $B[p_1,p_2]$, will be used for bisector 'segments' corresponding to geodesic segments.

2.3.3. Remark. Let $p\notin\overline{\mathrm B}V$ and $q\in\mathrm B\,V$. Then $\mathrm G[q,{}^pq]$ is the shortest geodesic from q to $\mathbb CPp^\perp\cap\mathrm B\,V$.

Let $S_1\rightleftharpoons\mathbb CPp_1^\perp$ and $S_2\rightleftharpoons\mathbb CPp_2^\perp$ be different projective lines with a nonisotropic intersection p. We assume S_1 and S_2 to be *nonorthogonal,* i.e., $\langle p_1,p_2\rangle\neq0$. Then, there exists a unique bisector B such that S_1 and S_2 are among its slices; p is the focus of B. We define $\mathrm{ta}(S_1,S_2)\rightleftharpoons\mathrm{ta}(p_1,p_2)$. If $p\notin\overline{\mathrm B}V$ and S_i's are of signature $+-$ (i.e., S_1 and S_2 are *ultraparallel*), then $\mathrm{ta}(S_1,S_2)$ is the minimum of the tance between the points in $S_1\cap\mathrm B\,V$ and the points in $S_2\cap\mathrm B\,V$. Indeed, let q_i denote the intersection of the complex spine $S(p_1,p_2)=\mathbb CPp^\perp$ with S_i, $i=1,2$. Then, the minimum in question equals $\mathrm{ta}(q_1,q_2)$. Since $q_1,q_2\in\mathrm G\wr p_1,p_2\wr$ are respectively orthogonal to $p_1,p_2\in\mathrm G\wr p_1,p_2\wr$, we have $\mathrm{ta}(q_1,q_2)=\mathrm{ta}(p_1,p_2)$. In particular, S_1 and S_2 are ultraparallel if and only if $\mathrm{ta}(p_1,p_2)>1$.



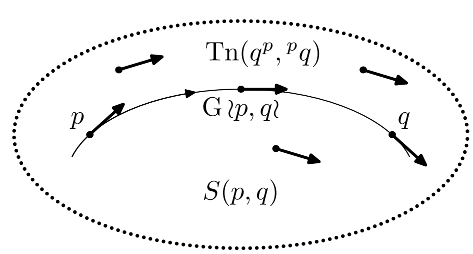


3. Displacements along Geodesics

3.1. Lemma. *Let $p\in\mathbb CPV\setminus\partial\,\mathrm B\,V$ and let $p\neq q\in\mathbb CPV\setminus\mathbb CPp^\perp$. The vector field $\mathrm{Tn}(q^p,{}^pq)$, defined out of $\mathbb CPp^\perp\cup\partial\,\mathrm B\,V$, has constant length over $S(p,q)$, that is, $\big\langle\,\mathrm{Tn}(q^p,{}^pq)(x),\mathrm{Tn}(q^p,{}^pq)(x)\big\rangle=1-\dfrac{\langle p,p\rangle\langle q,q\rangle}{\langle p,q\rangle\langle q,p\rangle}$ for $S(p,q)\ni x\notin\mathbb CPp^\perp\cup\partial\,\mathrm B\,V$, and it is tangent to $\mathrm G\wr p,q\wr$ if $\mathrm{ta}(p,q)\neq1$.*

Proof is a routine use of Lemma 2.1.1. From $\langle p,q\rangle\neq0$, it follows that $q^p\neq0$, and $p\neq q$ implies ${}^pq\neq0$.

Let $S(p,q)\ni x\notin\mathbb CPp^\perp\cup\partial\,\mathrm B\,V$. Then $x^p\neq0$ and



$$\big\langle\,\mathrm{Tn}(q^p,{}^pq)(x),\mathrm{Tn}(q^p,{}^pq)(x)\big\rangle=$$

$$=-\langle x,x\rangle\Big\langle{}^pq-\frac{\langle{}^pq,x\rangle}{\langle x,x\rangle}x,{}^pq-\frac{\langle{}^pq,x\rangle}{\langle x,x\rangle}x\Big\rangle\frac{1}{\langle x,q^p\rangle\langle q^p,x\rangle}=$$

$$=\frac{\big\langle\langle{}^pq,x\rangle x-\langle x,x\rangle\,{}^pq,{}^pq\big\rangle}{\langle x,q^p\rangle\langle q^p,x\rangle}=$$

$$=\frac{\big\langle\langle {}^pq,{}^px\rangle\,{}^px-\langle x^p,x^p\rangle\,{}^pq-\langle {}^px,{}^px\rangle\,{}^pq,{}^pq\big\rangle}{\langle x^p,q^p\rangle\langle q^p,x^p\rangle}=\frac{\langle {}^pq,{}^px\rangle\langle {}^px,{}^pq\rangle-\langle {}^px,{}^px\rangle\langle {}^pq,{}^pq\rangle}{\langle x^p,q^p\rangle\langle q^p,x^p\rangle}-$$
$$-\frac{\langle x^p,x^p\rangle\langle {}^pq,{}^pq\rangle}{\langle x^p,q^p\rangle\langle q^p,x^p\rangle}=-\frac{\langle x^p,x^p\rangle\langle {}^pq,{}^pq\rangle}{\langle x^p,q^p\rangle\langle q^p,x^p\rangle},$$

since ${}^px\in\mathbb C\,{}^pq$ implies that $\langle {}^pq,{}^px\rangle\langle {}^px,{}^pq\rangle-\langle {}^px,{}^px\rangle\langle {}^pq,{}^pq\rangle=0$. Using the fact that $0\neq x^p\in\mathbb Cq^p$, we can substitute x^p by q^p in the expression for $\big\langle\operatorname{Tn}(q^p,{}^pq)(x),\operatorname{Tn}(q^p,{}^pq)(x)\big\rangle$. Consequently,

$$\big\langle\operatorname{Tn}(q^p,{}^pq)(x),\operatorname{Tn}(q^p,{}^pq)(x)\big\rangle=-\frac{\langle {}^pq,{}^pq\rangle}{\langle q^p,q^p\rangle}=1-\frac{\langle p,p\rangle\langle q,q\rangle}{\langle p,q\rangle\langle q,p\rangle}.$$

The fact that $g\in\mathrm G\wr p,q\wr$ means that $b(g,p,q)=0$, that is, $\dfrac{\langle p,g\rangle\langle g,q\rangle}{\langle p,q\rangle}\in\mathbb R$. We need to verify that $t\Big(\dfrac{{}^g({}^pq)}{\langle q^p,g\rangle},g,p,q\Big)=0$. It follows from $\langle p,{}^pq\rangle=0$ that $\big\langle p,{}^g({}^pq)\big\rangle=-\dfrac{\langle g,{}^pq\rangle\langle p,g\rangle}{\langle g,g\rangle}$. Hence,

$$\frac{\Big\langle p,\dfrac{{}^g({}^pq)}{\langle q^p,g\rangle}\Big\rangle\langle g,q\rangle+\langle p,g\rangle\Big\langle\dfrac{{}^g({}^pq)}{\langle q^p,g\rangle},q\Big\rangle}{\langle p,q\rangle}=-\frac{\langle g,{}^pq\rangle\langle p,g\rangle}{\langle g,q^p\rangle\langle g,g\rangle\langle p,q\rangle}\langle g,q\rangle+\langle p,g\rangle\frac{\big\langle{}^g({}^pq),q\big\rangle}{\langle q^p,g\rangle\langle p,q\rangle}=$$
$$=-\frac{\langle g,{}^pq\rangle\langle p,g\rangle}{\langle g,q^p\rangle\langle g,g\rangle\langle p,q\rangle}\langle g,q\rangle-\langle p,g\rangle\frac{\langle {}^pq,g\rangle\langle g,q\rangle}{\langle q^p,g\rangle\langle p,q\rangle\langle g,g\rangle}+\langle p,g\rangle\frac{\big\langle{}^pq,q\big\rangle}{\langle q^p,g\rangle\langle p,q\rangle}=$$
$$=-\frac{\langle p,g\rangle\langle g,q\rangle}{\langle p,q\rangle\langle g,g\rangle}\Big(\frac{\langle g,{}^pq\rangle}{\langle g,q^p\rangle}+\frac{\langle {}^pq,g\rangle}{\langle q^p,g\rangle}\Big)+\langle p,g\rangle\frac{\langle {}^pq,q\rangle\langle p,p\rangle}{\langle q,p\rangle\langle p,g\rangle\langle p,q\rangle}\equiv\frac{\langle q,q\rangle\langle p,p\rangle}{\langle q,p\rangle\langle p,q\rangle}-1\equiv0,\quad\operatorname{mod}\mathbb R.$$

This implies that $t\Big(\dfrac{{}^g({}^pq)}{\langle q^p,g\rangle},g,p,q\Big)=0$ ■

3.2. Corollary. *Let R be a real plane and let $p,u\in R$, $p\notin\mathbb{CP}u^\perp\cap\partial\,\mathrm BV$. Then $P_u(x_p)=0$ for any $x_p\in\mathrm T_pR$.*

Proof. There exists $q\in R$ such that x_p is proportional to the tangent vector $\operatorname{Tn}(q^p,{}^pq)(p)$ to $\mathrm G\wr p,q\wr$ at p. Now, $P_u\big(\operatorname{Tn}(q^p,{}^pq)(p)\big)=-\dfrac12\operatorname{Im}\Big(\dfrac{\langle p,p\rangle\langle {}^pq,u\rangle}{\langle q^p,p\rangle\langle p,u\rangle}\Big)=0$, since $p,u,q\in R$ ■

Let $x,y\in V$, $x\neq0$. The vector field

$$\operatorname{Ct}(x,y)(*)\rightleftharpoons\frac{\langle *,x\rangle\,y_*}{\langle *,*\rangle\sqrt{\operatorname{ta}(x,*)}}$$

is defined out of $\mathbb{CP}x^\perp\cup\partial\,\mathrm BV$ (for x isotropic, it vanishes).

3.3. Lemma. *Let $p\in\mathbb{CP}V\setminus\partial\,\mathrm BV$ and let $0\neq\operatorname{ta}(p,q)\neq1$. Let $h,v\in p^\perp$ be such that $h\in\mathbb Cp+\mathbb Cq$ and $v\in(\mathbb Cp+\mathbb Cq)^\perp$. Then $\nabla_{\operatorname{Tn}(q^p,{}^pq)}\operatorname{Tn}\big(p,\langle p,p\rangle h\big)=0$ and $\nabla_{\operatorname{Tn}(q^p,{}^pq)}\operatorname{Ct}(p,v)=0$ over $\mathrm G\wr p,q\wr\setminus(\mathbb{CP}p^\perp\cup\partial\,\mathrm BV)$.*

Proof is basically a direct verification. Let $g\in\mathrm G\wr p,q\wr\setminus(\mathbb{CP}p^\perp\cup\partial\,\mathrm BV)$. Then $b(g,p,q)=0$, i.e., $\langle g,p\rangle\langle p,q\rangle\langle q,g\rangle=\langle g,q\rangle\langle q,p\rangle\langle p,g\rangle$, implying

$$\langle g,p\rangle\langle p,q^p\rangle\langle q^p,g\rangle+\langle g,p\rangle\langle p,q^p\rangle\langle {}^pq,g\rangle=\langle g,q^p\rangle\langle q^p,p\rangle\langle p,g\rangle+\langle g,{}^pq\rangle\langle q^p,p\rangle\langle p,g\rangle$$

which can be rewritten as

$$\langle g^p,p\rangle\langle p,q^p\rangle\langle q^p,g^p\rangle+\langle g,p\rangle\langle p,q^p\rangle\langle {}^pq,g\rangle=\langle g^p,q^p\rangle\langle q^p,p\rangle\langle p,g^p\rangle+\langle g,{}^pq\rangle\langle q^p,p\rangle\langle p,g\rangle.$$

It follows from $g^p\in\mathbb Cq^p$ that $\langle g^p,p\rangle\langle p,q^p\rangle\langle q^p,g^p\rangle=\langle g^p,q^p\rangle\langle q^p,p\rangle\langle p,g^p\rangle$. Therefore, $\langle g,p\rangle\langle p,q^p\rangle\langle {}^pq,g\rangle=\langle g,{}^pq\rangle\langle q^p,p\rangle\langle p,g\rangle$. Since $p\in\mathbb Cq^p$, we can substitute p by q^p in the last equality, getting

$$\langle g,q^p\rangle\langle {}^pq,g\rangle=\langle g,{}^pq\rangle\langle q^p,g\rangle.$$

For $t \rightleftharpoons \dfrac{{}^pq}{\langle q^p, g\rangle}$, we have $\mathrm{Tn}(q^p, {}^pq)(g) = t_g$. Being $\mathrm{Tn}\big(p, \langle p,p\rangle h\big)(*) = \dfrac{\langle p,p\rangle}{\langle p,*\rangle} h_*$, by Lemma 2.1.1,

$$\nabla_{\mathrm{Tn}(q^p,{}^pq)(g)} \mathrm{Tn}\big(p, \langle p,p\rangle h\big) = \nabla_{t_g}\Big(\frac{\langle p,p\rangle}{\langle p,*\rangle} h_*\Big) = t_g\Big(\frac{\langle p,p\rangle}{\langle p,*\rangle}\Big) h_g - \frac{\langle p,p\rangle}{\langle p,g\rangle}\langle h,g\rangle t_g =$$

$$= -\frac{\langle p,p\rangle}{\langle p,g\rangle^2}\langle p, {}^gt\rangle\langle g,g\rangle h_g - \frac{\langle p,p\rangle}{\langle p,g\rangle}\langle h,g\rangle t_g = -\frac{\langle p,p\rangle}{\langle p,g\rangle}\Big(\frac{\langle p, {}^gt\rangle\langle g,g\rangle}{\langle p,g\rangle} h_g + \langle h,g\rangle t_g\Big) =$$

$$= -\frac{\langle p,p\rangle}{\langle p,g\rangle}\Big(\frac{\big\langle p, {}^g({}^pq)\big\rangle\langle g,g\rangle}{\langle g,q^p\rangle\langle p,g\rangle} h_g + \frac{\langle h,g\rangle}{\langle q^p,g\rangle}({}^pq)_g\Big) = -\frac{\langle p,p\rangle}{\langle p,g\rangle}\Big(\frac{\big\langle p, \langle g,g\rangle\, {}^pq - \langle {}^pq, g\rangle g\big\rangle}{\langle g,q^p\rangle\langle p,g\rangle} h_g + \frac{\langle h,g\rangle}{\langle q^p,g\rangle}({}^pq)_g\Big) =$$

$$= \frac{\langle p,p\rangle}{\langle p,g\rangle}\Big(\frac{\langle g, {}^pq\rangle}{\langle g,q^p\rangle} h_g - \frac{\langle h,g\rangle}{\langle q^p,g\rangle}({}^pq)_g\Big) = 0,$$

since $h \in \mathbb{C}\,{}^pq$ and $\langle g,q^p\rangle\langle {}^pq,g\rangle = \langle g,{}^pq\rangle\langle q^p,g\rangle$.

By Lemma 2.1.1, $\nabla_{t_g} v_* = -\langle v,g\rangle t_g = 0$ and $t_g\Big(\dfrac{1}{\langle *,*\rangle}\Big) = 0$. Since $\mathrm{Ct}(p,v)(*) = \dfrac{\langle *,p\rangle v_*}{\langle *,*\rangle\sqrt{\mathrm{ta}(p,*)}}$, in order to prove that $\nabla_{\mathrm{Tn}(q^p,{}^pq)}\mathrm{Ct}(p,v) = 0$, it suffices to verify that $t_g\Big(\dfrac{\langle *,p\rangle}{\sqrt{\mathrm{ta}(p,*)}}\Big) = 0:$

$$t_g\Big(\frac{\langle *,p\rangle}{\sqrt{\mathrm{ta}(p,*)}}\Big) = \frac{\langle {}^gt,p\rangle\langle g,g\rangle}{\sqrt{\mathrm{ta}(p,g)}} - \frac{\langle g,p\rangle}{2\sqrt{\mathrm{ta}(p,g)}\cdot\mathrm{ta}(p,g)} t_g\Big(\frac{\langle p,*\rangle\langle *,p\rangle}{\langle p,p\rangle\langle *,*\rangle}\Big) =$$

$$= \frac{\langle {}^gt,p\rangle\langle g,g\rangle}{\sqrt{\mathrm{ta}(p,g)}} - \frac{\langle g,p\rangle}{2\sqrt{\mathrm{ta}(p,g)}\cdot\mathrm{ta}(p,g)}\cdot\frac{t_g\big(\langle p,*\rangle\langle *,p\rangle\big)}{\langle p,p\rangle\langle g,g\rangle} =$$

$$= \frac{\langle {}^gt,p\rangle\langle g,g\rangle}{\sqrt{\mathrm{ta}(p,g)}} - \frac{\langle g,p\rangle}{2\sqrt{\mathrm{ta}(p,g)}\cdot\mathrm{ta}(p,g)}\cdot\frac{\langle p,{}^gt\rangle\langle g,g\rangle\langle g,p\rangle + \langle p,g\rangle\langle {}^gt,p\rangle\langle g,g\rangle}{\langle p,p\rangle\langle g,g\rangle} =$$

$$= \frac{\langle g,g\rangle}{2\sqrt{\mathrm{ta}(p,g)}\cdot\langle p,g\rangle}\big(2\langle {}^gt,p\rangle\langle p,g\rangle - \langle p,{}^gt\rangle\langle g,p\rangle - \langle p,g\rangle\langle {}^gt,p\rangle\big) =$$

$$= \frac{\langle g,g\rangle}{2\sqrt{\mathrm{ta}(p,g)}\cdot\langle p,g\rangle}\big(\langle {}^gt,p\rangle\langle p,g\rangle - \langle p,{}^gt\rangle\langle g,p\rangle\big) = 0,$$

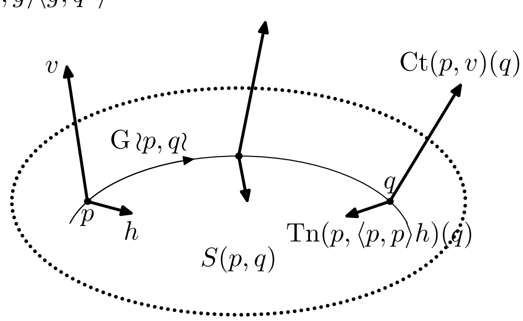
since

$$\langle {}^gt,p\rangle\langle p,g\rangle - \langle p,{}^gt\rangle\langle g,p\rangle = \frac{\big\langle {}^g({}^pq),p\big\rangle\langle p,g\rangle}{\langle q^p,g\rangle} - \frac{\big\langle p,{}^g({}^pq)\big\rangle\langle g,p\rangle}{\langle g,q^p\rangle} =$$

$$= -\frac{\langle {}^pq,g\rangle\langle g,p\rangle\langle p,g\rangle}{\langle g,g\rangle\langle q^p,g\rangle} + \frac{\langle g,{}^pq\rangle\langle p,g\rangle\langle g,p\rangle}{\langle g,g\rangle\langle g,q^p\rangle} = 0$$

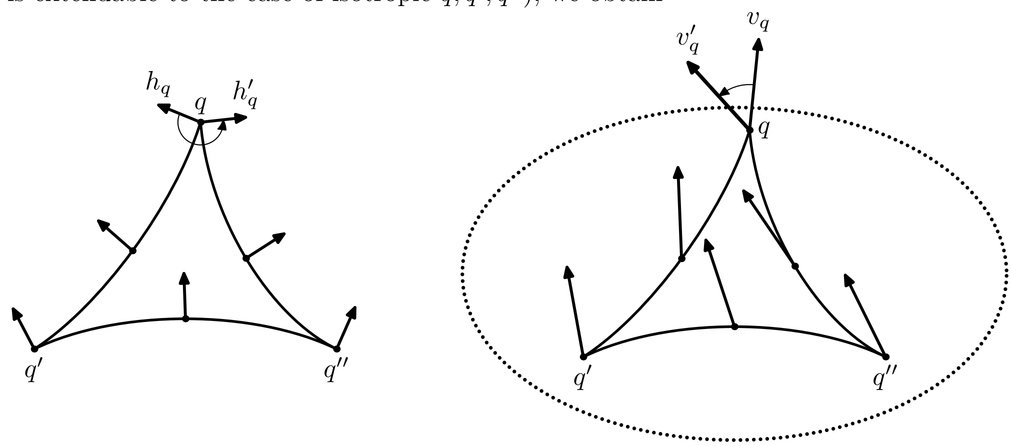
by the equality $\langle g,q^p\rangle\langle {}^pq,g\rangle = \langle g,{}^pq\rangle\langle q^p,g\rangle$ ■

3.4. Corollary. *The parallel displacement of any* $x_p \in \mathrm{T}_p\,\mathbb{CP}V$, *where* $x \rightleftharpoons h + v$, *along the geodesic* $\mathrm{G}\wr p,q\wr$ *yields* $\mathrm{Tn}\big(p,\langle p,p\rangle h\big)(q) + \mathrm{Ct}(p,v)(q)$ ■



Thus, making the parallel displacement along a geodesic, we can distinguish its *horizontal* (tangent to the projective line of the geodesic) and *vertical* (orthogonal to the projective line of the geodesic) components.

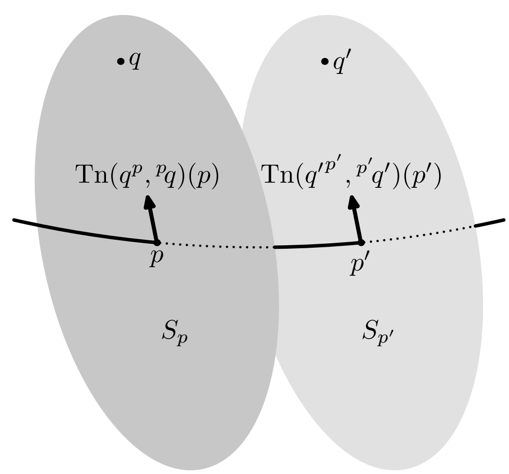
Let S be a projective line and let $q,q',q'' \in S \cap \mathrm{B}\,V$. For any $h_q \in \mathrm{T}_q\,S$, we parallelly displace h_q along $\mathrm{G}[q,q']$, then along $\mathrm{G}[q',q'']$, and finally, along $\mathrm{G}[q'',q]$, resulting in $h'_q \in \mathrm{T}_q\,S$. The angle from h_q to h'_q taken in the interval $(-\pi,\pi)$ is an additive measure of triangles and, therefore, it is proportional to the oriented area of the triangle $\Delta(q,q',q'')$. As is easy to see, the angle in question is equal to $\mp(\pi-\alpha-\alpha'-\alpha'')$ in the case of a triangle oriented in counterclockwise/clockwise sense, where α,α',α'' stand for the angles of the triangle. So, the coefficient of proportionality is $-\frac{1}{4}$, since $\operatorname{Area}\Delta(q,q',q'')=\pm\frac{1}{4}(\pi-\alpha-\alpha'-\alpha'')$ (we remember that our metric in a complex geodesic is $\frac{1}{4}$ of the usual one). The same arguments are applicable to a vertical vector. A straightforward calculation yields $v'_q=\dfrac{\langle q,q''\rangle\langle q'',q'\rangle\langle q',q\rangle v_q}{\langle q,q\rangle\langle q'',q''\rangle\langle q',q'\rangle\sqrt{\operatorname{ta}(q,q')\operatorname{ta}(q',q'')\operatorname{ta}(q'',q)}}$, implying that[7] $\angle(v_q,v'_q)=\arg\langle v'_q,v_q\rangle=\arg\big(-\langle q,q''\rangle\langle q'',q'\rangle\langle q',q\rangle\big)$. By considering a suitable ideal triangle (the formula is extendable to the case of isotropic q,q',q''), we obtain



3.5. Remark. (see [Gol], for instance). Let S be a projective line and let $q,q',q''\in S\cap\overline{\mathrm{B}}V$. Then $\arg\big(-\langle q,q'\rangle\langle q',q''\rangle\langle q'',q\rangle\big)=2\operatorname{Area}\Delta(q,q',q'')$.

3.6. Lemma. *Let $p,p'\in\mathbb{CP}V\setminus\partial\,\mathrm{B}\,V$ be such that $0\neq\operatorname{ta}(p,p')\neq 1$. We denote by S_p and $S_{p'}$ the slices of the bisector $B\wr p,p'\wr$ passing through p and p', respectively. For $q\in S_p$ different from the focus of $B\wr p,p'\wr$, let $t_p(q)\rightleftharpoons\operatorname{Tn}(q^p,{}^pq)(p)$ denote a tangent vector to the geodesic $\mathrm{G}\wr p,q\wr$ at p. We will use a similar notation $t_{p'}(q')\rightleftharpoons\operatorname{Tn}(q'^{p'},{}^{p'}q')(p')$ related to $q'\in S_{p'}$.*

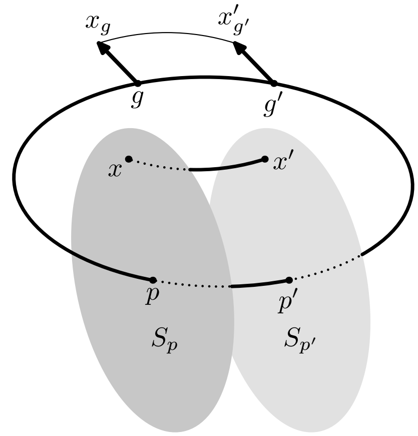
For any $q\in S_p$, there exists a unique $q'\in S_{p'}$, given explicitly by $q'=\langle p',p\rangle{}^pq+\langle q,p\rangle\sqrt{\operatorname{ta}(p,p')}\cdot p'$, such that the parallel displacement of $t_p(q)$ along the geodesic $\mathrm{G}\wr p,p'\wr$ is exactly $t_{p'}q'$.



Proof. By Corollary 3.4, the parallel displacement of $\operatorname{Tn}(q^p,{}^pq)(p)=\dfrac{q_p}{\langle q,p\rangle}=\operatorname{Ct}\Big(p,\dfrac{{}^pq}{\langle q,p\rangle}\Big)(p)$ along the geodesic $\mathrm{G}\wr p,p'\wr$ yields $\operatorname{Ct}\Big(p,\dfrac{{}^pq}{\langle q,p\rangle}\Big)(p')=\dfrac{\langle p',p\rangle({}^pq)_{p'}}{\langle p',p'\rangle\langle q,p\rangle\sqrt{\operatorname{ta}(p,p')}}$. We are looking for $q'\in S_{p'}$ such that $\dfrac{q'_{p'}}{\langle q',p'\rangle}=\dfrac{\langle p',p\rangle({}^pq)_{p'}}{\langle p',p'\rangle\langle q,p\rangle\sqrt{\operatorname{ta}(p,p')}}$. In $S_{p'}$, there exists a unique point q' with a given value of $\dfrac{{}^{p'}q'}{\langle q',p'\rangle}$ (the slice possesses a linear orthogonal basis formed by the focus of $B\wr p,p'\wr$ and p'). Explicitly,

[7] Here and in the sequel, we use the function arg taking values in $[-\pi,\pi]$.

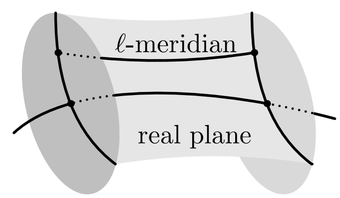
$q' = \langle p',p\rangle\, {}^p q + \langle q,p\rangle\sqrt{\mathrm{ta}(p,p')}\cdot p'$ ■



We call q' the *meridional displacement* of q. This meridional displacement identifies almost all slices of $B\wr p,p'\wr$, except those tangent to $\partial\,\mathrm{B}\,V$ if they exist. In this way, we obtain *meridional identification* of the slices that follow a given segment in the real spine. Thus, every point of the bisector (except of the vertices) generates a curve, called *ℓ-meridian,* over a segment of the real spine.[8] The real spine is one of the ℓ-meridians. By Lemma 3.6, every full ℓ-meridian passes through the vertices of the bisector (if they exist). Probably, an easier way to describe the meridional displacement is as follows. Let $g\in\mathrm{G}[p,p']$ be the polar point to S_p. To each tangent vector $x_g\in\mathrm{T}_g\,\mathbb{C}\mathbb{P}V$, we can associate the point $x\in S_p$. The parallel displacement of x_g along $\mathrm{G}[p,p']$ produces the ℓ-meridian of the associated points.

3.7. Corollary. *The ℓ-meridian $c(q,B)$ generated by $q\in B$ depends continuously on q and B* ■

3.8. Remark. Any real plane of a bisector can be obtained by the meridional displacement of a geodesic included into some slice and intersecting the real spine. Any ℓ-meridian is included into some real plane of the bisector.



Let $m\in\mathbb{C}\mathbb{P}V\setminus\partial\,\mathrm{B}\,V$. The reflection in $\mathbb{C}\mathbb{P}m^\perp$ is an isometry given by $R(m): x\mapsto 2\dfrac{\langle x,m\rangle}{\langle m,m\rangle}m-x$. As is easy to see, $R(m)\in\mathrm{SU}(2,1)$.

For given slices S_p and $S_{p'}$ of the bisector B, we call the slice $S_{m'}\eqqcolon\mathbb{C}\mathbb{P}m^\perp$ *middle* if $R(m)S_p=S_{p'}$.

3.9. Proposition. *For different slices S_p and $S_{p'}$ of the same signature, there are exactly two middle slices $S_m=\mathbb{C}\mathbb{P}m'^\perp$ and $S_{m'}=\mathbb{C}\mathbb{P}m^\perp$, where $\langle m,m'\rangle=0$. The meridional identification of S_p and $S_{p'}$ given by a segment $\mathrm{G}[p,p']$ of the real spine is induced by the reflection in S_m, where $m\in\mathrm{G}[p,p']$.*

Proof. We can assume that $\langle p,p'\rangle=\sigma r$ and $\langle p,p\rangle=\langle p',p'\rangle=\sigma$, where $\sigma=\pm1$ and $1\neq r>0$. Let $m=\alpha p+\alpha' p'$, where $\alpha,\alpha'\in\mathbb{R}$. The fact that $R(m)p=p'$ means that $2(\alpha+\alpha' r)(\alpha p+\alpha' p')=(\alpha^2+\alpha'^2+2\alpha\alpha' r)(p+xp')$ for some $x\in\mathbb{C}$. This implies $\alpha^2=\alpha'^2$. Hence, we obtain $m=p+p'$ and $m'=p-p'$.

Let q be the focus of the bisector $B\wr p,p'\wr$ and let $x\neq q$ be a point in the slice S_p passing through p. For some $c\in\mathbb{C}$, we can write $x=cq+p$. So, $\langle x,p\rangle=\sigma$ and $\langle x,p'\rangle=\sigma r$. By Lemma 3.6, $x'=\sigma r\,{}^p x+\sigma r p'=\sigma r\big((x-p)+p'\big)=\sigma r(cq+p')$. It is easy to see that $p-p'$ is the polar point to the middle slice $S_{p+p'}$ and that $p+p'\in\mathrm{G}[p,p']$. Now, $2\dfrac{\langle x,p-p'\rangle}{\langle p-p',p-p'\rangle}(p-p')-x=-(p'+cq)$ ■

3.10. Corollary. *The meridional identification is an isometry between slices of the same signature* ■

4. Angle between Bisectors with Common Slice. Transversality

4.1. Lemma. *Let $p\notin\overline{\mathrm{B}}V$ and $p_1,p_2\in\mathbb{C}\mathbb{P}V$ be such that $\mathrm{ta}(p_1,p),\mathrm{ta}(p_2,p)>1$ (in particular, $p_1,p_2\notin\overline{\mathrm{B}}V$ and $p_1,p_2\neq p$). Then $v_i\eqqcolon\Big(\mathrm{ta}(p_i,p)+\sqrt{\mathrm{ta}(p_i,p)\big(\mathrm{ta}(p_i,p)-1\big)}\Big)p_i-\dfrac{\langle p_i,p\rangle}{\langle p,p\rangle}p$ is the vertex of $B\wr p,p_i\wr$ that is closer to the slice $\mathbb{C}\mathbb{P}p_i^\perp$ than to the slice $\mathbb{C}\mathbb{P}p^\perp$ and*

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[8]We believe that our ℓ-meridians will not lead to any confusion, since we will not use the term 'meridian' in a generally accepted sense.

$$1-\frac{\langle v_1,v_2\rangle\langle p,p\rangle}{\langle v_1,p\rangle\langle p,v_2\rangle}=\frac{1}{\sqrt{1-\frac{1}{\operatorname{ta}(p_1,p)}}\sqrt{1-\frac{1}{\operatorname{ta}(p_2,p)}}}\Big(1-\frac{\langle p_1,p_2\rangle\langle p,p\rangle}{\langle p_1,p\rangle\langle p,p_2\rangle}\Big).$$

For any $q\in\mathbb{CP}p^\perp\setminus\partial\,\mathrm{B}\,V$,

$$\frac{\langle v_1,q\rangle\langle q,v_2\rangle\langle p,p\rangle}{\langle v_1,p\rangle\langle p,v_2\rangle\langle q,q\rangle}=\frac{1}{\sqrt{1-\frac{1}{\operatorname{ta}(p_1,p)}}\sqrt{1-\frac{1}{\operatorname{ta}(p_2,p)}}}\frac{\langle p_1,q\rangle\langle q,p_2\rangle\langle p,p\rangle}{\langle p_1,p\rangle\langle p,p_2\rangle\langle q,q\rangle}.$$

Proof is a direct verification. The fact that the projective lines $\mathbb{CP}p^\perp$ and $\mathbb{CP}p_i^\perp$ are ultraparallel and are slices of the bisector $B\wr p,p_i\wr$ follows from Remark 2.3.3.

We can assume that $\langle p_i,p_i\rangle=\langle p,p\rangle=1$ and denote $t_i\rightleftharpoons\sqrt{\operatorname{ta}(p_i,p)}$. Now, $v_i=t_i\big(t_i+\sqrt{t_i^2-1}\big)p_i-\langle p_i,p\rangle p$ and $t_i^2=\big\langle p_i,\langle p_i,p\rangle p\big\rangle$. Hence, $\langle v_i,v_i\rangle=t_i^2\big(t_i+\sqrt{t_i^2-1}\big)^2-2t_i\big(t_i+\sqrt{t_i^2-1}\big)t_i^2+t_i^2=0$. Since $\big\langle v_i,\langle p_i,p\rangle p\big\rangle=t_i\big(t_i+\sqrt{t_i^2-1}\big)t_i^2-t_i^2=t_i^2\big(t_i^2-1+t_i\sqrt{t_i^2-1}\big)>0$, therefore, for $\alpha\in[0,1]$, the point $p(\alpha)\rightleftharpoons(1-\alpha)v_i+\alpha\langle p_i,p\rangle p$ is the polar point to some slice of $B\wr p,p_i\wr\cap\mathrm{B}\,V$, being $p(0)=v_i$, $p(\frac12)\simeq p_i$, and $p(1)\simeq p$ (where $\simeq$ means $\mathbb{C}$-proportionality). This implies that v_i is the vertex of $B\wr p,p_i\wr$ that is closer to the slice $\mathbb{CP}p_i^\perp$ than to the slice $\mathbb{CP}p^\perp$. We have

$$1-\frac{\langle v_1,v_2\rangle\langle p,p\rangle}{\langle v_1,p\rangle\langle p,v_2\rangle}=$$

$$=1-\frac{\big(t_1^2+t_1\sqrt{t_1^2-1}\big)\big(t_2^2+t_2\sqrt{t_2^2-1}\big)\langle p_1,p_2\rangle+\big(1-t_1^2-t_1\sqrt{t_1^2-1}-t_2^2-t_2\sqrt{t_2^2-1}\big)\langle p_1,p\rangle\langle p,p_2\rangle}{\big(t_1^2+t_1\sqrt{t_1^2-1}-1\big)\big(t_2^2+t_2\sqrt{t_2^2-1}-1\big)\langle p_1,p\rangle\langle p,p_2\rangle}=$$

$$1-\frac{\big(t_1^2+t_1\sqrt{t_1^2-1}\big)\big(t_2^2+t_2\sqrt{t_2^2-1}\big)\Big(\frac{\langle p_1,p_2\rangle}{\langle p_1,p\rangle\langle p,p_2\rangle}-1\Big)+\big(t_1^2+t_1\sqrt{t_1^2-1}-1\big)\big(t_2^2+t_2\sqrt{t_2^2-1}-1\big)}{\big(t_1^2+t_1\sqrt{t_1^2-1}-1\big)\big(t_2^2+t_2\sqrt{t_2^2-1}-1\big)}$$

$$=\frac{1}{\sqrt{1-\frac{1}{t_1^2}}\sqrt{1-\frac{1}{t_1^2}}}\Big(1-\frac{\langle p_1,p_2\rangle\langle p,p\rangle}{\langle p_1,p\rangle\langle p,p_2\rangle}\Big),$$

since $\dfrac{1}{t_i+\sqrt{t_i^2-1}}=t_i-\sqrt{t_i^2-1}$ and $\dfrac{t_i^2+t_i\sqrt{t_i^2-1}}{t_i^2+t_i\sqrt{t_i^2-1}-1}=\dfrac{1}{1-\frac{t_i-\sqrt{t_i^2-1}}{t_i}}=\dfrac{1}{\sqrt{1-\frac{1}{t_i^2}}}$. For the same reasons,

$$\frac{\langle v_1,q\rangle\langle q,v_2\rangle\langle p,p\rangle}{\langle v_1,p\rangle\langle p,v_2\rangle\langle q,q\rangle}=\frac{\big(t_i^2+t_i\sqrt{t_i^2-1}\big)\big(t_i^2+t_i\sqrt{t_i^2-1}\big)\langle p_1,q\rangle\langle q,p_2\rangle\langle p,p\rangle}{\big(t_i^2-1+t_i\sqrt{t_i^2-1}\big)\big(t_i^2-1+t_i\sqrt{t_i^2-1}\big)\langle p_1,p\rangle\langle p,p_2\rangle\langle q,q\rangle}=$$

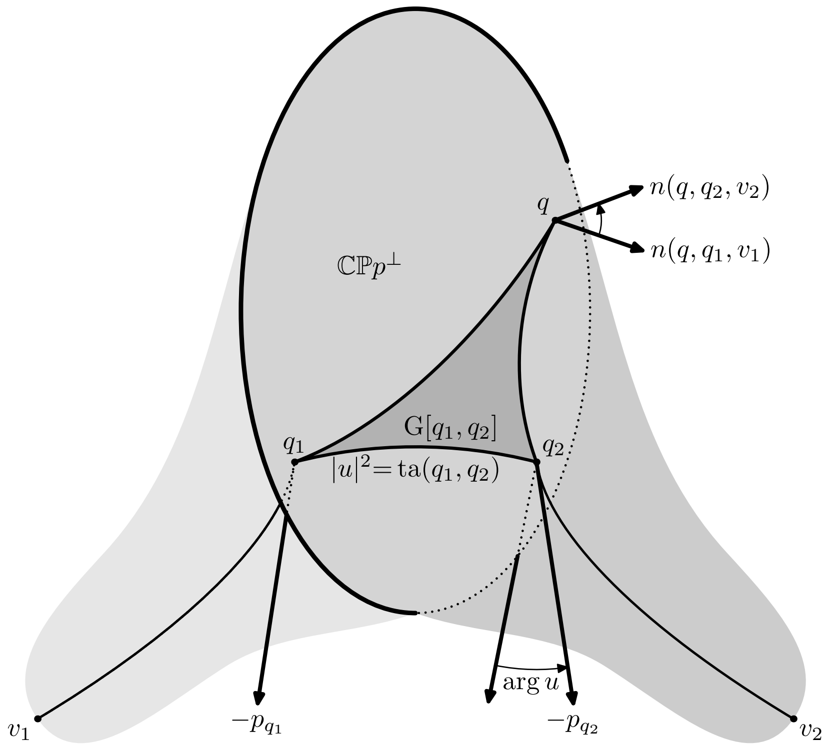
$$=\frac{1}{\sqrt{1-\frac{1}{t_1^2}}\sqrt{1-\frac{1}{t_2^2}}}\frac{\langle p_1,q\rangle\langle q,p_2\rangle\langle p,p\rangle}{\langle p_1,p\rangle\langle p,p_2\rangle\langle q,q\rangle}\ \blacksquare$$

4.2. Theorem. *Let* $p\notin\overline{\mathrm{B}}V$ *and let* $v_1,v_2\in\partial\,\mathrm{B}\,V\setminus\mathbb{CP}p^\perp$. *We put* $u\rightleftharpoons1-\dfrac{\langle v_1,v_2\rangle\langle p,p\rangle}{\langle v_1,p\rangle\langle p,v_2\rangle}$. *The bisectors* $B\wr p,v_1\wr$ *and* $B\wr p,v_2\wr$ *possess a common slice* $S\rightleftharpoons\mathbb{CP}p^\perp$. *We denote by* $q_i\in S\cap\mathrm{B}\,V$ *the point in the real spine of* $B\wr p,v_i\wr$. *Let* $q\in S\cap\mathrm{B}\,V$. *Then the angle* $\angle\big(q,B[q_1,v_1),B[q_2,v_2)\big)$ *from* $B[q_1,v_1)$ *to* $B[q_2,v_2)$ *at* q *can be calculated as follows:*

$$\angle\big(q,B[q_1,v_1),B[q_2,v_2)\big)=\operatorname{Arg}\frac{\langle v_1,q\rangle\langle q,v_2\rangle}{\langle v_1,p\rangle\langle p,v_2\rangle}=\operatorname{Arg}\big(-u\langle q,q_2\rangle\langle q_2,q_1\rangle\langle q_1,q\rangle\big)\equiv$$
$$\equiv\operatorname{Arg}u-2\operatorname{Area}\Delta(q,q_1,q_2)\mod 2\pi.$$

The number u *completely characterizes the geometrical configuration of* $B[q_1,v_1)$ *and* $B[q_2,v_2)$: $\operatorname{Arg}u$ *is the angle from* $\mathrm{G}[q_1,v_1)$ *to* $\mathrm{G}[q_2,v_2)$ *measured with the parallel displacement along* $\mathrm{G}[q_1,q_2]$

and $|u|^2=\mathrm{ta}(q_1,q_2)$ *is the tance between the complex spines of the bisectors. Any* $u\in\mathbb C$ *with* $|u|\geq 1$ *is possible.*



Proof. Clearly, we can take $q_i \rightleftharpoons {}^p v_i$. We can assume that $\langle p,p\rangle=\langle v_i,p\rangle=1$. Hence, $q_i=v_i-p$, $\langle v_i,q_i\rangle=-1$, $\langle q,q_i\rangle=\langle q,v_i\rangle$, $\langle q_i,q_i\rangle=-1$, ${}^{q_i}v_i=p$, $\langle q_1,q_2\rangle=\langle v_1,v_2\rangle-1=-u$, and $\mathrm{ta}(q_1,q_2)=|u|^2$. By Remark 2.3.3, $\mathrm{ta}(q_1,q_2)$ is the tance between the complex spines of the bisectors. By Lemma 2.3.1, $n(q,q_i,v_i)=\langle -,q\rangle\otimes i\cdot\Big(\dfrac{\langle q,v_i\rangle}{\langle q_i,v_i\rangle}q_i-\dfrac{\langle q,q_i\rangle}{\langle v_i,q_i\rangle}v_i\Big)=\langle -,q\rangle\otimes i\cdot\langle q,v_i\rangle p$ is a normal vector to $B[q_i,v_i)$ at q. Both such vectors are orthogonal to S and, therefore, are tangent to the projective line orthogonal to S at q. Thus, $\angle\big(q,B[q_1,v_1),B[q_2,v_2)\big)=\mathrm{Arg}\,\big\langle n(q,q_2,v_2),n(q,q_1,v_1)\big\rangle=\mathrm{Arg}\,\big(-\langle q,q\rangle\langle q,v_2\rangle\langle v_1,q\rangle\big)=\mathrm{Arg}\,\Big(\dfrac{\langle v_1,q\rangle\langle q,v_2\rangle}{\langle v_1,p\rangle\langle p,v_2\rangle}\Big)$. Since $-u\langle q,q_2\rangle\langle q_2,q_1\rangle\langle q_1,q\rangle=|u|^2\langle v_1,q\rangle\langle q,v_2\rangle$, it follows that $\mathrm{Arg}\,\Big(\dfrac{\langle v_1,q\rangle\langle q,v_2\rangle}{\langle v_1,p\rangle\langle p,v_2\rangle}\Big)=\mathrm{Arg}\,\big(-u\langle q,q_2\rangle\langle q_2,q_1\rangle\langle q_1,q\rangle\big)\equiv\mathrm{Arg}\,u-2\,\mathrm{Area}\,\Delta(q,q_1,q_2)\mod 2\pi$ by Remark 3.5.

By Corollary 3.4, the parallel displacement along $\mathrm{G}[q_1,q_2]$ of the vector $\dfrac{v_{1q_1}}{\langle v_1,q_1\rangle}=-v_{1q_1}=-p_{q_1}$ tangent to $\mathrm{G}[q_1,v_1)$ (by Lemma 3.1) is equal to $-\dfrac{\langle q_2,q_1\rangle p_{q_2}}{\langle q_2,q_2\rangle\sqrt{\mathrm{ta}(q_1,q_2)}}=-\dfrac{\overline u}{|u|}\,p_{q_2}$, whereas $-p_{q_2}$ is a tangent vector to $\mathrm{G}[q_2,v_2)$ by Lemma 3.1.

The determinant of the Gram matrix $\begin{pmatrix}1&1&1\\1&0&1-u\\1&1-\overline u&0\end{pmatrix}$ for p, v_1, and v_2 is equal to $1-|u|^2$, implying the rest ■

Since $\mathrm{Arg}\,u$ is independent of the choice of $q\in S\cap\mathrm{B}\,V$, we call it the *constant angle* between the bisectors: in fact, it is the angle between the bisectors at the points of $\mathrm{G}\wr q_1,q_2\wr\cap\mathrm{B}\,V$. The angle $\arg\big(-\langle q,q_2\rangle\langle q_2,q_1\rangle\langle q_1,q\rangle\big)=2\,\mathrm{Area}\,\Delta(q_1,q,q_2)$ depends only on the position of q in $S\cap\mathrm{B}\,V$. We call it the *nonconstant angle* between the bisectors.

4.3. Corollary (compare with [Gol, Corollary 9.1.3, p. 273] and with [Hsi])**.** *Let* $p\notin\overline{\mathrm{B}}V$ *and* $p_1,p_2\in\mathbb{CP}V$ *be such that* $\operatorname{ta}(p,p_1),\operatorname{ta}(p,p_2)>1$. *The bisectors* $B\wr p,p_1\wr$ *and* $B\wr p,p_2\wr$ *are transversal along their common slice* $\mathbb{CP}p^\perp\cap\overline{\mathrm{B}}V$ *if and only if*

$$\Big|\operatorname{Re}\frac{\langle p_1,p_2\rangle\langle p,p\rangle}{\langle p_1,p\rangle\langle p,p_2\rangle}-1\Big|<\sqrt{1-\frac{1}{\operatorname{ta}(p_1,p)}}\sqrt{1-\frac{1}{\operatorname{ta}(p_2,p)}}.$$

In this case, $B\wr p,p_1\wr\cap B\wr p,p_2\wr\cap\overline{\mathrm{B}}V=\mathbb{CP}p^\perp\cap\overline{\mathrm{B}}V$.

Proof. Due to Lemma 4.1, in terms of Theorem 4.2, the inequality means that $|\operatorname{Re}u|<1$. As in the proof of Theorem 4.2, we can assume that $\langle p,p\rangle=\langle v_i,p\rangle=1$, $q_i=v_i-p$, $\langle q_i,q_i\rangle=-1$, $\langle q_1,q_2\rangle=-u$, and $\langle v_1,v_2\rangle=1-u$.

Let us verify the fact that, for $q\in\mathbb{CP}p^\perp\cap\overline{\mathrm{B}}V$, the nonconstant angle $\arg\big(-\langle q,q_2\rangle\langle q_2,q_1\rangle\langle q_1,q\rangle\big)$ varies in $[-\alpha,\alpha]$, where $\cos\alpha=\dfrac{1}{|u|}$, $\alpha\in[0,\frac{\pi}{2})$. Indeed, assuming $q_1\neq q_2$, each point in $\mathbb{CP}p^\perp$, except of q_2 and q_1-uq_2 (the point orthogonal to q_2), can be written as $q(z)\coloneqq q_1+\Big(\dfrac{1}{\overline{z}}-u\Big)q_2$, $0\neq z\in\mathbb{C}$, being $q(z)\in\overline{\mathrm{B}}V$ if $|z|\leq\dfrac{1}{\sqrt{|u|^2-1}}$. For such $q(z)$, we obtain $\arg\Big(-\big\langle q(z),q_2\big\rangle\langle q_2,q_1\rangle\big\langle q_1,q(z)\big\rangle\Big)=\arg\dfrac{\overline{u}u}{\overline{z}z}\Big(1-\dfrac{|u|^2-1}{u}z\Big)$, implying the fact.

The bisectors $B\wr p,p_1\wr$ and $B\wr p,p_2\wr$ are transversal along $\mathbb{CP}p^\perp\cap\mathrm{B}\,V$ exactly when the angle between $B[q_1,v_1)$ and $B[q_2,v_2)$ along $\mathbb{CP}p^\perp\cap\mathrm{B}\,V$ is never 0 nor π, that is, if $\operatorname{Arg}u\in(\alpha,\pi-\alpha)\cup(\pi+\alpha,2\pi-\alpha)$. In other words, for the transversality, we require $|\cos\operatorname{Arg}u|<\cos\alpha$, i.e., $|\operatorname{Re}u|<1$. This inequality implies the transversality of the bisectors along $\mathbb{CP}p^\perp\cap\partial\mathrm{B}\,V$: let $v\in\mathbb{CP}p^\perp\cap\partial\mathrm{B}\,V$, let $\varphi\in\operatorname{Lin}_{\mathbb{C}}(\mathbb{C}v,V/\mathbb{C}v)\simeq\mathrm{T}_v\,\mathbb{CP}V$ be a tangent vector to both bisectors, and let $\hat\varphi\in\operatorname{Lin}_{\mathbb{C}}(\mathbb{C}v,V)$ be any lifting of φ. The fact that φ is tangent to $B\wr p,p_i\wr=B\wr p,v_i\wr$ means that $\dfrac{\big\langle p,\hat\varphi(v)\big\rangle\langle v,v_i\rangle+\langle p,v\rangle\big\langle\hat\varphi(v),v_i\big\rangle}{\langle p,v_i\rangle}-\dfrac{\big\langle v_i,\hat\varphi(v)\big\rangle\langle v,p\rangle+\langle v_i,v\rangle\big\langle\hat\varphi(v),p\big\rangle}{\langle v_i,p\rangle}=0$, that is, $\big\langle p,\hat\varphi(v)\big\rangle\langle v,v_i\rangle\in\mathbb{R}$. This is equivalent to $\big\langle p,\hat\varphi(v)\big\rangle\langle v,q_i\rangle\in\mathbb{R}$. If $\big\langle p,\hat\varphi(v)\big\rangle\neq0$, then $\langle v,q_1\rangle$ and $\langle v,q_2\rangle$ are $\mathbb{R}$-linearly dependent. Since v is a unique point in $\mathbb{CP}p^\perp$ orthogonal to v, we can take $v=q_1+\alpha q_2$, $\alpha\in\mathbb{R}$. Now, $\langle v,v\rangle=0$ implies $1+\alpha^2+2\alpha\operatorname{Re}u=0$. It follows that $1+(\alpha+\operatorname{Re}u)^2=(\operatorname{Re}u)^2$. A contradiction with $|\operatorname{Re}u|<1$. Thus, $\big\langle p,\hat\varphi(v)\big\rangle=0$, i.e., φ is tangent to $\mathbb{CP}p^\perp$.

Interchanging, if needed, the vertices of one of the bisectors, we can assume that $0\leq\operatorname{Re}u<1$. Every point in $\mathrm{G}\wr v_1,p\wr$ but v_1 has the form $g_1(\alpha_1)\coloneqq\dfrac{\alpha_1-1}{2}v_1+p$, $\alpha_1\in\mathbb{R}$, and $\big\langle g_1(\alpha_1),g_1(\alpha_1)\big\rangle=\Big\langle\dfrac{\alpha_1-1}{2}v_1+p,\dfrac{\alpha_1-1}{2}v_1+p\Big\rangle=\alpha_1$. Therefore, $g_1(\alpha_1)$, $\alpha_1>0$, runs over all the polar points to the slices of $B\wr p,v_1\wr\cap\mathrm{B}\,V$. The points $g_2(\alpha_2)\coloneqq\dfrac{\alpha_2-1}{2}v_2+p$, $\alpha_2>0$, play the same role for $B\wr p,v_2\wr\cap\mathrm{B}\,V$. Now, $\big\langle g_1(\alpha_1),g_2(\alpha_2)\big\rangle=\frac14\big((\alpha_1-1)(\alpha_2-1)(1-u)+2(\alpha_1+\alpha_2)\big)$ and $\operatorname{ta}\big(g_1(\alpha_1),g_2(\alpha_2)\big)=\dfrac{|u|^2(\alpha_1-1)^2(\alpha_2-1)^2+(\alpha_1+1)^2(\alpha_2+1)^2-2\operatorname{Re}u(\alpha_1^2-1)(\alpha_2^2-1)}{16\alpha_1\alpha_2}$. By Remark 2.3.3, we need to verify the condition $\operatorname{ta}\big(g_1(\alpha_1),g_2(\alpha_2)\big)>1$, which is equivalent to the inequality

$$\big(|u|^2-1\big)(\alpha_1-1)^2(\alpha_2-1)^2+4(\alpha_1-\alpha_2)^2+2(1-\operatorname{Re}u)(\alpha_1^2-1)(\alpha_2^2-1)>0.$$

Since $|u|>1$ and $\operatorname{Re}u\in[0,1)$, the inequality is valid if $\alpha_1,\alpha_2>1$ or if $0<\alpha_1,\alpha_2<1$. Therefore, excluding the case of $\alpha_1=\alpha_2=1$ (the case of the slice $\mathbb{CP}p^\perp$ for both bisectors), we can assume that $0<\alpha_1<1<\alpha_2$. In this case, we can take the minimum value of $\operatorname{Re}u$, verifying that $\big(|u|^2-1\big)(1-$

$\alpha_1)^2(\alpha_2-1)^2+4(\alpha_2-\alpha_1)^2-2(1-\alpha_1^2)(\alpha_2^2-1)=\big(|u|^2-1\big)(1-\alpha_1)^2(\alpha_2-1)^2+2(\alpha_2-\alpha_1)^2+2(\alpha_1\alpha_2-1)^2>0$, which follows from $|u|>1$. The bisectors cannot have a common vertex v: otherwise, they would coincide, having the same real spine $\mathrm{G}\wr v,p\wr$ ■

5. More about Bisectors

5.1. Lemma (compare with [Gol, Theorem 5.5.1, p.193])**.** *Let B be a bisector and let $v_1,v_2\in\partial\,\mathrm{B}\,V$, $v_1\ne v_2$, $\mathrm{G}(v_1,v_2)\not\subset B$. Then $\mathrm{G}(v_1,v_2)$ intersects $B\cap\mathrm{B}\,V$ at most twice. If $\mathrm{G}(v_1,v_2)$ is not transversal to $B\cap\mathrm{B}\,V$, they intersect only once and $\mathrm{G}(v_1,v_2)$ is all on the same side from $B\cap\mathrm{B}\,V$.*

Proof. We can assume that $\langle v_1,v_2\rangle=-\frac12$ so that every point in $\mathrm{G}(v_1,v_2)$ has the form $g(\alpha)\risingdotseq\alpha^{-1}v_1+\alpha v_2$, $\alpha>0$, with $\big\langle g(\alpha),g(\alpha)\big\rangle=-1$. By Lemma 3.1, $\dfrac{({}^{g(\alpha)}v_2)_{g(\alpha)}}{\big\langle v_2^{g(\alpha)},g(\alpha)\big\rangle}=(\alpha^{-1}v_1-\alpha v_2)_{g(\alpha)}$ is a tangent vector to $\mathrm{G}(v_1,v_2)$ at $g(\alpha)$, where $\alpha^{-1}v_1-\alpha v_2\in g(\alpha)^\perp$. Let $b(x,p_1,p_2)=0$ be an equation for B. Obviously, $t(x,x,p_1,p_2)=2b(x,p_1,p_2)$, therefore, $g(\alpha)\in B$ is equivalent to $t_{11}+2t_{12}\alpha^2+t_{22}\alpha^4=0$, where t_{ij} stands for $t(v_i,v_j,p_1,p_2)$. If $g(\alpha)\in B$, the nontransversality of this intersection means that $t\big(\alpha^{-1}v_1-\alpha v_2,g(\alpha),p_1,p_2\big)=0$, i.e., $t_{11}-t_{22}\alpha^4=0$. The condition $\mathrm{G}(v_1,v_2)\not\subset B$ implies that one of the t_{ij}'s does not vanish. Now, from $t_{11}+2t_{12}\alpha^2+t_{22}\alpha^4=0$ and $t_{11}-t_{22}\alpha^4=0$, it follows that $t_{11}t_{22}=t_{12}^2$. Thus, $t_{11}+2t_{12}\beta+t_{22}\beta^2=0$ has a unique solution in β. Furthermore, for $p\in\overline{\mathrm{B}}V$, the inequality $\operatorname{Im}b(p,p_1,p_2)>0$ determines if p is on the side of the normal vector to B (see Lemma 2.3.1). Since $t_{11},t_{12},t_{22}\in i\mathbb{R}$, from $t_{11}t_{22}=t_{12}^2$, we conclude that $\operatorname{Im}b(v_1,p_1,p_2)\cdot\operatorname{Im}b(v_2,p_1,p_2)>0$ ■

5.2. Lemma. *Let $p\notin\overline{\mathrm{B}}V$ and $q\in\mathrm{B}\,V$. Then $\mathrm{ta}(\mathbb{CP}p^\perp\cap\mathrm{B}\,V,q)=1-\mathrm{ta}(p,q)$. Let B be a bisector with positive focus, let $\hat S$ be some slice of $B\cap\mathrm{B}\,V$, and let $q\in\hat S\cap\mathrm{B}\,V$. Then the function $\mathrm{ta}(S\cap\mathrm{B}\,V,q)$ is increasing while S runs over the slices of $B\cap\mathrm{B}\,V$ on the same side from $\hat S$.*

Proof is straightforward. We can assume that $\langle p,p\rangle=1$, $\langle q,q\rangle=-1$, and $\langle q,p\rangle=a\in\mathbb{R}$. By Remark 2.3.3, $\mathrm{ta}(\mathbb{CP}p^\perp,q)=\mathrm{ta}({}^pq,q)=\mathrm{ta}(q-ap,q)=1+a^2=1-\mathrm{ta}(p,q)$.

Let $v_1,v_2\in\partial\,\mathrm{B}\,V$ be the vertices of B, $B=B(v_1,v_2)$. We assume that $\langle q,q\rangle=-1$, that $\langle v_1,v_2\rangle=\frac12$, and that v_1+v_2 is the polar point to $\hat S$, hence, $z=\langle v_1,q\rangle=-\langle v_2,q\rangle\ne0$. The polar point to any slice of $B\cap\mathrm{B}\,V$ has the form $p(\alpha)\risingdotseq\alpha^{-1}v_1+\alpha v_2$, $\alpha>0$, with $\big\langle p(\alpha),p(\alpha)\big\rangle=1$. By the above assertion, $\mathrm{ta}\big(\mathbb{CP}p(\alpha)^\perp\cap\mathrm{B}\,V,q\big)=1-\mathrm{ta}\big(p(\alpha),q\big)=1+(\alpha^{-2}+\alpha^2-2)\,|z|^2$ ■

For $v_1,v_2\in\partial\,\mathrm{B}\,V$ and $q\notin\partial\,\mathrm{B}\,V$ we denote $\eta(v_1,v_2,q)\risingdotseq\dfrac{\langle v_1,q\rangle\langle q,v_2\rangle}{\langle v_1,v_2\rangle\langle q,q\rangle}$ (see [Gol]).

5.3. Lemma (compare with [San])**.** *Let B be a bisector with positive focus and let $q\in\mathrm{B}\,V\setminus B$. Then there exists a unique slice $\hat S$ of $B\cap\mathrm{B}\,V$ such that $\mathrm{ta}(\hat S\cap\mathrm{B}\,V,q)=\mathrm{ta}(B\cap\mathrm{B}\,V,q)$. The function $\mathrm{ta}(S\cap\mathrm{B}\,V,q)$ is increasing while S runs over the slices of $B\cap\mathrm{B}\,V$ on the same side from $\hat S\cap\mathrm{B}\,V$. The shortest geodesic from q to $B\cap\mathrm{B}\,V$ is transversal to $B\cap\mathrm{B}\,V$. Let $v_1,v_2\in\partial\,\mathrm{B}\,V$ be the vertices of B. Then $\mathrm{ta}(B\cap\mathrm{B}\,V,q)=1-\operatorname{Re}\eta(v_1,v_2,q)+\big|\eta(v_1,v_2,q)\big|$.*

Proof is routine. We assume that $\langle q,q\rangle=-1$ and $\langle v_1,v_2\rangle=\frac12$. The polar point to any slice of $B\cap\mathrm{B}\,V$ has the form $p(\alpha)\risingdotseq\alpha^{-1}v_1+\alpha v_2$, $\alpha>0$, with $\big\langle p(\alpha),p(\alpha)\big\rangle=1$. We put $z_i\risingdotseq\langle q,v_i\rangle$. By Lemma 5.2, $\mathrm{ta}\big(\mathbb{CP}p(\alpha)^\perp,q\big)=1-\mathrm{ta}\big(p(\alpha),q\big)=1+2\operatorname{Re}(z_1\overline{z}_2)+\alpha^{-2}|z_1|^2+\alpha^2|z_2|^2$. For $\alpha>0$, this function has a unique minimum, exactly when $\alpha\risingdotseq\sqrt{\dfrac{|z_1|}{|z_2|}}$. Hence, $\mathrm{ta}(B\cap\mathrm{B}\,V,q)=1+2\operatorname{Re}(z_1\overline{z}_2)+2\,|z_1|\cdot|z_2|=1+2\operatorname{Re}\big(\langle v_1,q\rangle\langle q,v_2\rangle\big)+2\big|\,\langle v_1,q\rangle\langle q,v_2\rangle\big|=1-\operatorname{Re}\dfrac{\langle v_1,q\rangle\langle q,v_2\rangle}{\langle v_1,v_2\rangle\langle q,q\rangle}+\Big|\dfrac{\langle v_1,q\rangle\langle q,v_2\rangle}{\langle v_1,v_2\rangle\langle q,q\rangle}\Big|=1-\operatorname{Re}\eta(v_1,v_2,q)+\big|\eta(v_1,v_2,q)\big|$.

With this α, let $p \risingdotseq \alpha^{-1}v_1+\alpha v_2$. We can choose new representatives for v_1 and v_2 so that $p=v_1+v_2$ and $\langle v_1,v_2\rangle=\frac{1}{2}$. Now, for $z_i \risingdotseq \langle q,v_i\rangle$, we obtain $|z_1|=|z_2|$.

We have ${}^pq=q-(z_1+z_2)(v_1+v_2)$. By Lemma 3.1, $\dfrac{({}^{({}^pq)}q)_{({}^pq)}}{\langle q^{({}^pq)},{}^pq\rangle}=\Big(\dfrac{q}{\langle q,{}^pq\rangle}-\dfrac{{}^pq}{\langle {}^pq,{}^pq\rangle}\Big)_{({}^pq)}=\Big(\dfrac{\langle q,p\rangle p}{\langle p,p\rangle\langle q,q\rangle-\langle p,q\rangle\langle q,p\rangle}\Big)_{({}^pq)}=\Big(\dfrac{(z_1+z_2)(v_1+v_2)}{1+|z_1+z_2|^2}\Big)_{q-(z_1+z_2)(v_1+v_2)}$ is a tangent vector to $\mathrm{G}[{}^pq,q]$ at ${}^pq$. When verifying the transversality, we can take $\big((z_1+z_2)(v_1+v_2)\big)_{q-(z_1+z_2)(v_1+v_2)}$ instead. Now, $t\big((z_1+z_2)(v_1+v_2),\,q-(z_1+z_2)(v_1+v_2),\,v_1,v_2\big)=i\cdot\mathrm{Im}\,\big((\overline{z}_1+\overline{z}_2)(z_2-z_1)+(\overline{z}_1-\overline{z}_2)(z_1+z_2)\big)=4i\cdot\mathrm{Im}(\overline{z}_1z_2)=b(q,v_1,v_2)$ ■

5.4. Lemma. *Let B_1 and B_2 be bisectors with positive foci. Suppose that they possess a common slice S of signature $+-$ and that they are transversal along $S\cap\mathrm{B}\,V$. Then, for any $\varepsilon>0$, there exists some $\delta>0$ such that, for every $q\in B_2\cap\mathrm{B}\,V$, the inequality $\mathrm{ta}(B_1\cap\mathrm{B}\,V,q)<1+\delta^2$ implies the inequality $\mathrm{ta}(S\cap\mathrm{B}\,V,q)<1+\varepsilon^2$.*

Proof. Let p be the polar point to S and let $v_1,v_1'\in\partial\,\mathrm{B}\,V$ and $v_2,v_2'\in\partial\,\mathrm{B}\,V$ be the vertices of B_1 and B_2, respectively. As in the proof of Theorem 4.2, we assume that $\langle p,p\rangle=\langle v_i,p\rangle=1$ and that $\langle v_1,v_2\rangle=1-u$, where u stands for the invariant of B_1 and B_2 (see Theorem 4.2). By Theorem 4.2 and by Corollary 4.3, we know that $|u|>1$ and $|\,\mathrm{Re}\,u|<1$. Writing $u=u_0+iu_1$, $u_0,u_1\in\mathbb{R}$, and denoting $k\risingdotseq\sqrt{|u|^2-1}$, we obtain $|u_1|>k>0$. As is easy to see, $v_i'\risingdotseq 2p-v_i$ serves as the other vertex of B_i, $\langle v_i',p\rangle=1$, and $\langle v_i,v_i'\rangle=2$. Also, $\langle v_1,v_2'\rangle=\langle v_1',v_2\rangle=1+u$ and $\langle v_1',v_2'\rangle=1-u$. Furthermore, $d\risingdotseq v_1-uv_2+(u-1)p$ is the focus of B_2, $\langle d,v_2\rangle=\langle d,p\rangle=0$, $\langle d,d\rangle=\langle v_1,d\rangle=|u|^2-1=k^2$, and $\langle d,v_1'\rangle=1-|u|^2=-k^2$. Every point in $\mathrm{G}(v_2,v_2')$, the real spine of B_2, has the form $g(\alpha)\risingdotseq\dfrac{\alpha^{-1}v_2-\alpha v_2'}{2}$, $\big\langle g(\alpha),g(\alpha)\big\rangle=-1$, where $\alpha>0$. It follows that $\big\langle g(\alpha),p\big\rangle=\dfrac{\alpha^{-1}-\alpha}{2}$, $\big\langle v_1,g(\alpha)\big\rangle=\dfrac{\alpha^{-1}-\alpha}{2}-\dfrac{\alpha^{-1}+\alpha}{2}u$, and $\big\langle g(\alpha),v_1'\big\rangle=\dfrac{\alpha^{-1}-\alpha}{2}+\dfrac{\alpha^{-1}+\alpha}{2}\overline{u}$. Every point but d in the slice $S_{g(\alpha)}$ of B_2 passing through $g(\alpha)$ has the form

$$q(\alpha,z)\risingdotseq g(\alpha)+\frac{z}{k}\,d,\qquad z\in\mathbb{C},$$

$\big\langle q(\alpha,z),q(\alpha,z)\big\rangle=-\big(1-|z|^2\big)$, being $q(\alpha,z)\in\mathrm{B}\,V$ exactly if $|z|<1$. We derive that $\big\langle v_1,q(\alpha,z)\big\rangle=\dfrac{\alpha^{-1}-\alpha}{2}-\dfrac{\alpha^{-1}+\alpha}{2}u+k\overline{z}$ and $\big\langle q(\alpha,z),v_1'\big\rangle=\dfrac{\alpha^{-1}-\alpha}{2}+\dfrac{\alpha^{-1}+\alpha}{2}\overline{u}-kz$, which straightforwardly implies that $\dfrac{\big\langle v_1,q(\alpha,z)\big\rangle\big\langle q(\alpha,z),v_1'\big\rangle}{\langle v_1,v_1'\rangle\big\langle q(\alpha,z),q(\alpha,z)\big\rangle}=a+ib$, where

$$a\risingdotseq\frac{1+\frac{(\alpha^{-1}+\alpha)^2}{4}k^2+k^2|z|^2-(\alpha^{-1}+\alpha)k(u_0z_0-u_1z_1)}{2\big(1-|z|^2\big)},\qquad b\risingdotseq\frac{\alpha^{-1}-\alpha}{2}\cdot\frac{\frac{\alpha^{-1}+\alpha}{2}u_1+kz_1}{1-|z|^2},$$

and $z=z_0+iz_1$, $z_0,z_1\in\mathbb{R}$. By Lemma 5.3, $\mathrm{ta}\,\big(B_1\cap\mathrm{B}\,V,q(\alpha,z)\big)=1-a+|a+ib|$. Thus, for any $\delta>0$, the inequality $\mathrm{ta}\,\big(B_1\cap\mathrm{B}\,V,q(\alpha,z)\big)<1+\delta^2$ is equivalent to the inequality $b^2<\delta^4+2a\delta^2$.

On the other hand, by Lemma 5.2, $\mathrm{ta}\,\big(S\cap\mathrm{B}\,V,q(\alpha,z)\big)=1-\mathrm{ta}\,\big(p,q(\alpha,z)\big)=1+\dfrac{(\alpha^{-1}-\alpha)^2}{4\big(1-|z|^2\big)}=1+h^2$, where $h\risingdotseq\dfrac{|\alpha^{-1}-\alpha|}{2\sqrt{1-|z|^2}}$ varies in $[0,\infty)$. Denoting $f\risingdotseq\sqrt{1-|z|^2}\in(0,1]$, we have $b=\pm h\dfrac{u_1\sqrt{h^2f^2+1}+kz_1}{f}$ and $a=\dfrac{1+2k^2-k^2f^2+k^2h^2f^2-2k\sqrt{h^2f^2+1}(u_0z_0-u_1z_1)}{2f^2}$, since $\dfrac{\alpha^{-1}+\alpha}{2}=\sqrt{(hf)^2+1}$. Therefore, the inequality $b^2<\delta^4+2a\delta^2$ has the form

$$\big(u_1h\sqrt{h^2f^2+1}+khz_1\big)^2-\big(k\delta\sqrt{h^2f^2+1}+(u_1z_1-u_0z_0)\delta\big)^2<$$

$$<\delta^2\big(1+k^2+\delta^2f^2-k^2f^2-(u_0z_0-u_1z_1)^2\big). \tag{5.5}$$

The inequalities $|z_0|,|z_1|<1$ imply that $|u_1z_1-u_0z_0|\le|u_1|+|u_0|\eqqcolon c$. Using an inequality of the type $\big(|A|-|B|\big)^2-\big(|C|+|D|\big)^2\le(A-B)^2-(C+D)^2$ and the fact that $f^2\le1$, we deduce from (5.5) that

$$\big(|u_1|h\sqrt{h^2f^2+1}-kh|z_1|\big)^2-\big(k\delta\sqrt{h^2f^2+1}+c\delta\big)^2<\delta^2\big(1+k^2+\delta^2\big).$$

The last inequality (in view of $\sqrt{h^2f^2+1}>1$, $|u_1|>k$, and $|z_1|<1$) implies that $\big(|u_1|h\sqrt{h^2f^2+1}-kh\big)^2-\big(k\delta\sqrt{h^2f^2+1}+c\delta\big)^2<\delta^2\big(1+k^2+\delta^2\big)$ which can be converted into

$$\Big(\big(|u_1|h-k\delta\big)\sqrt{h^2f^2+1}-(kh+c\delta)\Big)\Big(\big(|u_1|h+k\delta\big)\sqrt{h^2f^2+1}-(kh-c\delta)\Big)<\delta^2\big(1+k^2+\delta^2\big). \tag{5.6}$$

Given $\varepsilon>0$, we have to find some $\delta>0$ such that the inequality $\operatorname{ta}(B_1\cap \mathrm{B}\,V,q)<1+\delta^2$ implies $h<\varepsilon$. First, we require that $\delta<\varepsilon$ and $\delta<\frac{k}{c}\varepsilon$. We can assume now that $|u_1|h-k\delta>0$ and $kh-c\delta>0$, for otherwise the inequality $h<\varepsilon$ follows in view of $|u_1|>k$. Next, we will require that $\delta<\dfrac{|u_1|-k}{c+k}\varepsilon$. Now, we can assume that $|u_1|h-k\delta>kh+c\delta$ (otherwise, the inequality $h<\varepsilon$ follows). Since $\sqrt{h^2f^2+1}>1$, this implies that $\big(|u_1|h-k\delta\big)\sqrt{h^2f^2+1}-(kh+c\delta)>0$. Assuming that $\delta<1$, we can deduce from the inequality (5.6) that $\Big(\big(|u_1|h-k\delta\big)\sqrt{h^2f^2+1}-(kh+c\delta)\Big)^2<\delta^2(2+k^2)$, which, in its turn, implies $|u_1|h-k\delta-kh-c\delta<\delta\sqrt{2+k^2}$ due to $\sqrt{h^2f^2+1}>1$ and $|u_1|h-k\delta>kh+c\delta$. We obtain $\big(|u_1|-k\big)h<\delta(\sqrt{2+k^2}+k+c)$ and, therefore, $h<\dfrac{\sqrt{2+k^2}+k+c}{|u_1|-k}\delta$. Finally, we require that $\delta<\dfrac{|u_1|-k}{\sqrt{2+k^2}+k+c}\varepsilon$ ■

6. General Construction

6.1. Cycle of Bisectors. Our general construction in $\mathbb{H}^2_{\mathbb{C}}$ mimics the following plane example.

Let us fix an integer $n\ge5$. Let P be a simply connected geodesic n-polygon in $\mathbb{H}^2_{\mathbb{R}}$ with the vertices $s_1,\dots,s_n$ and the angles $\alpha_1,\dots,\alpha_n$ such that $\alpha_1+\dots+\alpha_n=2\pi$. It is convenient to treat indices modulo n. We denote the edges by $b_1,\dots,b_n$, where b_i connects s_i and s_{i+1}. Let r_i denote the reflection in the middle point of b_i. By Poincaré's Polyhedron Theorem, P is a fundamental domain for the group H_n generated by r_i's and the defining relations are $r_n\dots r_1=1$ and $r_i^2=1$. For even n, by the same theorem, $P\cup r_1P$ is a fundamental domain for the subgroup G_n generated by r_1r_i, $2\le i\le n$, implying that G_n is the fundamental group of a closed orientable surface of genus $\frac{n}{2}-1$ (the polygon $P\cup r_1P$ has two cycles of vertices and $n-1$ pairs of edges to identify).

For odd n, we will observe that the polygon $Q\eqqcolon P\cup r_1P\cup r_2P\cup r_2r_1P$ is a fundamental domain for the group T_n generated by $a\eqqcolon r_2r_1r_2r_1$, $u\eqqcolon r_2r_1r_n$, $v\eqqcolon r_2r_nr_1$, $x_i\eqqcolon r_1r_i$, and $y_i\eqqcolon r_2r_1r_ir_2$, where $3\le i\le n-1$, implying that T_n is the fundamental group of a closed orientable surface of genus $n-3$. The polygon Q has the following vertices, angles, and edges:

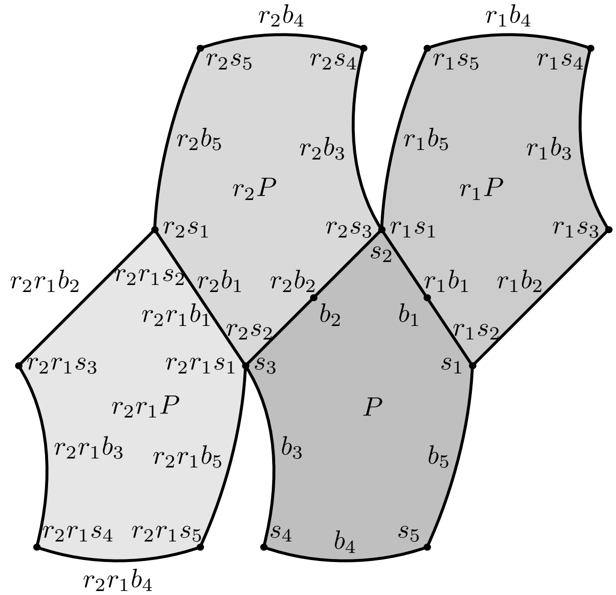
- the vertex s_3, whose angle is $\alpha_1+\alpha_2+\alpha_3$, connected by b_3 with s_4
- the vertex s_{i+1}, whose angle is α_{i+1}, connected by b_{i+1} with s_{i+2}, for $3\le i\le n-1$
- the vertex $s_1=r_1s_2$, whose angle is $\alpha_1+\alpha_2$, connected by r_1b_2 with r_1s_3
- the vertex r_1s_i, whose angle is α_i, connected by r_1b_i with r_1s_{i+1}, for $3\le i\le n$

• the vertex $r_1s_1 = r_2s_3$, whose angle is $\alpha_1 + \alpha_2 + \alpha_3$, connected by r_2b_3 with r_2s_4

• the vertex r_2s_{i+1}, whose angle is α_{i+1}, connected by r_2b_{i+1} with r_2s_{i+2}, for $3 \le i \le n-1$

• the vertex $r_2s_1 = r_2r_1s_2$, whose angle is $\alpha_1 + \alpha_2$, connected by $r_2r_1b_2$ with $r_2r_1s_3$

• the vertex $r_2r_1s_i$, whose angle is α_i, connected by $r_2r_1b_i$ with $r_2r_1s_{i+1}$, for $3 \le i \le n$, with $r_2r_1s_1 = s_3$



The edges and the vertices are identified as follows:

•• x_i identifies b_i and r_1b_i so that $x_is_i = r_1s_{i+1}$ and $x_is_{i+1} = r_1s_i$, for $3 \le i \le n-1$

•• u identifies b_n and $r_2r_1b_n$ so that $us_n = r_2r_1s_1$ and $us_1 = r_2r_1s_n$

•• a identifies r_1b_2 and $r_2r_1b_2$ so that $ar_1s_2 = r_2r_1s_3$ and $ar_1s_3 = r_2r_1s_2$

•• v identifies r_1b_n and r_2b_n so that $vr_1s_n = r_2s_1$ and $vr_1s_1 = r_2s_n$

•• y_i identifies r_2b_i and $r_2r_1b_i$ so that $y_ir_2s_i = r_2r_1s_{i+1}$ and $y_ir_2s_{i+1} = r_2r_1s_i$, for $3 \le i \le n-1$

The following are four cycles of vertices:

$$s_3 \xrightarrow{x_3} r_1s_4 \xleftarrow{x_4} s_5 \xrightarrow{x_5} r_1s_6 \xleftarrow{x_6} \dots \xrightarrow{x_{n-2}} r_1s_{n-1} \xleftarrow{x_{n-1}} s_n \xrightarrow{u} r_2r_1s_1 = s_3$$

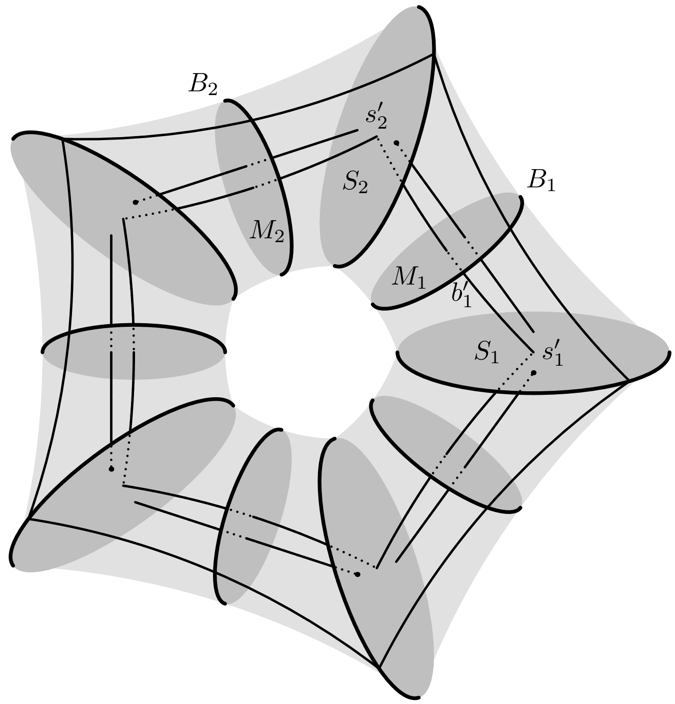
$$s_4 \xrightarrow{x_3} r_1s_3 \xrightarrow{a} r_2r_1s_2 = r_2s_1 \xleftarrow{v} r_1s_n \xleftarrow{x_{n-1}} s_{n-1} \xrightarrow{x_{n-2}} r_1s_{n-2} \xleftarrow{x_{n-3}} s_{n-3} \xrightarrow{x_{n-4}} \dots \xrightarrow{x_5} r_1s_5 \xleftarrow{x_4} s_4$$

$$s_1 \xrightarrow{u} r_2r_1s_n \xleftarrow{y_{n-1}} r_2s_{n-1} \xrightarrow{y_{n-2}} r_2r_1s_{n-2} \xleftarrow{y_{n-3}} r_2s_{n-3} \xrightarrow{y_{n-4}} \dots \xleftarrow{y_4} r_2s_4 \xrightarrow{y_3} r_2r_1s_3 \xleftarrow{a} r_1s_2 = s_1$$

$$r_2s_3 \xrightarrow{y_3} r_2r_1s_4 \xleftarrow{y_4} r_2s_5 \xrightarrow{y_5} r_2r_1s_6 \xleftarrow{y_6} \dots \xrightarrow{y_{n-2}} r_2r_1s_{n-1} \xleftarrow{y_{n-1}} r_2s_n \xleftarrow{v} r_1s_1 = r_2s_3$$

It is easy to see that total angle of each cycle is $\alpha_1 + \dots + \alpha_n = 2\pi$. (Notice that each generator x_i, u, a, y_i, and v appears twice and involves its both vertex identifications.)

Now, let $R_1, \dots, R_n$ be reflections in projective lines $M_1, \dots, M_n$ of signature $+-$ such that $R_n \dots R_1 = 1$ (in $\mathrm{PU}(2,1)$). Let S_1 be a projective line of signature $+-$. We define $S_{i+1} \rightleftharpoons R_iS_i$. Requiring that M_i and S_i are distinct, not orthogonal, and $M_i \cap S_i \not\subset \partial\,\mathrm{B}\,V$, there exists, by Remark 2.3.3, a unique bisector with slices M_i and S_i. Denote by B_i the closed oriented segment of this bisector starting at S_i, including M_i, and ending with S_{i+1}. We call $(B_1, \dots, B_n)$ a *cycle* of bisectors. We claim that almost any choice of S_1 meets the above requirements. Indeed, we have $M_i \rightleftharpoons \mathbb{C}\mathbb{P}m_i^\perp$ and $S_1 \rightleftharpoons \mathbb{C}\mathbb{P}p_1^\perp$ for some $m_i, p_1 \notin \overline{\mathrm{B}}V$. We can think of R_i as being $R(m_i) \in \mathrm{SU}(2,1)$. In this way, the relation $R_n \dots R_1 = 1$ in $\mathrm{PU}(2,1)$ takes the form $R_n \dots R_1 = \delta$, where $\delta \in \mathbb{C}$, $\delta^3 = 1$. In fact, we defined $S_i \rightleftharpoons \mathbb{C}\mathbb{P}p_i^\perp$, where $p_{i+1} \rightleftharpoons R(m_i)p_i$. (Notice that $p_{i+n} = \delta p_i$.) In these terms, the above requirements are $0 \neq \operatorname{ta}(m_i, p_i) \neq 1$, $i = 1, \dots, n$. Equivalently, $0 \neq \operatorname{ta}\big(R(m_1)\dots R(m_{i-2})R(m_{i-1})m_i, p_1\big) \neq 1$, $i = 1, \dots, n$, implying the claim.



Let $s'_1 \in S_1 \cap \operatorname{B}V$. This point generates an ℓ-meridian $b'_1 \subset \operatorname{B}V$ of B_1 which ends with some $s'_2 \in S_2 \cap \operatorname{B}V$. By Proposition 3.9, $R_1 b'_1 = b'_1$ and R_1 exchanges s'_1 and s'_2. In the same way, we obtain ℓ-meridians $b'_2, \dots, b'_n$ such that $R_i b'_i = b'_i$, $R_i s'_i = s'_{i+1}$, $R_i s'_{i+1} = s'_i$. From $R_n \dots R_1 = 1$, it follows that b'_n ends with s'_1. We call the closed curve $b' \eqqcolon b'_1 \cup \dots \cup b'_n$ a *meridian* of the cycle $(B_1, \dots, B_n)$. The arcs $b'_1, \dots, b'_n$ are *edges* and the points $s'_1, \dots, s'_n$ are *vertices* of the meridian. Also, we can start with $s'_1 \in S_1 \cap \partial \operatorname{B}V$, obtaining an *ideal meridian* $b' \eqqcolon b'_1 \cup \dots \cup b'_n \subset \partial \operatorname{B}V$. By Remark 3.8, the edges of any ideal meridian are segments of some $\mathbb{R}$-circles.

6.1.1. Proposition. *Let $(B_1, \dots, B_n)$ be a cycle of bisectors and let $R_i \in \operatorname{SU}(2,1)$ denote the reflection in the middle slice of B_i. We have $R_n \dots R_1 = \delta$, $\delta \in \mathbb{C}$, $\delta^3 = 1$. Let $\varrho : H_n \to \operatorname{PU}(2,1)$, $r_i \mapsto R_i$, be the induced representation. For odd n, the Toledo invariant of $\varrho|_{T_n}$ satisfies $\tau \equiv 4n - \frac{4 \operatorname{Arg}\delta}{\pi}$ mod 8. For even n, the Toledo invariant of $\varrho|_{G_n}$ satisfies $\tau \equiv -\frac{2\operatorname{Arg}\delta}{\pi}$ mod 4.*

Proof. Let $b' = b'_1 \cup \dots \cup b'_n$ be a meridian of the cycle $(B_1, \dots, B_n)$ with vertices $s'_1, \dots, s'_n$. We can assume that $R_i = R(m_i) : x \mapsto 2\langle x, m_i\rangle m_i - x$ with $\langle m_i, m_i\rangle = 1$. Taking $s'_1 \in V$, we define representatives as $s'_{i+1} = R_i s'_i \in V$, implying that $s'_{n+1} = \delta s'_1$. Since $\langle R_i x, x\rangle = 2\langle x, m_i\rangle\langle m_i, x\rangle - \langle x, x\rangle > 0$ for any $x \in \operatorname{B}V$, we obtain $\langle s'_{i+1}, s'_i\rangle > 0$.

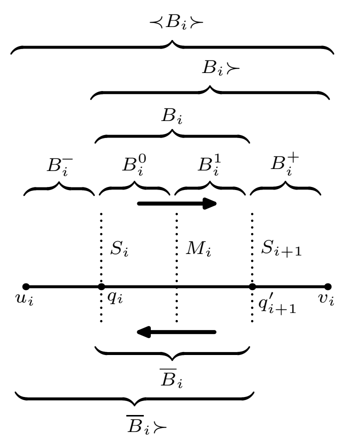
Let $P' \subset \operatorname{B}V$ be any disc with $\partial P' = b'$. It is easy to define a ϱ-equivariant continuous map $\varphi : \mathbb{H}^2_{\mathbb{R}} \to \mathbb{H}^2_{\mathbb{C}}$ such that $\varphi(P) = P'$, $\varphi(s_i) = s'_i$, and $\varphi(b_i) = b'_i$. The Toledo invariant of $\varrho|_{T_n}$ is defined as $\tau \eqqcolon 4\frac{1}{2\pi}\int_Q \varphi^*\omega$ [Tol]. Taking $u \in \operatorname{B}V$, we obtain (see Lemma 2.2.1) $\tau = \frac{8}{\pi}\int_{P'}\omega = \frac{8}{\pi}\int_{\partial P'} P_u = \frac{8}{\pi}\sum_i \int_{b'_i} P_u$ (for even n and $\varrho|_{G_n}$, $\tau \eqqcolon \frac{4}{\pi}\sum_i \int_{b'_i} P_u$). By Corollary 3.2 and by Lemma 2.2.1, $\int_{b'_i} P_u = \int_{b'_i}(P_u - P_{s'_i}) = \int_{b'_i} df_{u,s'_i}$. This number is the total variation of $\dfrac{1}{2}\operatorname{Arg}\dfrac{\langle u, p\rangle\langle p, s'_i\rangle}{\langle u, s'_i\rangle}$, while p runs over $b'_i \subset \mathrm{B}\,V$ from s'_i to s'_{i+1}. By Remark 2.3.2, $\dfrac{\langle u,p\rangle\langle p,s'_i\rangle}{\langle u,s'_i\rangle}$ is never real nonnegative. It follows that $\int_{b'_i} df_{u,s'_i} = \dfrac{1}{2}\operatorname{Arg}\dfrac{\langle u,s'_{i+1}\rangle\langle s'_{i+1},s'_i\rangle}{\langle u,s'_i\rangle} - \dfrac{1}{2}\operatorname{Arg}\dfrac{\langle u,s'_i\rangle\langle s'_i,s'_i\rangle}{\langle u,s'_i\rangle} = \dfrac{1}{2}\operatorname{Arg}\dfrac{\langle u,s'_{i+1}\rangle}{\langle u,s'_i\rangle} - \dfrac{\pi}{2}$, since $\langle s'_{i+1},s'_i\rangle > 0$ and $\langle s'_i,s'_i\rangle < 0$.

For odd n (similarly for even n), calculating $\mod 8$, we obtain

$$\tau \equiv \frac{4}{\pi}\sum_i \big(\operatorname{Arg}\langle u,s'_{i+1}\rangle - \operatorname{Arg}\langle u,s'_i\rangle - \pi\big) \equiv$$

$$\equiv \frac{4}{\pi}\big(-n\pi + \operatorname{Arg}\langle u,s'_{n+1}\rangle - \operatorname{Arg}\langle u,s'_1\rangle\big) \equiv 4n - \frac{4\operatorname{Arg}\delta}{\pi} \mod 8 \ \blacksquare$$

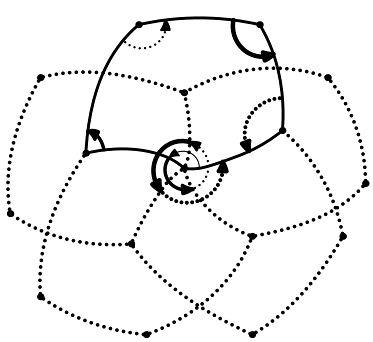
6.2. Poincaré's Polyhedron Theorem. Henceforth, we will assume that the focus of each bisector in our cycle is positive. As default, we will mean by B_i, S_i, etc. the former $B_i \cap \overline{\mathrm{B}}V$, $S_i \cap \overline{\mathrm{B}}V$, etc. and denote $\check B_i \rightleftharpoons B_i \cap \mathrm{B}\,V$, $\check S_i \rightleftharpoons S_i \cap \mathrm{B}\,V$, etc. We treat bisectors as oriented: B_i begins with S_i and ends with S_{i+1}, that is, $B_i = B[q_i, q'_{i+1}]$, where $q_i \in S_i$ and $q'_{i+1} \in S_{i+1}$ are in the real spine of B_i. By $\overline{B}_i$, we denote B_i with the opposite orientation, $\overline{B}_i \rightleftharpoons B[q'_{i+1}, q_i]$. We denote $B_i^- \rightleftharpoons B[u_i, q_i]$, $B_i^+ \rightleftharpoons B[q'_{i+1}, v_i]$, $B_i{\succ} \rightleftharpoons B[q_i, v_i]$, $\overline{B}_i{\succ} \rightleftharpoons B[q'_{i+1}, u_i]$, and ${\prec}B_i{\succ} \rightleftharpoons B[u_i, v_i]$, where u_i and v_i stand for the vertices of B_i which are closer to S_i and to S_{i+1}, respectively. So, $B_i{\succ} = B_i \cup B_i^+$, $\overline{B}_i{\succ} = B_i \cup B_i^-$, and ${\prec}B_i{\succ} = B_i^- \cup B_i \cup B_i^+$. We also denote by B_i^0 and B_i^1 the bisectors from S_i to M_i and from M_i to S_{i+1}, respectively, $B_i = B_i^0 \cup B_i^1$.



Unless otherwise stated, we consider all isometries as 'living' in $\mathrm{PU}(2,1)$.

We put $A_{ik} \rightleftharpoons R_iR_{i+1}\dots R_{k-1}$, where R_i denotes the reflection in M_i. For instance, $A_{i(i-1)} = R_iR_{i+1}\dots R_nR_1R_2\dots R_{i-3}R_{i-2} = R_{i-1}$ and $A_{ii} = 1$.

Let us fix some i. Since $A_{ik}S_k = S_i$, the bisectors $A_{ik}B_k$ (in particular, $A_{ii}B_i = B_i$ and $A_{i(i-1)}B_{i-1} = R_iR_{i+1}\dots R_{i-2}B_{i-1} = R_{i-1}B_{i-1} = \overline{B}_{i-1}$) have the common slice S_i: they all begin with S_i. Let $b = b_1 \cup \dots \cup b_n$ be a meridian of the cycle $(B_1,\dots,B_n)$ with vertices $s_1,\dots,s_n$. Denoting by α_k the angle from B_k to $\overline{B}_{k-1}$ at the point $s_k \in \check S_k$, we call $\alpha \rightleftharpoons \alpha_1 + \dots + \alpha_n$ the *total angle* of the cycle at the meridian b. For any k, $A_{ik}s_k = s_i$. Hence, the angle α_k is equal to the angle at s_i from $A_{ik}B_k$ to $A_{i(k-1)}B_{k-1}$. Now, it is easy to see that the total angle at the meridian is a multiple of 2π. Varying the meridian, we arrive at

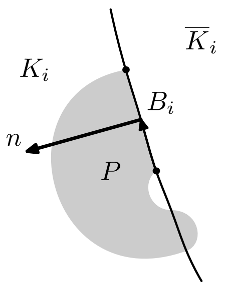


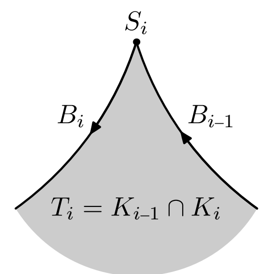
6.2.1. Remark. Modulo 2π, the total angle of any cycle of bisectors is zero. If, for any k, the bisectors ${\prec}B_{k-1}{\succ}$ and ${\prec}B_k{\succ}$ are transversal along their common slice S_k, then the total angle does not depend on the choice of a meridian and is an integer multiple of 2π.

We call a cycle of bisectors *transversal* if, for any k, ${\prec}B_{k-1}{\succ}$ and ${\prec}B_k{\succ}$ are transversal along S_k. Our intention is to study the discreteness of the group H generated by the R_i's. We will take as a fundamental domain the polyhedron bounded by a cycle of bisectors. It seems reasonable to require the transversality of the cycle, otherwise, we will not have a good tessellation around any S_i (see Corollary 4.3 and its proof). Clearly, the transversality of the cycle is equivalent to the transversality of $A_{i(k-1)}{\prec}B_{k-1}{\succ}$ and $A_{ik}{\prec}B_k{\succ}$ along S_i for all k (an arbitrary i is fixed).

For a transversal cycle of bisectors, we can change the cyclic order of the bisectors and their orientation so that the total angle of the new cycle will be $2n\pi - \alpha$. Therefore, without loss of generality, we can assume that $0 \le \alpha \le n\pi$ (meaning that we deal with the 'inside angles'). Since we are going to prove the discreteness by showing that some transversal cycle of bisectors bounds a fundamental domain for H, we do not have hope, if the total angle is different from 2π.

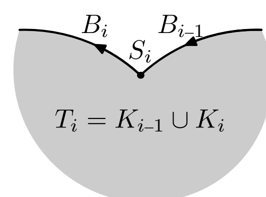
We call a cycle *simple* if $B_i \cap B_k \ne \varnothing$ implies that $k = i-1$ or $k = i$ or $k = i+1$.





Let $(B_1, \dots, B_n)$ be a simple transversal cycle of bisectors. The solid torus $\partial_0 P \rightleftharpoons B_1 \cup \dots \cup B_n$ is fibred in two ways: the fibres are the meridians (including ideal ones) and the slices, $\partial_0 P \simeq \mathbb{S}^1 \times \mathbb{B}^2$. The torus $T \rightleftharpoons \partial_0 P \cap \partial\,\mathrm{B}\,V = \partial\partial_0 P$ divides $\partial\,\mathrm{B}\,V \simeq \mathbb{S}^3$ into two closed connected pieces. Let F be either of them. Since $\partial_0 P$ is transversal to $\partial\,\mathrm{B}\,V$, the orientable connected closed 3-manifold $\partial_0 P \cup F$ divides the 4-ball $\overline{\mathrm{B}}V$ into two closed connected polyhedra. We denote by P the closed polyhedron on the side of the normal vector to each B_i.

Every $\prec B_i \succ$ divides $\overline{\mathrm{B}}V$ into two closed 4-balls K_i and $\overline{K}_i$, being K_i on the side of the normal vector to $\prec B_i \succ$. (Notice that, in general, $P \not\subset K_i$.) By Corollary 4.3, the bisectors $\prec B_{i-1} \succ$ and $\prec B_i \succ$ divide $\overline{\mathrm{B}}V$ into four 4-balls. To every S_i, we associate the *inside sector* T_i from $B_i \succ$ to $\overline{B}_{i-1} \succ$: if $\alpha_i < \pi$ (it does not matter at which point in $\check{S}_i$ we measure α_i), then we put $T_i \rightleftharpoons K_{i-1} \cap K_i$; otherwise, we put $T_i \rightleftharpoons K_{i-1} \cup K_i$. We define $\partial_0 T_i \rightleftharpoons B_i \succ \cup\, \overline{B}_{i-1} \succ$.

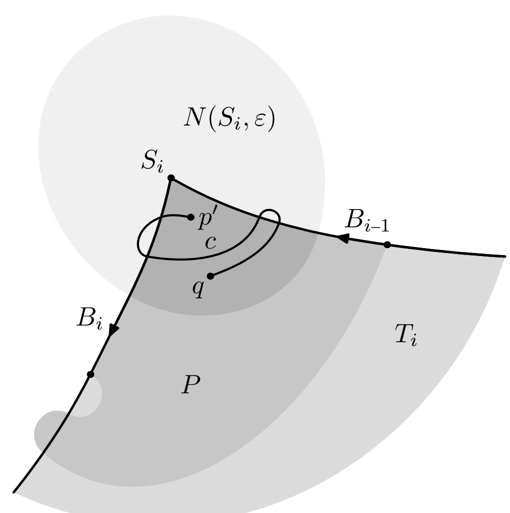


The polyhedron P 'includes' the inside sectors, i.e., for any $p \in S_i$, we can find a point $p' \in \mathring{P} \cap \mathring{T}_i$ arbitrarily close to p, where $\mathring{P}$ and $\mathring{T}_i$ stand for the interior of P and for the interior of T_i, respectively.

Clearly, $B_i^0 \cap B_i^+ = B_i^1 \cap B_i^- = \varnothing$. Since $\prec B_{i-1} \succ$ and $\prec B_i \succ$ intersect only in S_i, we obtain $S_i \cap B_{i-1}^- = S_i \cap B_i^+ = \varnothing$. The cycle is simple, hence, $B_i \cap B_k = \varnothing$ for $k \ne i-1, i, i+1$. Since $\prec B_i \succ$ and $\prec B_{i+1} \succ$ intersect only in S_{i+1} and the cycle is simple, $B_i^0 \cap B_k = \varnothing$ for $k \ne i-1, i$. For similar reasons, $B_i^1 \cap B_k = \varnothing$ for $k \ne i, i+1$. Consequently, there exists $d > 0$ such that

$$\mathrm{dist}(\check{B}_i^0, \check{B}_i^+), \mathrm{dist}(\check{B}_i^1, \check{B}_i^-), \mathrm{dist}(\check{S}_i, \check{B}_{i-1}^-), \mathrm{dist}(\check{S}_i, \check{B}_i^+) \ge d,$$

$\mathrm{dist}(\check{B}_i, \check{B}_k) \ge d$ for $k \ne i-1, i, i+1$, $\mathrm{dist}(\check{B}_i^0, \check{B}_k) \ge d$ for $k \ne i-1, i$, and $\mathrm{dist}(\check{B}_i^1, \check{B}_k) \ge d$ for $k \ne i, i+1$.



For $p \in \mathrm{B}\,V$ and $\varepsilon > 0$, we denote by $N(p, \varepsilon)$ the open ball of radius ε centred in p. For $X \subset \mathrm{B}\,V$, we denote by $N(X, \varepsilon) \rightleftharpoons \bigcup_{p \in X} N(p, \varepsilon)$ the ε-neighborhood of X.

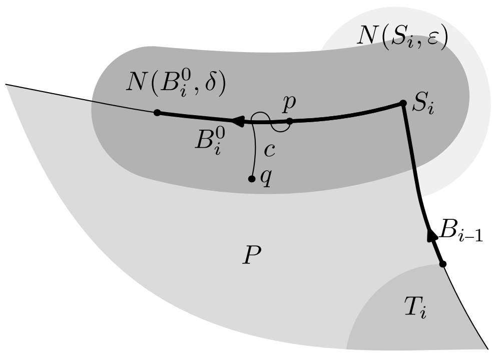
Let $0 < \varepsilon \le d/3$ be fixed.

6.2.2. Lemma. $\check{P} \cap N(\check{S}_i, \varepsilon) = \check{T}_i \cap N(\check{S}_i, \varepsilon)$.

Proof. Let $q \in N(\check{S}_i, \varepsilon) \setminus \partial_0 T_i$. There is a geodesic of length $< d/3$ connecting q with some $p \in \check{S}_i$. We can find a point $p' \in N(p, d/3) \cap \mathring{P} \cap \mathring{T}_i$ and connect q and p' with a curve $c \subset \mathrm{B}\,V$ of length $< 2d/3$ transversal to $\prec \check{B}_{i-1} \succ$ and to $\prec \check{B}_i \succ$ and not passing through $\check{S}_i$. So, $\ell(c) + \mathrm{dist}(p', p) < d$. If $k \ne j-1, j, j+1$, then $\mathrm{dist}(\check{B}_j, \check{B}_k) \ge d$ by the choice of d. Hence, in $\partial_0 P$, c

can meet only $(\check B_{i-1}\setminus\check S_{i-1})\cup(\check B_i\setminus\check S_{i+1})$. The parity of the number of such intersections indicates if $q\in\check P$. Also, $\operatorname{dist}(\check S_i,\check B_{i-1}^-)\geq d$ and $\operatorname{dist}(\check S_i,\check B_i^+)\geq d$ by the choice of d. Therefore, in ∂_0T_i, c can meet only $(\check B_{i-1}\setminus\check S_{i-1})\cup(\check B_i\setminus\check S_{i+1})$. Consequently, the same parity verifies if $q\in\check T_i$. For $q\in N(\check S_i,\varepsilon)\cap\check B_i\succ$, it follows from $\operatorname{dist}(\check S_i,\check B_i^+)\geq d$ that $q\in\check B_i$. The same reason works for $q\in N(\check S_i,\varepsilon)\cap\overline{B}_{i-1}\succ$ ■

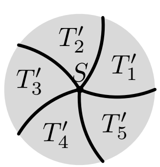
By Lemma 5.4, there exists some δ, $0<\delta\leq\varepsilon/2$, such that $\prec\check B_{i-1}\succ\cap N(\check B_i,\delta)\subset N(\check S_i,\varepsilon/2)$ and $\prec\check B_{i+1}\succ\cap N(\check B_i,\delta)\subset N(\check S_{i+1},\varepsilon/2)$ for all i. In what follows, we will refer to these inclusions as related to the choice of δ. We fix δ and put $D_i^0\rightleftharpoons\check P\cap N(\check B_i^0,\delta)$, $D_i^1\rightleftharpoons\check P\cap N(\check B_i^1,\delta)$, and $D_i\rightleftharpoons\check P\cap N(\check B_i,\delta)$. So, $D_i=D_i^0\cup D_i^1$.



6.2.3. Lemma. *$D_i^0\subset\check T_i$ and $D_i^1\subset\check T_{i+1}$.*

Proof. By symmetry, it suffices to show that $D_i^0\subset\check T_i$. Let $q\in D_i^0$. We can assume that $q\notin N(\check S_i,\varepsilon)$, otherwise, the result follows from Lemma 6.2.2. Let $p\in\check B_i^0$ be a point closest to q. Since $\delta\leq\varepsilon$, we obtain $p\notin\check S_i$. We can connect q and p with a curve $c\subset\mathrm BV$ of length $<\delta$ transversal to $\prec\check B_{i-1}\succ$ and to $\prec\check B_i\succ$ and not passing through $\check S_i$. Since $\prec\check B_{i-1}\succ\cap N(\check B_i,\delta)\subset N(\check S_i,\varepsilon/2)$, $\delta\leq\varepsilon/2$, and $q\notin N(\check S_i,\varepsilon)$, the curve c cannot meet $\prec\check B_{i-1}\succ$. Now, the inequalities $\operatorname{dist}(\check B_i^0,\check B_k)\geq d>\delta$, $k\neq i-1,i$, imply that, in ∂_0P, c can meet only $\check B_i\setminus\check S_{i+1}$ and the inequalities $\operatorname{dist}(\check B_i^0,\check B_i^+)\geq d>\delta$ imply that, in ∂_0T_i, c can meet only $\check B_i\setminus\check S_{i+1}$. Since $q\in\check P$, we conclude that $q\in\check T_i$ ■

6.2.4. Lemma. *Let $B'_k\succ$, $1\leq k\leq n$, be bisectors with positive foci. Suppose that they all have a common slice S of signature $+-$ (they all begin with S) such that $\prec B'_{k-1}\succ$ and $\prec B'_k\succ$ are transversal along S for all k and that the sum of the angles from $B'_k\succ$ to $B'_{k-1}\succ$ at some and the same point in $\check S$ equals 2π. Then $B'_k\succ$'s divide $\overline{\mathrm B}V$ into n sectors T'_k (from $B'_k\succ$ to $B'_{k-1}\succ$) such that $T'_k\cap T'_j=S$ if $j\neq k-1,k,k+1$ and $T'_k\cap T'_{k+1}=B'_k\succ$.*[9]



Proof easily follows by viewing the projective line orthogonal to S and passing through the 'extra' point of the intersection in question (see also the Corollary 4.3 and its proof)[10] ■

6.2.5. Lemma. *Suppose that the total angle of the cycle equals 2π. Then $N(\check S_i,\varepsilon)\cup N(\check B_i,\delta)\cup N(\check S_{i+1},\varepsilon)=N(\check S_i,\varepsilon)\cup D_i\cup R_iD_i\cup N(\check S_{i+1},\varepsilon)$ and $D_i\cap R_iD_i=\check B_i$.*

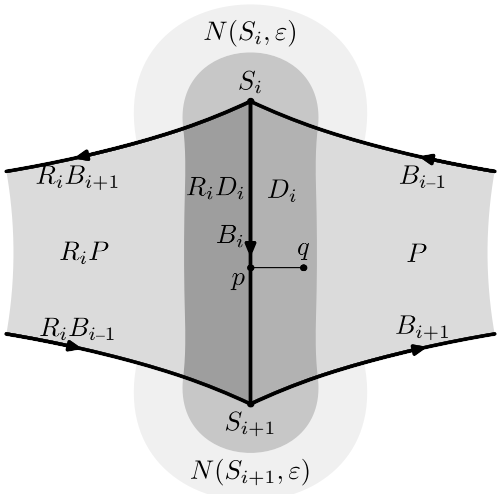
Proof. Let $q\in N(\check B_i,\delta)$. We have two tasks: to prove that $q\in N(\check S_i,\varepsilon)\cup\check P\cup R_i\check P\cup N(\check S_{i+1},\varepsilon)$ and that $q\in\check P\cap R_i\check P$ implies $q\in\check B_i$.

For the first task, we can obviously assume that $q\notin N(\check S_i,\varepsilon)\cup N(\check S_{i+1},\varepsilon)$. So can we do for the second one by applying twice Lemmas 6.2.4 and 6.2.2 to the bisectors $A_{ik}B_k\succ$'s and to the bisectors $A_{(i+1)k}B_k\succ$'s. Also, we assume that $q\notin\check B_i$. Let $\mathrm G[q,p]$ be the shortest geodesic (of length $<\delta$) connecting q with some point $p\in\check B_i$. By Lemma 5.2 and by the inequality $\delta<\varepsilon$, we obtain $q\notin\prec\check B_i\succ$. Now, by Lemma 5.3, p is the point in $\prec\check B_i\succ$ closest to q. By the choice of δ, we have $\prec\check B_{i-1}\succ\cap N(\check B_i,\delta)\subset N(\check S_i,\varepsilon/2)$, $\prec\check B_{i+1}\succ\cap N(\check B_i,\delta)\subset N(\check S_{i+1},\varepsilon/2)$, $R_i\prec\check B_{i-1}\succ\cap N(\check B_i,\delta)\subset N(\check S_{i+1},\varepsilon/2)$, and $R_i\prec\check B_{i+1}\succ\cap N(\check B_i,\delta)\subset N(\check S_i,\varepsilon/2)$, since $R_i\check B_i=\check B_i$, $R_i\check S_i=\check S_{i+1}$, and $R_i\check S_{i+1}=\check S_i$.

[9]In other words, the intersections of the sectors are 'prescribed.'

[10]Nevertheless, notice that the bisectors are not necessarily all transversal, some pair of them can even form a full bisector.

Let us show that $\mathrm{G}[q,p]$ cannot meet $\prec\check B_{i-1}\succ\cup\prec\check B_{i+1}\succ$. Indeed, if $x\in\mathrm{G}[q,p]\cap(\prec\check B_{i-1}\succ\cup\prec\check B_{i+1}\succ)$ then $x\in N(\check S_i,\varepsilon/2)\cup N(\check S_{i+1},\varepsilon/2)$ by the choice of δ. Now, it follows from $\delta\le\varepsilon/2$ that $q\in N(S_i,\varepsilon)\cup N(S_{i+1},\varepsilon)$. By symmetry, $\mathrm{G}[q,p]$ does not meet $R_i\prec\check B_{i-1}\succ\cup R_i\prec\check B_{i+1}\succ$.



By the inequalities $\operatorname{dist}(\check B_i,\check B_k)\ge d>\delta$ and $\operatorname{dist}(\check B_i,\check R_iB_k)\ge d>\delta$ for $k\ne i-1,i,i+1$, in ∂_0P and in ∂_0R_iP, the geodesic $\mathrm{G}[q,p]$ can meet only B_i. Since $\mathrm{G}[q,p]$ meets B_i only in $p\in\check P\cap R_i\check P$, we conclude that $q\in\check P\cup R_i\check P$. Suppose that $q\in\check P\cap R_i\check P$ and $q\notin B_i$, that is, $q\notin\partial_0P\cup\partial_0R_iP$. Then we slightly extend $\mathrm{G}[q,p]$ to $\mathrm{G}[q,p']$. By Lemma 5.3, $\mathrm{G}[q,p']$ is transversal to B_i. Hence, by Lemma 5.1, $p'\notin\check P\cup R_i\check P$. A contradiction ■

We put $N_k\rightleftharpoons\check P\cap N(\check S_k,\varepsilon)$, $E_i\rightleftharpoons N_i\cup D_i\cup N_{i+1}$, and $U_i\rightleftharpoons\check B_i\cap N(\check S_i,\varepsilon)$.

6.2.6. Lemma. *Suppose that the total angle of the cycle equals 2π. Then the set $N\rightleftharpoons\check P\cup\bigcup\limits_{i=1}^n R_iE_i\cup\bigcup\limits_{i=1}^n\bigcup\limits_{\substack{k\ne i,\\ k\ne i\pm1}}A_{ik}N_k=\check P\cup\bigcup\limits_{i=1}^n N(\check B_i,\delta)\cup\bigcup\limits_{i=1}^n N(\check S_i,\varepsilon)$ is open and $N\supset N(\check P,\delta)$. The following are the only nonempty intersections between $\check P$, R_iE_i, and $A_{ik}N_k$, $k\ne i-1,i,i+1$:*

- $\check P\cap R_iE_i=\check B_i$
- $\check P\cap A_{ik}N_k=\check S_i$
- $R_{i-1}E_{i-1}\cap R_iE_i=\check S_i$
- $R_{i-1}E_{i-1}\cap A_{i(i-2)}N_{i-2}=A_{i(i-2)}U_{i-2}$
- $R_iE_i\cap A_{i(i+2)}N_{i+2}=A_{i(i+1)}U_{i+1}$
- $R_{i-1}E_{i-1}\cap A_{ik}N_k=\check S_i$ *for* $k\ne i-2$
- $R_iE_i\cap A_{ik}N_k=\check S_i$ *for* $k\ne i+2$
- $A_{ik}N_k\cap A_{i(k+1)}N_{k+1}=A_{ik}U_k$
- $A_{ik}N_k\cap A_{ij}N_j=\check S_i$ *for* $k\ne j-1,j+1$

In other words, all these intersections are 'prescribed.'

Proof is routine and straightforward. The inclusion $N\supset N(\check S_i,\varepsilon)$ follows from Lemmas 6.2.4 and 6.2.2. Now, by Lemma 6.2.5, we obtain $N\supset N(\check B_i,\delta)$. By Lemmas 6.2.4, 6.2.2, and 6.2.5, $\check P\cap R_iE_i=\check B_i$. By Lemmas 6.2.4 and 6.2.2, $\check P\cap A_{ik}N_k=\check S_i$. By the choice of d, we have $\operatorname{dist}(\check B_k,\check B_l)\ge\delta$, hence, due to $2\varepsilon<d$, $N(\check S_i,\varepsilon)\cap N(\check S_j,\varepsilon)=\varnothing$ for $i\ne j$. Therefore, $A_{ik}N_k\cap A_{jl}N_l=\varnothing$ for $i\ne j$. By Lemmas 6.2.4 and 6.2.2, $A_{ik}N_k\cap A_{i(k+1)}N_{k+1}=A_{ik}U_k$ and $A_{ik}N_k\cap A_{ij}N_j=\check S_i$ for $k\ne j-1,j+1$. By the choice of d and due to $2\delta<d$, we obtain $N(\check B_i,\delta)\cap N(\check B_j,\delta)=\varnothing$ for $j\ne i-1,i,i+1$. With the use of $N(\check S_l,\varepsilon)\cap N(\check S_m,\varepsilon)=\varnothing$ for $l\ne m$, this implies that $R_iE_i\cap R_jE_j=\varnothing$ for $j\ne i-1,i,i+1$. By the choice of d, $d\le\operatorname{dist}(\check B_i^0,\check B_j)\le\operatorname{dist}(\check S_i,\check B_j)$ for $j\ne i-i,i$. Since $d>\varepsilon+\delta$, $N(\check S_i,\varepsilon)\cap N(\check B_j,\delta)=\varnothing$ for $j\ne i-1,i$. Hence, $R_jE_j\cap A_{ik}N_k=\varnothing$ for $j\ne i-1,i$.

By the choice of d, $N(\check B_{i-1}^0,\delta)\cap N(\check S_i,\varepsilon)=\varnothing$, $N(\check B_{i-1}^0,\delta)\cap N(\check B_i,\delta)=\varnothing$, $N(\check B_{i-1},\delta)\cap N(\check S_{i+1},\varepsilon)=\varnothing$, $N(\check B_i^1,\delta)\cap N(\check S_i,\varepsilon)=\varnothing$, $N(\check B_i^1,\delta)\cap N(\check B_{i-1},\delta)=\varnothing$, and $N(\check B_i,\delta)\cap N(\check S_{i-1},\varepsilon)=\varnothing$. Since $N(\check S_l,\varepsilon)\cap N(\check S_m,\varepsilon)=\varnothing$ for $l\ne m$, we obtain $R_{i-1}E_{i-1}\cap R_iE_i=R_{i-1}(D_{i-1}^0\cup N_{i-1})\cap R_i(D_i^1\cup N_{i+1})$ and $R_iE_i\cap A_{ik}N_k=R_i(D_i^1\cup N_{i+1})\cap A_{ik}N_k$. By Lemma 6.2.3, $D_{i-1}^0\cup N_{i-1}\subset T_{i-1}$ and $D_i^1\cup N_{i+1}\subset\check T_{i+1}$. Now, from $R_i=A_{i(i+1)}$ and $R_{i-1}=A_{i(i-1)}$, it follows that $R_{i-1}E_{i-1}\cap R_iE_i=\check S_i$, $R_iE_i\cap A_{i(i+2)}N_{i+2}\subset A_{i(i+1)}\check B_{i+1}\succ$, and $R_iE_i\cap A_{ik}N_k=\check S_i$ for $k\ne i+2$ by Lemma 6.2.4. By the choice of δ, $\prec\check B_{i+1}\succ\cap N(\check B_i,\delta)\subset N(\check S_{i+1},\varepsilon)$. Therefore, $(D_i^1\cup N_{i+1})\cap\check B_{i+1}\succ=U_{i+1}$, which implies that $R_iE_i\cap A_{i(i+2)}N_{i+2}=A_{i(i+1)}U_{i+1}$. If we change the orientation of the cycle, the conditions related to the choice of d and δ will remain valid. By this symmetry, $R_{i-1}E_{i-1}\cap A_{i(i-2)}N_{i-2}=A_{i(i-2)}U_{i-2}$ and $R_{i-1}E_{i-1}\cap A_{ik}N_k=\check S_i$ for $k\neq i-2$ ■

In literature, we have found no convenient version of Poincaré's Polyhedron Theorem. Some versions require that $\check P$ and $A_{ik}\check P$ have no extra intersection (which leads to a difficult problem of proving that two bisectors have no intersection), other ones have gaps in the proof. Probably, [Bea, p. 246] could be applied in our case. Unfortunately, there is a mistake in the description of condition (A6) there. Our proof essentially uses some specific properties of bisectors (see Lemmas 5.1, 5.2, 5.3, 5.4, and 6.2.4).

6.2.7. Theorem (Poincaré's Polyhedron Theorem). *Let $(B_1,\dots,B_n)$ be a simple transversal cycle of bisectors with the total angle 2π. Then H is discrete and $\check P$ is a fundamental domain for $H\simeq H_n$.*

Proof is standard. The group H_n naturally acts on $H_n\times\check P$, the disjoint union of H_n-copies of $\check P$. There are a homomorphism $h:H_n\to H$, $r_i\mapsto R_i$, and a continuous H_n-equivariant map $\psi:H_n\times\check P\to \mathrm{B}\,V$, $(g,p)\mapsto h(g)(p)$. For all $g\in H_n$ and i, we identify $(g,\check B_i)\subset(g,\check P)$ and $(gr_i,\check B_i)\subset(gr_i,\check P)$ with the help of R_i. We obtain a topological quotient space J and continuous H_n-equivariant maps $\pi:H_n\times\check P\to J$ and $\varphi:J\to\mathrm{B}\,V$ such that $\psi=\varphi\circ\pi$. We put $\pi(g,p)\eqqcolon gp\in J$, $a_{ik}\eqqcolon r_ir_{i+1}\dots r_{k-1}$, and $\widetilde N\eqqcolon\check P\cup\bigcup\limits_{i=1}^n r_iE_i\cup\bigcup\limits_{i=1}^n\bigcup\limits_{\substack{k\neq i,\\ k\neq i\pm1}}a_{ik}N_k\subset J$. It follows from the choice of δ that $D_i\cap\check B_{i-1}\subset N(\check S_i,\varepsilon)$ and $D_i\cap\check B_{i+1}\subset N(\check S_{i+1},\varepsilon)$. Now, it is easy to see that $\widetilde N$ is an open neighborhood of $\check P$ in J. Lemma 6.2.6 can be read as follows: $\varphi|_{\widetilde N}$ is a bijection between $\widetilde N$ and N. Taking into account that bisectors are smooth hypersurfaces and that $A_{ik}\prec B_k\succ$'s are transversal along S_i, we conclude that $\varphi|_{\widetilde N}$ is open. Hence, φ is open. In particular, $\varphi(J)$ is open in $\mathrm{B}\,V$. Let $q\in\mathrm{B}\,V$ be in the closure of $\varphi(J)$. Then $h(g)\check P\cap N(q,\delta)\neq\varnothing$ for some $g\in H_n$. Hence, $q\in N\big(h(g)\check P,\delta\big)\subset h(g)N=\varphi(g\widetilde N)\subset\varphi(J)$. Thus, φ is surjective.

Let $q\in\mathrm{B}\,V$. We define $W_q\eqqcolon\big\{g\in H_n\mid N(q,\delta/2)\cap h(g)\check P\neq\varnothing\big\}$ and, for every $g\in W_q$, we put $N_g\eqqcolon\varphi^{-1}\big(N(q,\delta/2)\big)\cap g\widetilde N$. Since $N(q,\delta/2)\cap h(g)\check P\neq\varnothing$ implies $N(q,\delta/2)\subset h(g)N$, we conclude that $\varphi:N_g\to N(q,\delta/2)$ is a homeomorphism. Let $p\in\varphi^{-1}\big(N(q,\delta/2)\big)$. Then $p\in g\check P\subset J$ for some $g\in H_n$. Hence, $\varphi(p)\in N(q,\delta/2)\cap h(g)\check P$, $g\in W_q$, and $p\in N_g$. In other words, $\varphi^{-1}\big(N(q,\delta/2)\big)=\bigcup\limits_{g\in W_p}N_g$. If $N_{g_1}\cap N_{g_2}\neq\varnothing$ for some $g_1,g_2\in W_p$, then $N_{g_1}=N_{g_2}$. Indeed, N_{g_1} is connected and $N_{g_1}\cap N_{g_2}$ is open and closed[11] in N_{g_1}. We proved that φ is a covering and, therefore, a homeomorphism ■

6.3. Simplicity and Transversality. We remember that the initial projective line S_1 was chosen almost arbitrarily. In H, there are many collections of reflections with product 1. For instance, let us fix an arbitrary i. We denote by $R'_k\eqqcolon R_k^{A_{ik}}$ (where X^Y stands for YXY^{-1}) the reflection in the projective line $M'_k\eqqcolon A_{ik}M_k$. It is easy to verify that $R'_1R'_2\dots R'_n=1$. We can view a cycle of bisectors in a simpler way as being given by some projective line S ($\eqqcolon S_i$) of signature $+-$ and by a collection of projective lines $M'_1,M'_2,\dots,M'_n$ ($M'_k\eqqcolon A_{ik}M_k$) of signature $+-$, each ultraparallel to S and different from S, such that $R'_1R'_2\dots R'_n=1$, where R'_k stands for the reflection in M'_k. These data suffice to restitute the cycle: we define $A'_{ik}\eqqcolon R'_iR'_{i+1}\dots R'_{k-1}$ and $R_k\eqqcolon {R'_k}^{A'_{ik}}$.

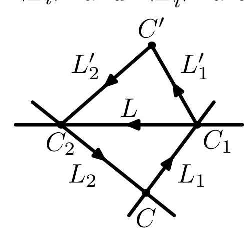
Let $S'_k\eqqcolon R'_kS$ and let B'_k ($=A_{ik}B_k$ in the terms of the first description of the cycle) denote the bisector starting at S, including M'_k, and ending with S'_k. Then $S_k=A'_{ik}S$, $M_k=A'_{ik}M'_k$, and $B_k=A'_{ik}B'_k$. As was noticed below Remark 6.2.1, the transversality of the cycle is equivalent to the

[11] Since $N(q,\delta/2)$ is Hausdorff, the diagonal $\Delta_{N(q,\delta/2)}$ is closed in $N(q,\delta/2)\times N(q,\delta/2)$. Hence, $\Delta_{N_{g_1}\cap N_{g_2}}$, being the preimage of $\Delta_{N(q,\delta/2)}$ with respect to $\varphi\times\varphi:N_{g_1}\times N_{g_2}\to N(q,\delta/2)\times N(q,\delta/2)$, is closed in $N_{g_1}\times N_{g_2}$. The projection $N_{g_1}\times N_{g_2}\to N_{g_1}$ is a closed map, implying that $N_{g_1}\cap N_{g_2}$ is closed in N_{g_1}.

transversality of $\prec B'_{k-1}\succ$ and $\prec B'_k\succ$ along S for all k. As was observed above Remark 6.2.1, the total angle of the cycle at a meridian is equal to the sum of the angles from $B'_{k-1}\succ$ to $B'_k\succ$, $1\le k\le n$, at the respective point in $\check S$.

Let $(B_1,\dots,B_n)$ be a cycle of bisectors and let C be a projective line of signature $+-$ different from all M_k's and ultraparallel to each M_k. We call C a *centre* of the cycle. By the above considerations, the condition that the cycle possesses a centre does not seem too restrictive. By Remark 2.3.3, we can connect C and M_k with some bisector B'_k (oriented from C to M_k). Let $c\in\check C$. Denoting by β_k the angle from $B'_{k-1}\succ$ to $B'_k\succ$ at c, it is easy to see that $\beta\rightleftharpoons\beta_1+\dots+\beta_n$, the *central angle* of the cycle at C, is zero modulo 2π. If, for any k, the bisectors $\prec B'_{k-1}\succ$ and $\prec B'_k\succ$ are transversal along their common slice C, then the total angle does not depend on the choice of $c\in\check C$ and is an integer multiple of 2π.

Let C,C_1,C_2 be pairwise ultraparallel different projective lines of signature $+-$. By Remark 2.3.3, we can connect them with three oriented segments of bisectors which form the *oriented triangle* $\Delta(C,C_1,C_2)$ of bisectors. We call the triangle *transversal* if these bisectors are transversal along their common slices. We denote by L the bisector from C_1 to C_2, by L_1 the bisector from C to C_1, and by L_2 the bisector from C_2 to C. Since $\prec L_1\succ$ and $\prec L_2\succ$ are transversal along C, either the angle from $L_1\succ$ to $\overline{L}_2\succ$ is less than π or the angle from $\overline{L}_2\succ$ to $L_1\succ$ is less than π. We denote by T the sector including the smaller angle. So, $T=K_1\cap K_2$ or $T=\overline{K}_1\cap\overline{K}_2$, respectively (we denote by K_i or by K the part of $\overline{\mathrm{B}}V$ on the side of the normal vector to $\prec L_i\succ$ or to $\prec L\succ$). We call T the *angle* at C and L, L_1, and L_2, the *sides*. Similarly, we define the angle T_i between L and L_i. By changing the orientation of the triangle, if needed, we can always assume that $T=K_1\cap K_2$. Let C' be one more projective line of signature $+-$ ultraparallel to C_1 and to C_2 such that the triangle $\Delta(C',C_2,C_1)$ is also transversal. We denote by L'_1 the bisector from C_1 to C' and by L'_2, the bisector from C' to C_2. We say that the triangle $\Delta(C',C_2,C_1)$ *suits* the triangle $\Delta(C,C_1,C_2)$ if some point in C' belongs to T and the bisectors $\prec L_i\succ$ and $\prec L'_i\succ$ are transversal along C_i for $i=1,2$.



6.3.1. Lemma. *Let $\Delta(C,C_1,C_2)$ be a transversal triangle oriented so that $T=K_1\cap K_2$, where T stands for the angle at C. Then the side L from C_1 to C_2 is included into T. Moreover, for the angle T_i at C_i, we have $T_i=K\cap K_i$. Suppose that some transversal triangle $\Delta(C',C_2,C_1)$ suits $\Delta(C,C_1,C_2)$. Then*[12] *$\Delta(C',C_2,C_1)\subset T$.*

Proof. Since L is connected and intersects $\prec L_i\succ$'s only in the C_i's, the side L is included into one of the four sectors formed by the $\prec L_i\succ$'s. The only sector that includes both C_i's is T.

Suppose that $T_1=\overline{K}\cap\overline{K}_1$. Then, by the above statement, we have $L_2\subset T_1$ which implies $L_2\subset\overline{K}_1$. On the other hand, $L_2\subset T\subset K_1$, a contradiction. The same works for T_2.

The bisector $\prec L'_1\succ$ intersects $\prec L_1\succ$ only in C_1. Hence, either $L'_1\succ\subset K_1$ or $L'_1\succ\subset\overline{K}_1$. Since some point of C' belongs to $T\subset K_1$ and $C'\subset L'_1\succ$, we conclude that $L'_1\succ\subset K_1$. By the same arguments, $\overline{L}'_2\succ\subset K_2$. From $L\subset T$, we conclude that $L\succ\subset K_1$ and $\overline{L}\succ\subset K_2$. Consequently, $T'_1\subset K_1$ and $T'_2\subset K_2$, where T'_i stands for the angle of $\Delta(C',C_2,C_1)$ at C_i. Since $L'_2\subset T'_1$ and $L'_1\subset T'_2$, we obtain $L'_2\subset K_1$ and $L'_1\subset K_2$. Now, from $L'_1\succ\subset K_1$ and $\overline{L}'_2\succ\subset K_2$, we deduce that $L'_i\subset T$ ■

If the cycle is transversal, then M_{k-1} and M_k are ultraparallel. In this case, assuming that the cycle possesses a centre C, we can form two triangles $\Delta_k\rightleftharpoons\Delta(C,M_{k-1},M_k)$ and $\Delta'_k\rightleftharpoons\Delta(S_k,M_k,M_{k-1})$.

It is difficult to decide whether two bisectors intersect (see, for instance, [San]). So, there is no good numerical criterion verifying the simplicity of a cycle. To a certain extent, transversality implies simplicity:

[12]Notice that $\Delta(C',C_2,C_1)$ can be 'inside' or 'outside' of $\Delta(C,C_1,C_2)$.

6.3.2. Criterion. *Let* $(B_1,\dots,B_n)$ *be a transversal cycle of bisectors possessing a centre* C. *For every* k, *we suppose that the triangles* Δ_k *and* Δ'_k *are transversal and that* Δ'_k *suits* Δ_k. *If the central angle is* 2π *and the angle from* $B'_{k-1}\succ$ *to* $B'_k\succ$ *is less than* π *for every* k, *then the cycle is simple.*[13]

Proof. By Lemma 6.2.4, the bisectors $B'_k\succ$ divide $\overline{\mathrm{B}}V$ into n sectors T'_k's. Clearly, $B_{k-1}\cap B'_{k-1}\succ=M_{k-1}$ and $B_k\cap B'_k\succ=M_k$. By Lemma 6.3.1, $B^1_{k-1}\cup B^0_k\subset T'_k$ ■

6.4. Fibred Polyhedra, Euler Number

We can weaken the condition $R_n\dots R_1=1$ to $R_n\dots R_1S_1=S_1$, obtaining the notion of a *configuration* of bisectors instead of the notion of a cycle of bisectors. A (transversal) triangle is an important example of a configuration. We can also define meridians for a configuration, however, they can be nonclosed. A configuration $(B_1,\dots,B_n)$ is said to be *simple* if $B_i\cap B_k=\varnothing$ for $k\ne i-1,i,i+1$ and $B_{i-1}\cap B_i=S_i$. (Notice that the latter requirement is weaker than the transversality of $\prec B_{i-1}\succ$ and $\prec B_i\succ$ along S_i.) Henceforth, we will also apply this notion of simplicity (which differs from the first one) to cycles. As above, if the configuration is simple, it divides $\mathrm{B}\,V$ into two closed connected polyhedra. We denote by P the polyhedron on the side of the normal vector to B_i's and introduce the solid torus ∂_0P fibred into slices and the torus T in the above way. We put $\partial_1P\rightleftharpoons P\cap\partial\,\mathrm{B}\,V$ and $\partial P\rightleftharpoons\partial_0P\cup\partial_1P$. Clearly, $T=\partial_0P\cap\partial_1P$.

6.4.1. Lemma. *Suppose that the polyhedron* P *related to a simple configuration of bisectors is a 4-ball,* $P\simeq\mathbb{B}^4$. *Then* P *is a disc bundle over a disc with* ∂_0P *being a union of entire fibres, i.e.,* $P\simeq\mathbb{B}^2\times\mathbb{B}^2$ *with* $\partial_0P\simeq\mathbb{S}^1\times\mathbb{B}^2$, *if and only if* ∂_1P *is a solid torus. In this case, the slice bundle of* ∂_0P *is extendable to a disc bundle of* P *over a disc.*

Proof is standard. If $P\simeq\mathbb{B}^2\times\mathbb{B}^2$, then $P\simeq\mathbb{B}^4$ and ∂P is a sphere $\mathbb{S}^3$ decomposed into two solid tori $\partial\mathbb{B}^2\times\mathbb{B}^2$ and $\mathbb{B}^2\times\partial\mathbb{B}^2$ glued along the torus $\partial\mathbb{B}^2\times\partial\mathbb{B}^2$. Hence, $\partial_0P\simeq\partial\mathbb{B}^2\times\mathbb{B}^2$ implies that $\partial_1P\simeq\mathbb{B}^2\times\partial\mathbb{B}^2$.

Conversely, if ∂_1P is a solid torus, then $\partial P\simeq\mathbb{S}^3$ is decomposed into two solid tori glued along the torus T. As is well known, such a decomposition of $\mathbb{S}^3$ is topologically unique and, arbitrarily fibering one of the solid tori into discs and extending the fibration for the other one, we obtain compatible decompositions $T\simeq\mathbb{S}^1\times\mathbb{S}^1$, $\partial_0P\simeq\mathbb{S}^1\times\mathbb{B}^2$, and $\partial_1P\simeq\mathbb{B}^2\times\mathbb{S}^1$. Since $P\simeq\mathbb{B}^4$ is a cone over $\partial P\simeq\mathbb{S}^3$, we can readily extend these decompositions to a compatible decomposition $P\simeq\mathbb{B}^2\times\mathbb{B}^2$ ■

In the situation described in Lemma 6.4.1, we will say that the polyhedron P is *fibred.* The Dehn Lemma immediately implies

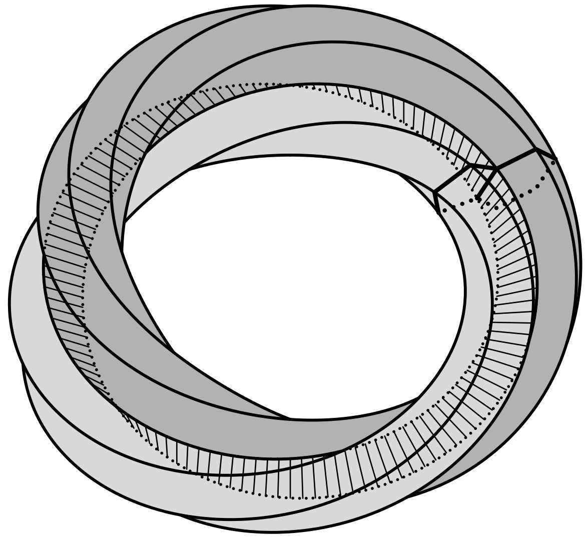
6.4.2. Remark. Let P be a polyhedron related to some simple configuration of bisectors. Then P is fibred if and only if there exists some simple closed curve $c\subset T$ contractible in ∂_1P such that c intersects each slice of ∂_0P exactly once.

We call such a curve c *trivializing.*

Let $(B_1,\dots,B_l,B_{l+1},\dots,B_n)$, $1\le l<n$, and $(B'_m,\dots,B'_{l+1}\overline{B}_l,\dots,\overline{B}_1)$, $1\le l<m$, be simple configurations of bisectors with a common sequence of bisectors (oriented in the opposite way in the other configuration) such that the polyhedra P_1 and P_2 related to the configurations intersect only in $B_1\cup\dots\cup B_l$. Then we can glue P_1 and P_2 along $B_1\cup\dots\cup B_l$, obtaining a *gluing* $P\rightleftharpoons P_1\cup P_2$, the polyhedron related to the simple configuration $(B_{l+1},\dots,B_n,B'_m,\dots,B'_{l+1})$. The polyhedra P and R_iP in Theorem 6.2.7 yield an example of such a gluing: they are glued along B_i. Suppose that P_1 and P_2 are fibred. Then the solid tori ∂_1P_1 and ∂_1P_2 intersect in the annulus $A\rightleftharpoons\partial B_1\cup\dots\cup\partial B_l$ which is an annular neighbourhood of a simple curve generating the fundamental group of each solid torus. Hence,

[13] This condition is not too restrictive, since only one of these angles can be more than π.

we can choose a trivializing curve c_i contractible in $\partial_1 P_i$ so that c_1 and c_2 coincide along A. Thus, we arrive at



6.4.3. Remark. Let P_1 and P_2 be fibred and let $P = P_1 \cup P_2$ be their gluing. Then P is fibred and a trivializing curve for P can be obtained by gluing (and removing the common part) some trivializing curves c_1 and c_2 for P_1 and for P_2 which coincide over the common sequence of bisectors.

Let P be a fibred polyhedron related to a simple cycle and let b be a meridian of the cycle. Clearly, there exists a simple disc $D \subset P$ which intersects each fibre in P exactly once and such that $b = \partial D$ and $\mathring{D} \subset \mathring{P}$. We assume that meridians and trivializing curves are oriented with respect to the orientation of the cycle and we also equip T with the following orientation: the first coordinate is the standardly oriented boundary of a slice and the second is an ideal meridian already oriented. The orientation of b orients D. We call D an *equivariant section* of the fibred polyhedron P. For another meridian b', $b \cap b' = \varnothing$, we can find an equivariant section D' with $\partial D' = b'$ and choose D and D' to be transversal. We call the number $eP \rightleftharpoons \#D \cap D'$ (the signs are taken into account) the *Euler number*[14] of the fibred polyhedron P. In fact, eP measures the difference between two decompositions of the slice bundle $\partial_0 P$ into the product $\mathbb{S}^1 \times \mathbb{B}^2$: the meridian decomposition and the decomposition induced by the trivialization $P \simeq \mathbb{B}^2 \times \mathbb{B}^2$. Obviously, the difference can be measured in terms of T :

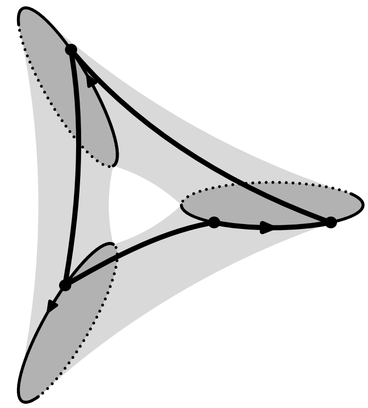
6.4.4. Remark. Let P be a fibred polyhedron related to a simple cycle. Then $eP = \#b \cap c$, where b stands for an ideal meridian and c, for a trivializing curve. In other words, $[b] = eP[s]$ in $\pi_1 \partial_1 P$ equipped with the generator $[s]$ represented by the boundary of a slice oriented naturally.

Summarizing, we obtain

6.4.5. Proposition. *Let P be a fibred polyhedron related to a simple transversal cycle of bisectors $(B_1, \dots, B_n)$ whose total angle equals 2π. Then $\mathrm{B}\,V/T_n$ for odd n ($\mathrm{B}\,V/G_n$ for even n) is a complex hyperbolic manifold homeomorphic to a disc bundle over a closed orientable surface of genus $n-3$ (for even n and for $\mathrm{B}\,V/G_n$, the genus equals $\frac{n}{2}-1$). The Euler number of the bundle equals $4eP$ (for even n and for $\mathrm{B}\,V/G_n$, the Euler number equals $2eP$).*

Proof summarizes previous results. We assume n to be odd. (The same arguments work for even n.) By Theorem 6.2.7, P is a fundamental domain for the discrete group $H \simeq H_n$ generated by R_i's. Using the arguments similar to those in the beginning of this section, we can see that $Q \rightleftharpoons P \cup R_1 P \cup R_2 P \cup R_2 R_1 P$ is a fundamental domain for T_n. By Remark 6.4.3, Q is a fibred polyhedron related to a suitable simple cycle $\mathcal{C}$ of bisectors. By Proposition 3.9, the meridians in P, $R_1 P$, $R_2 P$, and $R_2 R_1 P$ are glued along the ℓ-meridians of B_1, B_2, and $R_2 B_1$ forming (the common parts removed after gluing) the meridians of $\mathcal{C}$. Since T_n identifies the bisectors in $\mathcal{C}$ according to their slice fibration, by Lemma 6.4.1, $\mathrm{B}\,V/T_n$ is a disc bundle over a surface. These identifications glue any meridian with itself, hence, any equivariant section of Q generates some section of the bundle. Clearly, for an equivariant section D of P, the simple disc $D \cup R_1 D \cup R_2 D \cup R_2 R_1 D$ is an equivariant section of Q ■

[14]Standard arguments show that this number does not depend on the choice of the meridians b and b' and the equivariant sections D and D'.



6.5. Transversal Triangles. Let (B_1,B_2,B_3) be a transversal triangle of bisectors. By Lemma 6.3.1, we can assume that the triangle is oriented so that its angles are of the form $T_i=K_{i-1}\cap K_i$. We call such an orientation *counterclockwise.* By Corollary 3.10, the meridional identifications, first, along B_1, then along B_2, and, finally, along B_3, induce some isometry I_1 in S_1. We say that the triangle is *elliptic, parabolic, hyperbolic,* or *trivial* if I_1 is elliptic, parabolic, hyperbolic, or $I_1=1$. In the elliptic case, for any $s_1\in S_1$, following the counterclockwise orientation of the triangle, we can draw a meridian b beginning with s_1 and ending with some $s'_1\in S_1$. So, $I_1s_1=s'_1$. Let $s_1\in\partial S_1$. Then $s'_1\in\partial S_1$. Since ∂S_1 is a naturally oriented circle, following this orientation, from s'_1 to s_1, we can draw in ∂S_1 an arc a, obtaining a closed oriented curve $c\rightleftharpoons b\cup a\subset T$. We call this curve *standard.* In the case of a hyperbolic triangle, there are two fixed points for I_1 in ∂S_1. They divide ∂S_1 into two I_1-invariant parts: the R-*part* where I_1 moves the points in counterclockwise sense and the L-*part* where I_1 moves the points in the clockwise sense. Let $s_1\in\partial S_1$ be a point in the interior of the L-part. As above, we can draw a meridian b beginning with s_1 and ending with $s'_1=I_1s_1\in\partial S_1$ which is also in the L-part. We can draw an arc c from s'_1 to s_1 in the counterclockwise sense. Clearly, $c\subset\partial S_1$ is included in the L-part. We call the closed oriented curve $c\rightleftharpoons b\cup a\subset T$ *standard* as well. We notice that there are two closed meridians in T, both isotopic to a standard curve. For a parabolic I_1, we distinguish R-*parabolic* and L-*parabolic* cases. Exactly one point in ∂S_1 is fixed for I_1. The isometry I_1 moves all the other points in ∂S_1 in the same sense, counterclockwise for I_1 R-parabolic and clockwise for I_1 L-parabolic. As above, for an L-parabolic triangle, we define a *standard* curve. In T, this curve is isotopic to a closed meridian. In the case of an R-parabolic or trivial triangle, there is no standard curve.

We will also define L-*part* of ∂S_1 for the cases of I_1 elliptic, parabolic, or trivial. For elliptic I_1, it is all the ∂S_1. For L-parabolic I_1, it is ∂S_1 minus the fixed point. For the other two cases, it is empty.

In order to be able to prove that some polyhedron P is fibred and to calculate the Euler number eP, we need the following

6.5.1. Theorem. *Let (B_1,B_2,B_3) be a transversal triangle of bisectors oriented in counterclockwise sense. Then the triangle cannot be trivial nor* R-*parabolic. The polyhedron P is fibred and its standard curve is trivializing. In particular, any closed meridian in T is trivializing if it exists.*

6.5.2. Lemma. *Theorem* 6.5.1 *holds for any triangle of bisectors with common complex spine.*

Proof. In fact, any triangle of bisectors with common complex spine S is generated by a usual triangle $\Delta_0\rightleftharpoons\Delta(c_1,c_2,c_3)\subset\check S\simeq\mathbb H^2_{\mathbb R}$. By our convention, Δ_0 is oriented in counterclockwise sense, therefore, $\operatorname{Area}\Delta_0\in(0,\frac\pi4)$ (we remind that our metric in a complex geodesic is $\frac14$ of the usual one). By Remark 3.5, by the considerations above Remark 3.5, and by Proposition 3.9, I_1 is a rotation about c_1 by the angle $-2\operatorname{Area}\Delta_0$. Hence, the angular measure of the arc a (with respect to the centre c_1) is $\ell(a)=2\operatorname{Area}\Delta_0\in(0,\frac\pi2)$. We can contract Δ_0 inside of Δ_0. For instance, we can choose a point p in the interior of Δ_0 and define a triangle $\Delta_t\rightleftharpoons\Delta\big(c_1(t),c_2(t),c_3(t)\big)$, $t\in[0,1]$, where $c_i(t)\in\mathrm G[c_i,p]$, $c_i(0)=c_i$, and $c_i(1)=p$. We define $s_1(t)$ as obtained by the meridional displacement of s_1 along $B\big(c_1,c_1(t)\big)$ and generate with $s_1(t)$ a curve $c_t\rightleftharpoons b_t\cup a_t\subset T_t$. Considering the polyhedron P_t built over Δ_t, it is easy to see that $P_t\subset P_0$ and, hence, $\partial_1P_t\subset\partial_1P_0$. Since, by Corollary 3.7, the meridional displacement continuously depends on the choice of geodesics and $\ell(a_t)\to0$, the result follows ■

6.5.3. Criterion. *Let $p_i\notin\overline{\mathrm B}V$, $i=1,2,3$, be such that $\langle p_i,p_j\rangle\ne0$. We put $C_i\rightleftharpoons\mathbb{CP}p_i^\perp$, $t_{ij}^2\rightleftharpoons\operatorname{ta}(p_i,p_j)$, $t_{ij}>0$, $k\rightleftharpoons\dfrac{\langle p_1,p_2\rangle\langle p_2,p_3\rangle\langle p_3,p_1\rangle}{\langle p_1,p_1\rangle\langle p_2,p_2\rangle\langle p_3,p_3\rangle}$, $\varepsilon\rightleftharpoons\dfrac{k}{|k|}$, $\varepsilon_0\rightleftharpoons\operatorname{Re}\varepsilon$, $\varepsilon_1\rightleftharpoons\operatorname{Im}\varepsilon$, and $d\rightleftharpoons$*

$t_{12}^2+t_{23}^2+t_{31}^2-2t_{12}t_{23}t_{31}\varepsilon_0-1$. *Suppose*[15] *that* $1<t_{12}\le t_{23},t_{31}$. *Then* $d\ge0$, *where the equality means that* p_i*'s are in the same projective line. The triangle* $\Delta(C_1,C_2,C_3)$ *is transversal if and only if* $t_{12}^2\varepsilon_0^2+t_{23}^2+t_{31}^2<1+2t_{12}t_{23}t_{31}\varepsilon_0$. *The transversal triangle is oriented in counterclockwise sense if and only if* $\varepsilon_1<0$.

Proof is a straightforward verification. We can assume that $\langle p_i,p_i\rangle=1$, $\langle p_1,p_2\rangle=t_{12}$, $\langle p_2,p_3\rangle=t_{23}$, and $\langle p_3,p_1\rangle=t_{31}\varepsilon$. Now, $G=\begin{pmatrix}1&t_{12}&t_{31}\overline\varepsilon\\t_{12}&1&t_{23}\\t_{31}\varepsilon&t_{23}&1\end{pmatrix}$ is the Gram matrix for p_1, p_2, p_3. The first assertion follows from $\det G=-d$ by the Sylvester Criterion.

By Corollary 4.3, the transversality of the bisectors along C_3 can be written as $\Big|\operatorname{Re}\dfrac{t_{12}}{t_{31}\overline\varepsilon t_{23}}-1\Big|<\dfrac{\sqrt{t_{23}^2-1}\sqrt{t_{31}^2-1}}{t_{23}t_{31}}$ which is equivalent to $t_{12}^2\varepsilon_0^2+t_{23}^2+t_{31}^2<1+2t_{12}t_{23}t_{31}\varepsilon_0$. Similarly, the other two transversalities can be written as $t_{12}^2+t_{23}^2\varepsilon_0^2+t_{31}^2<1+2t_{12}t_{23}t_{31}\varepsilon_0$ and as $t_{12}^2+t_{23}^2+t_{31}^2\varepsilon_0^2<1+2t_{12}t_{23}t_{31}\varepsilon_0$. The last two inequalities follow from $t_{12}^2\varepsilon_0^2+t_{23}^2+t_{31}^2<1+2t_{12}t_{23}t_{31}\varepsilon_0$, since $t_{12}^2\le t_{23}^2,t_{31}^2$ and $0\le1-\varepsilon_0^2$.

By Lemma 6.3.1 and by Corollary 4.3 (and its proof), it suffices to measure the constant angle from $B[q_1,v_1)$ to $B[q_2,v_2)$, where v_i stands for the vertex of $B\wr p_3,p_i\wr$ which is closer to C_i than to C_3 and $q_i\in C_3$ stands for the point in the real spine of $B\wr p_3,p_i\wr$, $i=1,2$. By Theorem 4.2 and by Lemma 4.1, the angle in question equals $\operatorname{Arg}\Big(1-\dfrac{\langle p_1,p_2\rangle\langle p_3,p_3\rangle}{\langle p_1,p_3\rangle\langle p_3,p_2\rangle}\Big)=\operatorname{Arg}\Big(1-\dfrac{t_{12}\varepsilon}{t_{23}t_{31}}\Big)$ with $\dfrac{t_{12}}{t_{23}t_{31}}>0$ ■

6.5.4. Lemma. *Let* $p_i\notin\overline{\mathrm B}V$, $i=1,2,3$, *be such that* $\operatorname{ta}(p_i,p_j)>1$ *for* $i\ne j$. *Let* C_i, t_{ij}, k, ε, ε_0, *and* d *be defined as in Criterion* 6.5.3. *Denoting by* $R_i\in\mathrm{SU}(2,1)$ *the reflection in the middle slice of the respective side of the triangle* $\Delta(C_1,C_2,C_3)$, *we obtain* $\operatorname{tr}(R_3R_2R_1)=\varepsilon-(1+\overline\varepsilon)\Big(1+\dfrac{d}{(t_{12}+1)(t_{23}+1)(t_{31}+1)}\Big)$ *and* $R_3R_2R_1p_1=\varepsilon p_1$. *For the isometry* $I_1\in\mathrm{SU}(1,1)$ *of the slice* C_1 *induced by* $R_3R_2R_1$, *we have* $|\operatorname{tr}I_1|=\sqrt{2(1+\varepsilon_0)}\Big(1+\dfrac{d}{(t_{12}+1)(t_{23}+1)(t_{31}+1)}\Big)$.

Proof is straightforward. In terms of the proof of Criterion 6.5.3, we put $m_1\rightleftharpoons\dfrac{p_1+p_2}{\sqrt{2(t_{12}+1)}}$, $m_2\rightleftharpoons\dfrac{p_2+p_3}{\sqrt{2(t_{23}+1)}}$, and $m_3\rightleftharpoons\dfrac{\varepsilon p_1+p_3}{\sqrt{2(t_{31}+1)}}$. It is easy to verify that m_i is the polar point to the middle slice of the respective side of $\Delta(C_1,C_2,C_3)$ and that $\langle m_i,m_i\rangle=1$. Hence, $R_i=\varphi_i-1$, where $\varphi_ix\rightleftharpoons2\langle x,m_i\rangle m_i$. Clearly, $R_1p_1=p_2$, $R_2p_2=p_3$, and $R_3p_3=\varepsilon p_1$. Since $\varphi_j\varphi_ix=4\langle x,m_i\rangle\langle m_i,m_j\rangle m_j$ and $\varphi_k\varphi_j\varphi_ix=8\langle x,m_i\rangle\langle m_i,m_j\rangle\langle m_j,m_k\rangle m_k$, by considering the orthogonal decomposition $V=\mathbb Cm_i\dotplus m_i^\perp$, we obtain $\operatorname{tr}\varphi_i=2$, $\operatorname{tr}(\varphi_j\varphi_i)=4\langle m_i,m_j\rangle\langle m_j,m_i\rangle$, and $\operatorname{tr}(\varphi_k\varphi_j\varphi_i)=8\langle m_i,m_j\rangle\langle m_j,m_k\rangle\langle m_k,m_i\rangle$. It follows from $R_3R_2R_1=\varphi_3\varphi_2\varphi_1-\varphi_3\varphi_2-\varphi_3\varphi_1-\varphi_2\varphi_1+\varphi_3+\varphi_2+\varphi_1-1$ that $\operatorname{tr}(R_3R_2R_1)=8\langle m_1,m_2\rangle\langle m_2,m_3\rangle\langle m_3,m_1\rangle-4\langle m_2,m_3\rangle\langle m_3,m_2\rangle-4\langle m_1,m_3\rangle\langle m_3,m_1\rangle-4\langle m_1,m_2\rangle\langle m_2,m_1\rangle+3$. With a straightforward calculation, we derive

$$\langle m_1,m_2\rangle=\frac{1+t_{12}+t_{23}+t_{31}\overline\varepsilon}{2\sqrt{(t_{12}+1)(t_{23}+1)}},\ \langle m_2,m_3\rangle=\frac{1+t_{23}+t_{31}+t_{12}\overline\varepsilon}{2\sqrt{(t_{23}+1)(t_{31}+1)}},\ \langle m_3,m_1\rangle=\frac{\varepsilon+t_{31}\varepsilon+t_{12}\varepsilon+t_{23}}{2\sqrt{(t_{31}+1)(t_{12}+1)}},$$

$$8\langle m_1,m_2\rangle\langle m_2,m_3\rangle\langle m_3,m_1\rangle=\frac{(1+t_{12}+t_{23}+t_{31}\overline\varepsilon)(1+t_{23}+t_{31}+t_{12}\overline\varepsilon)(\varepsilon+t_{31}\varepsilon+t_{12}\varepsilon+t_{23})}{(t_{12}+1)(t_{23}+1)(t_{31}+1)}=$$

$$=2+\frac{(t_{12}^2+t_{23}^2+t_{31}^2-1)(2+t_{12}+t_{23}+t_{31})+t_{12}t_{23}t_{31}(1+\overline\varepsilon^2)}{(t_{12}+1)(t_{23}+1)(t_{31}+1)}+$$

[15] By Remark 2.3.3, this means that the C_i's are pairwise ultraparallel.

$$+\frac{(t_{12}t_{23}+t_{23}t_{31}+t_{31}t_{12}+t_{12}^2t_{23}+t_{12}t_{23}^2+t_{23}^2t_{31}+t_{23}t_{31}^2+t_{31}^2t_{12}+t_{31}t_{12}^2)\overline{\varepsilon}}{(t_{12}+1)(t_{23}+1)(t_{31}+1)}+$$
$$+\frac{(1+t_{12}+t_{23})(1+t_{23}+t_{31})(1+t_{31}+t_{12})\varepsilon}{(t_{12}+1)(t_{23}+1)(t_{31}+1)}=$$
$$=2+\frac{d(2+t_{12}+t_{23}+t_{31})+t_{12}t_{23}t_{31}(2+t_{12}+t_{23}+t_{31})(\varepsilon+\overline{\varepsilon})+t_{12}t_{23}t_{31}(1+\overline{\varepsilon}^2)}{(t_{12}+1)(t_{23}+1)(t_{31}+1)}+\varepsilon-\overline{\varepsilon}+$$
$$+(t_{12}+t_{23}+t_{31})(\varepsilon+\overline{\varepsilon})+\frac{(1-t_{12}^2-t_{23}^2-t_{31}^2)\overline{\varepsilon}-t_{12}t_{23}t_{31}(2+t_{12}+t_{23}+t_{31})(\varepsilon+\overline{\varepsilon})}{(t_{12}+1)(t_{23}+1)(t_{31}+1)}=$$
$$=2+2(t_{12}+t_{23}+t_{31})\varepsilon_0+\frac{d(t_{12}+t_{23}+t_{31}+3)}{(t_{12}+1)(t_{23}+1)(t_{31}+1)}+\varepsilon-\overline{\varepsilon}+$$
$$+\frac{-d+t_{12}t_{23}t_{31}(1+\overline{\varepsilon}^2)+(1-t_{12}^2-t_{23}^2-t_{31}^2)\overline{\varepsilon}}{(t_{12}+1)(t_{23}+1)(t_{31}+1)}=$$
$$=2+2(t_{12}+t_{23}+t_{31})\varepsilon_0+\frac{d(t_{12}+t_{23}+t_{31}+3)}{(t_{12}+1)(t_{23}+1)(t_{31}+1)}+\varepsilon-\overline{\varepsilon}-\frac{d(1+\overline{\varepsilon})}{(t_{12}+1)(t_{23}+1)(t_{31}+1)},$$
$$4\langle m_2,m_3\rangle\langle m_3,m_2\rangle-1=\frac{(t_{12}+1)\big((1+t_{23}+t_{31})^2+t_{12}^2+2t_{12}(1+t_{23}+t_{31})\varepsilon_0\big)}{(t_{12}+1)(t_{23}+1)(t_{31}+1)}-1=$$
$$=\frac{(t_{12}+1)\big(t_{23}+t_{31}+t_{23}t_{31}+t_{12}^2+t_{23}^2+t_{31}^2+2t_{12}(1+t_{23}+t_{31})\varepsilon_0\big)}{(t_{12}+1)(t_{23}+1)(t_{31}+1)}=$$
$$=1+\frac{(t_{12}+1)\big(d+2t_{12}(1+t_{23}+t_{31}+t_{23}t_{31})\varepsilon_0\big)}{(t_{12}+1)(t_{23}+1)(t_{31}+1)}=1+2t_{12}\varepsilon_0+\frac{d(t_{12}+1)}{(t_{12}+1)(t_{23}+1)(t_{31}+1)},$$
$$4\langle m_1,m_3\rangle\langle m_3,m_1\rangle=1+2t_{23}\varepsilon_0+\frac{d(t_{23}+1)}{(t_{12}+1)(t_{23}+1)(t_{31}+1)},$$
$$4\langle m_1,m_2\rangle\langle m_2,m_1\rangle=1+2t_{31}\varepsilon_0+\frac{d(t_{31}+1)}{(t_{12}+1)(t_{23}+1)(t_{31}+1)}.$$

This implies that $\operatorname{tr}(R_3R_2R_1)=\varepsilon-(1+\overline{\varepsilon})\Big(1+\dfrac{d}{(t_{12}+1)(t_{23}+1)(t_{31}+1)}\Big)$. Since $\det R_3R_2R_1=1$, $R_3R_2R_1p_1=\varepsilon p_1$, and $C_1=\mathbb{CP}p_1^\perp$, we arrive at $\operatorname{tr}I_1=\pm(\sqrt{\varepsilon}+\overline{\sqrt{\varepsilon}})\Big(1+\dfrac{d}{(t_{12}+1)(t_{23}+1)(t_{31}+1)}\Big)$ ∎

6.5.5. Lemma. *In terms of Lemma* 6.5.4*, let I_1 be parabolic or trivial, let $\varepsilon_1<0$, and let $t_{12}\varepsilon_0>1$. Then the triangle is* L*-parabolic.*

Proof. Let $q\in\partial C_1$ be a fixed point of $\varphi\rightleftharpoons R_3R_2R_1$ and let $p\rightleftharpoons{}^{p_1}p_2$. It follows from $t_{12}>1$ that $p\in\mathrm{B}\,V\cap C_1$. We can assume that $\langle p,q\rangle=1$. Since φ is parabolic or trivial on $\mathbb{C}p+\mathbb{C}q$, we have $\varphi(p)=up+uivq$ and $\varphi(q)=uq$ for some $u,v\in\mathbb{C}$, $u\neq0$. Since $\varphi\in\mathrm{SU}(2,1)$, $\langle p,q\rangle=1$, and $\langle q,q\rangle=0$, we obtain $1=\langle p,q\rangle=\big\langle\varphi(p),\varphi(q)\big\rangle=|u|^2$ and $\langle p,p\rangle=\big\langle\varphi(p),\varphi(p)\big\rangle=|u|^2\big(\langle p,p\rangle-i\overline{v}+iv\big)=\langle p,p\rangle-2\operatorname{Im}v$. Hence, $v\in\mathbb{R}$.

For $p'\rightleftharpoons\langle p,p\rangle q-p$, we have $\langle p',p'\rangle=-\langle p,p\rangle$ and $\langle p,p'\rangle=0$. Therefore, $p(z)\rightleftharpoons zp'+p\in\overline{\mathrm{B}}V$, $z\in\mathbb{C}$, if and only if $|z|\le1$. Clearly, $p(1)\simeq q$ (where $\simeq$ means $\mathbb{C}$-proportionality). In terms of z,

$$\varphi\big(p(z)\big)\simeq z\langle p,p\rangle q+(1-z)(p+ivq)=\frac{z\langle p,p\rangle+(1-z)iv}{\langle p,p\rangle}\big(\langle p,p\rangle q-p\big)+\frac{\langle p,p\rangle+(1-z)iv}{\langle p,p\rangle}p\simeq p(z'),$$

where $z'\rightleftharpoons\dfrac{z\langle p,p\rangle+(1-z)iv}{\langle p,p\rangle+(1-z)iv}$. (Since $|z|\le1$, we have $\langle p,p\rangle+(1-z)iv\neq0$.) In particular, $\varphi\big(p(-1)\big)\simeq p(z_0)$, where $z_0\rightleftharpoons\dfrac{2iv-\langle p,p\rangle}{2iv+\langle p,p\rangle}$ and $\operatorname{Im}z_0=\dfrac{4v\langle p,p\rangle}{\big|2iv+\langle p,p\rangle\big|^2}$. Thus, the triangle is L-parabolic (trivial) if and only if $v<0$ $(v=0)$.

As is easy to see, $v=\frac{1}{i}\Big(u^{-1}\big\langle\varphi(p),p\big\rangle-\langle p,p\rangle\Big)$. So, we have $v=\operatorname{Im}\Big(u^{-1}\big\langle\varphi(p),p\big\rangle\Big)$. By Lemma 6.5.4, $\varepsilon+2u=\varepsilon-(1+\overline{\varepsilon})\Big(1+\frac{d}{(t_{12}+1)(t_{23}+1)(t_{31}+1)}\Big)$. Since $d\geq 0$, the triangle is L-parabolic (trivial) if and only if $\operatorname{Im}\Big((1+\varepsilon)\big\langle\varphi(p),p\big\rangle\Big)>0$ $(=0)$.

Using the expressions for m_i's obtained in the proof of Lemma 6.5.4, we have

$$R_1(x)=\langle x,p_1+p_2\rangle\frac{p_1+p_2}{t_{12}+1}-x,\ R_2(x)=\langle x,p_2+p_3\rangle\frac{p_2+p_3}{t_{23}+1}-x,\ R_3(x)=\langle x,\varepsilon p_1+p_3\rangle\frac{\varepsilon p_1+p_3}{t_{31}+1}-x.$$

Hence, $R_1(p_2)=p_1$, $R_2(p_1)=\frac{t_{12}+t_{31}\overline{\varepsilon}}{t_{23}+1}\,(p_2+p_3)-p_1$, and

$$\varphi(p_2)=\Big\langle\frac{t_{12}+t_{31}\overline{\varepsilon}}{t_{23}+1}\,(p_2+p_3)-p_1,\,\varepsilon p_1+p_3\Big\rangle\frac{\varepsilon p_1+p_3}{t_{31}+1}-\frac{t_{12}+t_{31}\overline{\varepsilon}}{t_{23}+1}\,(p_2+p_3)+p_1=$$
$$=\Big(\frac{t_{12}+t_{31}\overline{\varepsilon}}{t_{23}+1}\,(t_{12}\overline{\varepsilon}+t_{23}+t_{31}+1)-\overline{\varepsilon}-t_{31}\overline{\varepsilon}\Big)\frac{\varepsilon p_1+p_3}{t_{31}+1}-\frac{t_{12}+t_{31}\overline{\varepsilon}}{t_{23}+1}\,(p_2+p_3)+p_1=$$
$$=\frac{(t_{12}+t_{31}\overline{\varepsilon})(t_{12}\overline{\varepsilon}+t_{31})+(t_{23}+1)(t_{12}-\overline{\varepsilon})}{(t_{23}+1)(t_{31}+1)}\,(\varepsilon p_1+p_3)-\frac{t_{12}+t_{31}\overline{\varepsilon}}{t_{23}+1}\,(p_2+p_3)+p_1.$$

Since $p=p_2-t_{12}p_1\in p_1^{\perp}$ and $\varphi(p_1)=\varepsilon p_1$, we obtain

$$(t_{23}+1)(t_{31}+1)\big\langle\varphi(p),p\big\rangle=(t_{23}+1)(t_{31}+1)\big\langle\varphi(p_2),p_2-t_{12}p_1\big\rangle=$$
$$=(t_{23}+1)(t_{31}+1)\Big\langle\frac{(t_{12}+t_{31}\overline{\varepsilon})(t_{12}\overline{\varepsilon}+t_{31})+(t_{23}+1)(t_{12}-\overline{\varepsilon})}{(t_{23}+1)(t_{31}+1)}\,p_3-\frac{t_{12}+t_{31}\overline{\varepsilon}}{t_{23}+1}\,(p_2+p_3),\,p_2-t_{12}p_1\Big\rangle=$$
$$=(t_{12}+t_{31}\overline{\varepsilon})(t_{12}\overline{\varepsilon}+t_{31})(t_{23}-t_{12}t_{31}\varepsilon)+(t_{23}+1)(t_{12}-\overline{\varepsilon})(t_{23}-t_{12}t_{31}\varepsilon)-$$
$$-(t_{12}+t_{31}\overline{\varepsilon})(t_{31}+1)(1-t_{12}^2+t_{23}-t_{12}t_{31}\varepsilon)=$$
$$=t_{12}t_{23}t_{31}\overline{\varepsilon}^2+(t_{12}^2t_{23}+t_{12}^2t_{31}-t_{23}^2-t_{23}t_{31}-t_{31}^2-t_{23}-t_{31})\overline{\varepsilon}+$$
$$+t_{12}^3+t_{12}t_{23}^2+t_{12}t_{23}t_{31}+t_{12}t_{31}^2-t_{12}-t_{12}^2t_{23}t_{31}\varepsilon.$$

Hence,

$$(t_{23}+1)(t_{31}+1)(1+\varepsilon)\big\langle\varphi(p),p\big\rangle=t_{12}t_{23}t_{31}\overline{\varepsilon}^2+(t_{12}^2t_{23}+t_{12}^2t_{31}+t_{12}t_{23}t_{31}-t_{23}^2-t_{23}t_{31}-t_{31}^2-$$
$$-t_{23}-t_{31})\overline{\varepsilon}+t_{12}^3+t_{12}^2t_{23}+t_{12}^2t_{31}+t_{12}t_{23}^2+t_{12}t_{23}t_{31}+t_{12}t_{31}^2-t_{23}^2-t_{23}t_{31}-t_{31}^2-t_{12}-t_{23}-t_{31}+$$
$$+(t_{12}^3+t_{12}t_{23}^2+t_{12}t_{23}t_{31}+t_{12}t_{31}^2-t_{12}-t_{12}^2t_{23}t_{31})\varepsilon-t_{12}^2t_{23}t_{31}\varepsilon^2$$

and, since $\operatorname{Im}\varepsilon^2=2\varepsilon_0\varepsilon_1$,

$$\operatorname{Im}\Big((t_{23}+1)(t_{31}+1)(1+\varepsilon)\big\langle\varphi(p),p\big\rangle\Big)=$$
$$=(t_{12}+1)\varepsilon_1\big(t_{12}^2+t_{23}^2+t_{31}^2-2t_{12}t_{23}t_{31}\varepsilon_0-1-(t_{12}-1)(t_{23}+1)(t_{31}+1)\big)=$$
$$=(t_{12}+1)\varepsilon_1\big(d-(t_{12}-1)(t_{23}+1)(t_{31}+1)\big).$$

By Lemma 6.5.4, $\sqrt{2(1+\varepsilon_0)}\Big(1+\frac{d}{(t_{12}+1)(t_{23}+1)(t_{31}+1)}\Big)=2$, implying that

$$d=\Big(\frac{2}{\sqrt{2(1+\varepsilon_0)}}-1\Big)(t_{12}+1)(t_{23}+1)(t_{31}+1).$$

We conclude that the triangle is L-parabolic (trivial) if and only if $\varepsilon_1\Big(\frac{t_{12}+1}{\sqrt{2(1+\varepsilon_0)}}-t_{12}\Big)>0$ $(=0)$. This follows from $\varepsilon_1<0$, $t_{12}\varepsilon_0>1$, and $t_{12}>1$ ■

6.5.6. Lemma. *The inequalities*

$$\textbf{(6.5.7)}\qquad 1<t_1\leq t_2\leq t_3,\qquad t_1^2e^2+t_2^2+t_3^2<2t_1t_2t_3e+1\leq t_1^2+t_2^2+t_3^2$$

imply the inequalities $t_1e>1>e>0$. Every point in the region Z of $\mathbb{R}^4$ given by the inequalities (6.5.7) *is path-connected with the part of Z given by the equality $2t_1t_2t_3e+1=t_1^2+t_2^2+t_3^2$.*

Proof. Clearly, the inequalities (6.5.7) imply $e^2<1$ and, hence, $t_1e<t_2t_3$. Now, the inequality $t_1^2e^2+t_2^2+t_3^2<2t_1t_2t_3e+1$ can be rewritten as $(t_1e-t_2t_3)^2<(t_2^2-1)(t_3^2-1)$ and thus is equivalent to the inequalities $t_2t_3-\sqrt{(t_2^2-1)(t_3^2-1)}<t_1e\le t_2t_3$. It follows from $t_2t_3>1$ that $1\le t_2t_3-\sqrt{(t_2^2-1)(t_3^2-1)}$ which implies that $t_1e>1$ and, hence, $1>e>0$.

Dealing with the inequalities

$$(\mathbf{6.5.8})\qquad 1<t_1\le t_2\le t_3,\qquad t_2t_3-\sqrt{(t_2^2-1)(t_3^2-1)}<t_1e\le t_2t_3,\qquad 2t_1t_2t_3e+1\le t_1^2+t_2^2+t_3^2$$

equivalent to (6.5.7), we will increase t_1. Our inequalities imply that $t_1e>1$. Consequently, $t_2<t_2t_3e$, which means that the function $f(x)\rightleftharpoons x^2+t_2^2+t_3^2-2xt_2t_3e-1$ is decreasing for $x\in[t_1,t_2]$. Increasing t_1 and preserving the inequalities (6.5.8), we can reach a position where either $t_1=t_2$ or $f(t_1)=0$. The latter means that $2t_1t_2t_3e+1=t_1^2+t_2^2+t_3^2$.

In the first case, our inequalities are $1<t_1=t_2\le t_3$ and $2t_1^2(t_3e-1)\le t_3^2-1<t_1^2(2t_3e-e^2-1)$. As we know, they imply that $t_1e>1>e>0$. It follows that $t_3e-1>0$ and $2t_3e-e^2-1>0$. Now, we will increase $t_1=t_2$ preserving our inequalities. We will come to a position where either $t_1=t_2=t_3$ or $2t_1^2(t_3e-1)=t_3^2-1$. The latter means that $2t_1t_2t_3e+1=t_1^2+t_2^2+t_3^2$.

For the case $t_1=t_2=t_3$, our inequalities are $1<t_1=t_2=t_3$, $(t_1^2-t_1e)^2<(t_1^2-1)^2$, and $e\le\dfrac{3t_1^2-1}{2t_1^3}$ and they imply $t_1>e>0$. Therefore, they are equivalent to $1<t_1=t_2=t_3>e>0$ and $1<t_1e\le\dfrac{3t_1^2-1}{2t_1^2}$. It follows from $\dfrac{3t_1^2-1}{2t_1^3}<t_1$ that, preserving our inequalities, we can increase e until $e=\dfrac{3t_1^2-1}{2t_1^3}$. Again, $2t_1t_2t_3e+1=t_1^2+t_2^2+t_3^2$ ■

Proof of Theorem 6.5.1. For given numbers $t_{12},t_{23},t_{31},\varepsilon_0\in\mathbb{R}$ satisfying the inequalities $1<t_{12}\le t_{23},t_{31}$ and $t_{12}^2\varepsilon_0^2+t_{23}^2+t_{31}^2<1+2t_{12}t_{23}t_{31}\varepsilon_0\le t_{12}^2+t_{23}^2+t_{31}^2$, by the Sylvester Criterion, by Criterion 6.5.3, and by Lemma 6.5.6, there always exists a transversal triangle of bisectors oriented in counterclockwise sense with the invariants $t_{12},t_{23},t_{31},\varepsilon\rightleftharpoons\varepsilon_0-i\sqrt{1-\varepsilon_0^2}$ (this triangle is geometrically unique) and vice versa: these invariants of an arbitrary transversal triangle of bisectors oriented in counterclockwise sense (with a suitable choice of t_{12}) satisfy the above inequalities. Moreover, the equality $1+2t_{12}t_{23}t_{31}\varepsilon_0=t_{12}^2+t_{23}^2+t_{31}^2$ means that the triangle is one of those we dealt with in Lemma 6.5.2. By Lemmas 6.5.6 and 6.5.5, the triangle cannot be trivial nor R-parabolic. Therefore, it possesses a standard curve.

It is easy to see that, when continuously varying the parameters $t_{12},t_{23},t_{31},\varepsilon_0$ and preserving the above inequalities, we can continuously change the corresponding triangle. Moreover, by Lemmas 6.5.5 and 6.5.6, we can supply each triangle with some standard curve that will vary continuously during the deformation. In other words, we obtain a continuous deformation of a simple torus $T\subset\partial\,\mathrm{B}\,V$ equipped with a simple curve. By Lemmas 6.5.6 and 6.5.2, we can assume that, for the final torus, its standard curve is contractible in ∂_1P. By standard topological arguments, the initial polyhedron ∂_1P is a solid torus and its standard curve is contractible.

One can detail the above deformation in more explicit terms. For p_1, p_2, and p_3 of an initially given triangle, we will assume that $\langle p_i,p_i\rangle=1$, $\langle p_1,p_2\rangle=t_{12}$, $\langle p_2,p_3\rangle=t_{23}$, and $\langle p_3,p_1\rangle=t_{31}\varepsilon$. We put $u\rightleftharpoons\dfrac{\big(\sqrt{t_{12}^2-1}-t_{12}\big)p_1+p_2}{\sqrt{t_{12}^2-1}}$. It is easy to see that $\langle u,u\rangle=0$, $\langle p_1,u\rangle=1$, and $p_2=\big(t_{12}-\sqrt{t_{12}^2-1}\big)p_1+\sqrt{t_{12}^2-1}\,u$. We choose $p\in V$ such that $\langle p,p_1\rangle=0$, $\langle p,u\rangle=0$, and $\langle p,p\rangle=1$. It follows from $\langle p_2,p_3\rangle = t_{23}$ and $\langle p_3,p_1\rangle = t_{31}\varepsilon$ that $p_3 = \Big(t_{31}\varepsilon + \dfrac{t_{23}-t_{12}t_{31}\varepsilon}{\sqrt{t_{12}^2-1}}\Big)p_1 + \dfrac{t_{12}t_{31}\varepsilon - t_{23}}{\sqrt{t_{12}^2-1}}u + rp$ for some $r\in\mathbb{C}$. We can take a representative for p so that $r\ge 0$ and $\langle p,p\rangle = 1$. During the deformation, p_1, u, and p will be fixed.

For given parameters e,t_1,t_2,t_3 satisfying inequalities (6.5.7), we define $\varepsilon' \risingdotseq e - i\sqrt{1-e^2}$, $p_1' \risingdotseq p_1$, $p_2' \risingdotseq \big(t_1 - \sqrt{t_1^2-1}\big)p_1 + \sqrt{t_1^2-1}\,u$, and $p_3' \risingdotseq \Big(t_3\varepsilon + \dfrac{t_2 - t_1t_3\varepsilon'}{\sqrt{t_1^2-1}}\Big)p_1 + \dfrac{t_1t_3\varepsilon' - t_2}{\sqrt{t_1^2-1}}u + r'p$, where $r'\ge 0$ can be determined from the equality $\langle p_3',p_3'\rangle = 1$. The existence of such an r' is guaranteed by the inequality $2t_1t_2t_3e + 1 \le t_1^2+t_2^2+t_3^2$. While varying t_1,t_2,t_3,e subject to inequalities (6.5.7), the triangle determined by p_1', p_2', and p_3' changes continuously. So do the isometry I_1 of C_1, the nonordered pair of its fixed points, and the L-part of ∂C_1 (nonempty by Lemmas 6.5.5 and 6.5.6). In order to define a standard curve continuously varying during the deformation, due to Corollary 3.7, we only need to choose the initial point $s_1\in\partial C_1$ of a standard curve varying continuously during the deformation. In the initial triangle, we choose s_1 in the interior of the L-part. All we need to be careful with is to avoid s_1 being a fixed point of I_1 during the deformation. Then it will automatically be in the interior of the L-part during all the deformation ■

7. Examples

7.1. Basic Examples. In this section, we will construct an explicit series of discrete groups with defining relations $U^n = V^n = (V^{-1}U)^2 = 1$. Taking a subgroup of index $4n$ (or of index $2n$ if n is even), we will arrive at a disc bundle over a closed orientable surface of genus $n-3$ ($\frac{n}{2}-1$, respectively).

From now on, all isometries will 'live' in $\mathrm{SU}(2,1)$. We will look for two rotations $U,V\in\mathrm{SU}(2,1)$ such that $V^{-1}U = R$ is a reflection in a projective line of signature $+-$, assuming that each rotation has pairwise different eigenvalues. In order to diminish the number of conditions, we will explore the symmetry of interchanging U and V.

In some orthonormal basis, $V = \begin{pmatrix} v_1^2 & 0 & 0\\ 0 & v_2^2 & 0\\ 0 & 0 & v_3^2\end{pmatrix}$, where $|v_i| = 1$ and $v_1v_2v_3 = 1$. We denote by u_1^2,u_2^2,u_3^2 the eigenvalues of U subject to $|u_i| = 1$ and $u_1u_2u_3 = 1$. Let q_i denote the eigenvector of V corresponding to v_i^2 and s_i, the eigenvector of U corresponding to u_i^2. We will assume that $q_1\in\mathrm{B}\,V$. So, we require

$$q_1\in\mathrm{B}\,V,\qquad |u_i| = |v_i| = u_1u_2u_3 = v_1v_2v_3 = 1,\qquad u_1^2\ne u_2^2\ne u_3^2\ne u_1^2,\qquad v_1^2\ne v_2^2\ne v_3^2\ne v_1^2. \quad (1)$$

The reflection $R(m) = V^{-1}U$ is given by some $m\notin\overline{\mathrm{B}}V$, $\langle m,m\rangle = 1$. Multiplying, if needed, the elements of the orthonormal basis for V by unitary complex numbers, we can assume that the coordinates of m are real nonnegative: $m = \begin{pmatrix} m_1\\ m_2\\ m_3\end{pmatrix}$, $m_i\ge 0$, and $m_1^2 = m_2^2 + m_3^2 - 1$. As is easy to see, $R(m) = \begin{pmatrix} -2m_1^2-1 & 2m_1m_2 & 2m_1m_3\\ -2m_1m_2 & 2m_2^2-1 & 2m_2m_3\\ -2m_1m_3 & 2m_2m_3 & 2m_3^2-1\end{pmatrix}$ in the basis $q_1,q_2,q_3\in V$. By [Gol, Theorem 6.2.4, p. 204],[16] the condition that $VR(m)$ is a rotation with eigenvalues u_1^2,u_2^2,u_3^2 is equivalent to the equality $\mathrm{tr}\,\big(VR(m)\big) = u_1^2+u_2^2+u_3^2$, i.e., to the equality $(v_2^2 - v_1^2)m_2^2 + (v_3^2 - v_1^2)m_3^2 = w - v_1^2$, where $w \risingdotseq \dfrac{u_1^2+u_2^2+u_3^2+v_1^2+v_2^2+v_3^2}{2}$.

Since the v_i^2's are pairwise distinct, the equation $(v_2^2-v_1^2)y_2 + (v_3^2-v_1^2)y_3 = w - v_1^2$ in y_2 and y_3 has a unique real solution (y_2 or y_3 can however be negative). Let us verify that $y_2 \risingdotseq \dfrac{\mathrm{Re}\,\big(v_2(v_1^2-w)\big)}{\mathrm{Re}\,\big(v_2(v_1^2-v_2^2)\big)}$ and $y_3 \risingdotseq \dfrac{\mathrm{Re}\,\big(v_3(v_1^2-w)\big)}{\mathrm{Re}\,\big(v_3(v_1^2-v_3^2)\big)}$ yield a solution. Indeed,

[16] The curve in [Gol, p. 205] is a deltoid.

$$y_2=\frac{v_2(w-v_1^2)+\overline{v}_2(\overline{w}-\overline{v}_1^2)}{v_2(v_2^2-v_1^2)+\overline{v}_2(\overline{v}_2^2-\overline{v}_1^2)}=\frac{\big(v_2^2(w-v_1^2)+(\overline{w}-\overline{v}_1^2)\big)v_1^2v_2^2}{v_1^2v_2^4(v_2^2-v_1^2)+(v_1^2-v_2^2)}=\frac{v_1^2v_2^4(w-v_1^2)+v_1^2v_2^2(\overline{w}-\overline{v}_1^2)}{(v_1^2v_2^4-1)(v_2^2-v_1^2)}.$$

Similarly, $y_3=\dfrac{v_1^2v_3^4(w-v_1^2)+v_1^2v_3^2(\overline{w}-\overline{v}_1^2)}{(v_1^2v_3^4-1)(v_3^2-v_1^2)}$. Since $v_1v_2v_3=1$ implies $\dfrac{v_1^2v_2^2}{v_1^2v_2^4-1}+\dfrac{v_1^2v_3^2}{v_1^2v_3^4-1}=0$ and $\dfrac{v_1^2v_2^4}{v_1^2v_2^4-1}+\dfrac{v_1^2v_3^4}{v_1^2v_3^4-1}=1$, we obtain $(v_2^2-v_1^2)y_2+(v_3^2-v_1^2)y_3=w-v_1^2$.

We require

$$\operatorname{Re}(v_2w)<\operatorname{Re}v_2^3<\operatorname{Re}(v_2v_1^2),\qquad \operatorname{Re}v_3^3<\operatorname{Re}(v_3v_1^2)\geq\operatorname{Re}(v_3w). \tag{2}$$

It follows from (2) that $y_2>1$ and $y_3\geq 0$. Hence, conditions (1–2) guarantee the existence and the uniqueness of $m\notin\overline{\mathrm{B}}V$ and U such that $VR(m)=U$. Moreover, by (2), $\operatorname{ta}(m,q_2)>1$ and $\langle m,m\rangle=1$ for $m_1,m_2,m_3\geq 0$ given by

$$m_2\risingdotseq\sqrt{\frac{\operatorname{Re}\big(v_2(v_1^2-w)\big)}{\operatorname{Re}\big(v_2(v_1^2-v_2^2)\big)}},\qquad m_3\risingdotseq\sqrt{\frac{\operatorname{Re}\big(v_3(v_1^2-w)\big)}{\operatorname{Re}\big(v_3(v_1^2-v_3^2)\big)}},\qquad m_1\risingdotseq\sqrt{m_2^2+m_3^2-1}.$$

For the reason of symmetry between U and V, we will also require

$$\operatorname{Re}(u_2w)<\operatorname{Re}u_2^3<\operatorname{Re}(u_2u_1^2),\qquad \operatorname{Re}u_3^3<\operatorname{Re}(u_3u_1^2). \tag{3}$$

Since $V^{-1}U=R(m)$ implies $U^{-1}V=R(m)$, assuming that $s_1\in\mathrm{B}V$, we deduce from (1–3) that $\operatorname{ta}(m,s_2)>1$ and that $\operatorname{Re}(u_3u_1^2)\geq\operatorname{Re}(u_3w)$. Consequently, conditions (1–3) are symmetric under the assumption that $s_1\in\mathrm{B}V$.

We require the conditions

$$u_1^2+v_3^2\neq 0,\qquad m_2^2\frac{\operatorname{Re}\big(u_1^{-2}(v_2^2-v_1^2)\big)}{1+\operatorname{Re}(u_1^{-2}v_2^2)}+m_3^2\frac{\operatorname{Re}\big(u_1^{-2}(v_3^2-v_1^2)\big)}{1+\operatorname{Re}(u_1^{-2}v_3^2)}\geq 1. \tag{4}$$

7.1.1. Lemma. *If $u_i^2+v_j^2\neq 0$ for some i and for all j, then $s_i=\begin{pmatrix}\frac{m_1}{u_i^2v_1^{-2}+1}\\ \frac{m_2}{u_i^2v_2^{-2}+1}\\ \frac{m_3}{u_i^2v_3^{-2}+1}\end{pmatrix}$. The point s_1 belongs to $\mathrm{B}V$. If $u_i^2+v_j^2=0$ for some i and j, then either $i=j=1$, $m_1=0$, and $s_1=\begin{pmatrix}1\\0\\0\end{pmatrix}$ or $i=j=3$, $m_3=0$, and $s_3=\begin{pmatrix}0\\0\\1\end{pmatrix}$.*

Proof. Until some moment, we will not use conditions (4). Let $x=\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}\neq 0$. We fix some $i=1,2,3$. The condition that x is the eigenvector of $VR(m)$ corresponding to u_i^2 is given by the equalities $2v_j^2m_jf(x)=(u_i^2+v_j^2)x_j$, $j=1,2,3$, where $f(x)\risingdotseq -m_1x_1+m_2x_2+m_3x_3$. Thus, if $u_i^2+v_j^2\neq 0$ for all j, then $f(x)\neq 0$ and we can take $x_j=\dfrac{m_j}{u_i^2v_j^{-2}+1}$, $j=1,2,3$.

If $u_i^2+v_j^2=0$ for some j and $f(x)\neq 0$, then $m_j=0$, $f(x)$ does not depend on x_j, and we can arbitrarily choose x_j, keeping x an eigenvector of U. Would x_k be different from 0 for some $k\neq j$, we could obtain too many eigenvectors corresponding to u_i^2. The eigenvalues of U are pairwise different. Hence, $x_k=0$ for any $k\neq j$, implying $f(x)=0$, a contradiction. Therefore, $u_i^2+v_j^2=0$ implies $f(x)=0$. In this case, $u_i^2+v_k^2\neq 0$ for any $k\neq j$. Also, we have $x_k=0$ for any $k\neq j$ and $x_j\neq 0$. Now, $f(x)=0$ implies $m_j=0$. In other words, $u_i^2+v_j^2=0$ implies $s_i=q_j$ and $m_j=0$. Since, by (2), $m_2>1$, this can only happen for $j=1,3$.

Now, with the use of conditions (4), we will prove that $s_1 \in \mathrm{B}V$. Suppose that $u_1^2 + v_j^2 = 0$ for some j. It follows from (4) that $j \neq 3$. Hence, $j = 1$ and $s_1 = q_1 \in \mathrm{B}V$. So, we can assume that $u_1^2 + v_j^2 \neq 0$ for all j and take $s_1 = \begin{pmatrix} \frac{m_1}{u_1^2 v_1^{-2}+1} \\ \frac{m_2}{u_1^2 v_2^{-2}+1} \\ \frac{m_3}{u_1^2 v_3^{-2}+1} \end{pmatrix}$. The condition that $s_1 \in \mathrm{B}V$ (it suffices to prove only that $s_1 \in \overline{\mathrm{B}}V$) has the form $-\dfrac{m_1^2}{|u_1^2 v_1^{-2}+1|^2} + \dfrac{m_2^2}{|u_1^2 v_2^{-2}+1|^2} + \dfrac{m_3^2}{|u_1^2 v_3^{-2}+1|^2} \leq 0$. In view of the equalities $m_1^2 = m_2^2 + m_3^2 - 1$ and $|u_1^2 v_k^{-2} + 1|^2 = 2\big(1 + \mathrm{Re}(u_1^{-2} v_k^2)\big)$ for $k = 1,2,3$, the last inequality is equivalent to $m_2^2 \dfrac{\mathrm{Re}\,\big(u_1^{-2}(v_2^2 - v_1^2)\big)}{1 + \mathrm{Re}(u_1^{-2} v_2^2)} + m_3^2 \dfrac{\mathrm{Re}\,\big(u_1^{-2}(v_3^2 - v_1^2)\big)}{1 + \mathrm{Re}(u_1^{-2} v_3^2)} \geq 1$.

The conditions symmetric to (1–3) follows from (1–4), since the latter imply $s_1 \in \mathrm{B}V$. We derived from (1–3) that the equality $u_i^2 + v_j^2 = 0$ implies $s_i = q_j$ and $j \neq 2$. By symmetry, it implies $i \neq 2$. Since q_k or s_k belong to $\mathrm{B}V$ only for $k = 1$, the proof is complete ■

We claim that conditions (1–4) are symmetric (in particular, they imply that $u_2^2 + v_j^2 \neq 0$ for $j = 1,2,3$). Indeed, by Lemma 7.1.1, $v_1^2 + u_3^2 \neq 0$. In the proof of Lemma 7.1.1, we deduced the conditions symmetric to (1–3). The second inequality in (4) expresses the fact that $s_1 \in \mathrm{B}V$ in the case of $u_1^2 + v_j^2 \neq 0$ for all j. By Lemma 7.1.1, the symmetric inequality is valid unless $u_1^2 + v_1^2 = 0$. If $u_1^2 + v_1^2 = 0$, then $m_1 = 0$, i.e., $m_2^2 + m_3^2 = 1$, and the last inequality in (4) is automatically valid.

By (2–3) and by Remark 2.3.3, $M_1 \rightleftharpoons \mathbb{CP}m^\perp$ is ultraparallel to $C \rightleftharpoons \mathbb{CP}q_2^\perp$ and M_1 is ultraparallel to $S_2 \rightleftharpoons \mathbb{CP}s_2^\perp$. It follows from $R(m)M_1 = M_1$, $VC = C$, and $US_2 = S_2$ that

$$M_2 \rightleftharpoons VM_1 = UM_1$$

is ultraparallel to C and to S_2. Clearly, $\mathrm{ta}(C, M_1) = \mathrm{ta}(C, M_2) = m_2^2$ and $\mathrm{ta}(S_2, M_1) = \mathrm{ta}(S_2, M_2)$.

We require

$$\mathrm{Re}(v_2^2 \overline{w}) > 1, \qquad \mathrm{Re}(u_2^2 \overline{w}) > 1. \tag{5}$$

Since $\langle Vm, m\rangle = -v_1^2 m_1^2 + v_2^2 m_2^2 + v_3^2 m_3^2 = v_1^2 + (v_2^2 - v_1^2) m_2^2 + (v_3^2 - v_1^2) m_3^2 = w$, the inequality $|w| > 1$, implied by (5), means that M_1 and M_2 are ultraparallel. We have $\langle q_2, m\rangle\langle m, Vm\rangle\langle Vm, q_2\rangle = m_2 \overline{w} v_2^2 m_2$, hence, $\varepsilon = \dfrac{v_2^2 \overline{w}}{|w|}$. By Criterion 6.5.3 and its proof, the triangle $\Delta(C, M_1, M_2)$ is transversal if $\big(\mathrm{Re}(v_2^2 \overline{w})\big)^2 + 2m_2^2 < 1 + 2m_2^2 \,\mathrm{Re}(v_2^2 \overline{w})$ and $\dfrac{m_2^2 \big(\mathrm{Re}(v_2^2 \overline{w})\big)^2}{|w|^2} + m_2^2 + |w|^2 < 1 + 2m_2^2 \,\mathrm{Re}(v_2^2 \overline{w})$. We will show that the second inequality implies the first one. Indeed, the first inequality is equivalent to $\big(\mathrm{Re}(v_2^2 \overline{w}) + 1\big)\big(\mathrm{Re}(v_2^2 \overline{w}) - 1\big) < 2m_2^2 \big(\mathrm{Re}(v_2^2 \overline{w}) - 1\big)$, i.e., to $\mathrm{Re}(v_2^2 \overline{w}) + 1 < 2m_2^2$. The function $\dfrac{m_2^2 x^2}{|w|^2} + m_2^2 + |w|^2 - 1 - 2m_2^2 x$ is decreasing in x while $x < |w|^2$. Hence, substituting $\mathrm{Re}(v_2^2 \overline{w})$ by $|w|$ in the second inequality, we obtain $\big(|w| + 1\big)\big(|w| - 1\big) < 2m_2^2 \big(|w| - 1\big)$. Therefore, $|w| + 1 < 2m_2^2$, which implies $\mathrm{Re}(v_2^2 \overline{w}) + 1 < 2m_2^2$.

The second inequality can be written as $\Big(|w| - \dfrac{\mathrm{Re}(v_2^2 \overline{w})}{|w|}\Big)^2 m_2^2 < \big(|w|^2 - 1\big)\big(m_2^2 - 1\big)$. We require

$$\Big(|w| - \frac{\mathrm{Re}(v_2^2 \overline{w})}{|w|}\Big)^2 \,\mathrm{Re}\,\big(v_2(v_1^2 - w)\big) < \big(|w|^2 - 1\big)\,\mathrm{Re}\,\big(v_2(v_2^2 - w)\big), \tag{6}$$

$$\Big(|w| - \frac{\mathrm{Re}(u_2^2 \overline{w})}{|w|}\Big)^2 \,\mathrm{Re}\,\big(u_2(u_1^2 - w)\big) < \big(|w|^2 - 1\big)\,\mathrm{Re}\,\big(u_2(u_2^2 - w)\big), \tag{7}$$

which imply that the triangles $\Delta(C, M_1, M_2)$ and $\Delta(S_2, M_2, M_1)$ are transversal.

We will prove that the bisectors $B\wr m,s_2\wr$ and $B\wr m,q_2\wr$ are transversal along their common slice M_1. By Lemma 7.1.1, we obtain

$$\frac{\langle s_2,q_2\rangle\langle m,m\rangle}{\langle s_2,m\rangle\langle m,q_2\rangle}=\frac{m_2}{(u_2^2v_2^{-2}+1)\big(-\frac{m_1^2}{u_2^2v_1^{-2}+1}+\frac{m_2^2}{u_2^2v_2^{-2}+1}+\frac{m_3^2}{u_2^2v_3^{-2}+1}\big)m_2}=$$

[since $m_1^2=m_2^2+m_3^2-1$]

$$=\frac{(u_2^2v_1^{-2}+1)(u_2^2v_3^{-2}+1)}{(u_2^2v_2^{-2}+1)(u_2^2v_3^{-2}+1)+m_2^2u_2^2(v_1^{-2}-v_2^{-2})(u_2^2v_3^{-2}+1)+m_3^2u_2^2(v_1^{-2}-v_3^{-2})(u_2^2v_2^{-2}+1)}=$$

[since $m_2^2(v_1^{-2}-v_2^{-2})v_3^{-2}+m_3^2(v_1^{-2}-v_3^{-2})v_2^{-2}=m_2^2(v_2^2-v_1^2)v_1^{-2}v_2^{-2}v_3^{-2}+m_3^2(v_3^2-v_1^2)v_1^{-2}v_3^{-2}v_2^{-2}$, $m_2^2(v_1^{-2}-v_2^{-2})+m_3^2(v_1^{-2}-v_3^{-2})=m_2^2(\overline{v}_1^2-\overline{v}_2^2)+m_3^2(\overline{v}_1^2-\overline{v}_3^2)$, $v_1v_2v_3=u_1u_2u_3=1$, and $m_2^2(v_2^2-v_1^2)+m_3^2(v_3^2-v_1^2)=w-v_1^2$]

$$=\frac{u_2^2(\overline{u}_2^2+\overline{v}_1^2+\overline{v}_3^2)+u_2^4v_2^2}{u_2^2(\overline{u}_2^2+\overline{v}_2^2+\overline{v}_3^2)+u_2^4v_1^2+u_2^4(w-v_1^2)-u_2^2(\overline{w}-\overline{v}_1^2)}=\frac{\overline{u}_2(\overline{u}_2^2+\overline{v}_1^2+\overline{v}_3^2)+u_2v_2^2}{u_2w+\overline{u}_2(\overline{u}_2^2+\overline{v}_1^2+\overline{v}_2^2+\overline{v}_3^2-\overline{w})}=$$

[since $\overline{u}_2^2+\overline{v}_1^2+\overline{v}_2^2+\overline{v}_3^2-\overline{w}=\overline{w}-\overline{u}_1^2-\overline{u}_3^2$]

$$=\frac{\overline{u}_2(\overline{u}_2^2+\overline{v}_1^2+\overline{v}_3^2)+u_2v_2^2}{u_2w+\overline{u}_2\overline{w}-(\overline{u}_1^2\overline{u}_2+\overline{u}_2\overline{u}_3^2)}=\frac{\overline{u}_2(\overline{u}_1^2+\overline{u}_2^2+\overline{u}_3^2+\overline{v}_1^2+\overline{v}_3^2-\overline{w})+u_2(v_2^2-w)}{u_2w+\overline{u}_2\overline{w}-(u_2u_3^2+\overline{u}_2\overline{u}_3^2)}+1=$$

[since $\overline{u}_1^2+\overline{u}_2^2+\overline{u}_3^2+\overline{v}_1^2+\overline{v}_3^2-\overline{w}=\overline{w}-\overline{v}_2^2$]

$$=\frac{u_2(v_2^2-w)-\overline{u}_2(\overline{v}_2^2-\overline{w})}{u_2w+\overline{u}_2\overline{w}-(u_2u_3^2+\overline{u}_2\overline{u}_3^2)}+1\in 1+i\mathbb{R}.$$

By Corollary 4.3, $B\wr m,s_2\wr$ and $B\wr m,q_2\wr$ are transversal along M_1. This implies that $B\wr Um,s_2\wr=B\wr Vm,s_2\wr$ and $B\wr Vm,q_2\wr$ are transversal along M_2.

Let $n\geq 3$. By Theorem 4.2 and Lemma 4.1, the angle of the triangle $\Delta(C,M_1,M_2)$ at $q_1\in C$ equals $\operatorname{Arg}\dfrac{\langle m,q_1\rangle\langle q_1,Vm\rangle}{\langle m,q_2\rangle\langle q_2,Vm\rangle}=\operatorname{Arg}\dfrac{(-m_1)(-\overline{v}_1^2m_1)}{m_2\overline{v}_2^2m_2}=\operatorname{Arg}(v_2^2v_1^{-2})$. By symmetry, the angle of the triangle $\Delta(S_2,M_2,M_1)$ at $s_1\in S_2$ equals $\operatorname{Arg}(u_1^2u_2^{-2})$. We require

$$u_1^2u_2^{-2}=v_2^2v_1^{-2}=\exp\frac{2\pi i}{n},\qquad u_1^{6n}=v_1^{6n}=1. \tag{8}$$

Conditions (1) and (8) imply that both the angle of $\Delta(C,M_1,M_2)$ at $q_1\in C$ and the angle of $\Delta(S_2,M_2,M_1)$ at $s_1\in S_2$ equal $\dfrac{2\pi}{n}$ and that $U^n=V^n=1$ in $\mathrm{PU}(2,1)$. By Lemma 6.3.1, the triangles $\Delta(C,M_1,M_2)$ and $\Delta(S_2,M_2,M_1)$ are oriented in counterclockwise sense.

We require

$$\operatorname{Re}(u_1^{-1}v_1)\cdot\operatorname{Re}(u_1^{-1}v_2)\cdot\operatorname{Re}(u_1^{-1}v_3)\cdot\operatorname{Im}\Big(u_1^{-1}v_2^2\big((2m_2^2-1)(u_1^2+v_1^2-v_2^2+v_3^2)+u_2^2+u_3^2\big)\Big)\geq 0. \tag{9}$$

Let us verify that $\Delta(S_2,M_2,M_1)$ suits $\Delta(C,M_1,M_2)$. We have already proven that $B\wr m,s_2\wr$ and $B\wr m,q_2\wr$ are transversal along M_1 and that $B\wr Vm,s_2\wr$ and $B\wr Vm,q_2\wr$ are transversal along M_2. So, by Lemma 2.3.1, it suffices to verify the inequalities $\operatorname{Im}\dfrac{\langle q_2,s_1\rangle\langle s_1,m\rangle}{\langle q_2,m\rangle}\geq 0$ and $\operatorname{Im}\dfrac{\langle Vm,s_1\rangle\langle s_1,q_2\rangle}{\langle Vm,q_2\rangle}\geq 0$.

Since the case of $u_1^2+v_1^2=0$ is trivial, we can assume that $s_1=\begin{pmatrix}\frac{m_1}{u_1^2v_1^{-2}+1}\\ \frac{m_2}{u_1^2v_2^{-2}+1}\\ \frac{m_3}{u_1^2v_3^{-2}+1}\end{pmatrix}$. As is easy to see, each inequality is equivalent to the inequality

$$\operatorname{Im}\Big(\frac{1}{u_1^2+v_2^2}\Big(-\frac{m_1^2}{u_1^{-2}+v_1^{-2}}+\frac{m_2^2}{u_1^{-2}+v_2^{-2}}+\frac{m_3^2}{u_1^{-2}+v_3^{-2}}\Big)\Big)\geq 0,$$

i.e., to the inequality $\operatorname{Im}\Big(\frac{1}{u_1^2+v_2^2}\Big(\frac{1-m_2^2}{u_1^{-2}+v_1^{-2}}+\frac{m_3^2(v_1^{-2}-v_3^{-2})}{(u_1^{-2}+v_1^{-2})(u_1^{-2}+v_3^{-2})}\Big)\Big)\geq 0$. This inequality can be rewritten as $\operatorname{Im}\frac{u_1^{-1}v_2^{-1}u_1v_1u_1v_3\big((1-m_2^2)(u_1^{-2}+v_3^{-2})+m_3^2(v_1^{-2}-v_3^{-2})\big)}{(u_1v_2^{-1}+u_1^{-1}v_2)(u_1^{-1}v_1+u_1v_1^{-1})(u_1^{-1}v_3+u_1v_3^{-1})}\geq 0$ which is equivalent to $\operatorname{Re}(u_1^{-1}v_1)\cdot\operatorname{Re}(u_1^{-1}v_2)\cdot\operatorname{Re}(u_1^{-1}v_3)\cdot\operatorname{Im}\Big(u_1v_2^{-2}\big((1-m_2^2)(u_1^{-2}+v_3^{-2})+m_2^2(v_2^{-2}-v_1^{-2})-\overline{w}+v_1^{-2}\big)\Big)\geq 0$ due to $m_3^2(v_1^{-2}-v_3^{-2})=m_2^2(v_2^{-2}-v_1^{-2})-\overline{w}+v_1^{-2}$. All we need to arrive at (9) is to conjugate the terms inside of Im.

7.1.2. Theorem. *Let $n\geq 3$, $0\leq k\leq l\leq n-3$, and $p=1,2$ be such that $4np+3n-2k+4l+6\not\equiv 0 \mod 6n$. We put*

$$u_1\rightleftharpoons\exp\frac{(2np-k)\pi i}{3n},\qquad u_2\rightleftharpoons\exp\frac{(2np-k-3)\pi i}{3n},\qquad u_3\rightleftharpoons\exp\frac{(2np+2k+3)\pi i}{3n},$$

$$v_1\rightleftharpoons\exp\frac{l\pi i}{3n},\ v_2\rightleftharpoons\exp\frac{(l+3)\pi i}{3n},\ v_3\rightleftharpoons\exp\frac{-(2l+3)\pi i}{3n},\ w\rightleftharpoons\frac{u_1^2+u_2^2+u_3^2+v_1^2+v_2^2+v_3^2}{2},$$

$$m_2\rightleftharpoons\sqrt{\frac{\operatorname{Re}\big((v_1^2-w)v_2\big)}{\operatorname{Re}\big((v_1^2-v_2^2)v_2\big)}},\qquad m_3\rightleftharpoons\sqrt{\frac{\operatorname{Re}\big((v_1^2-w)v_3\big)}{\operatorname{Re}\big((v_1^2-v_3^2)v_3\big)}},\qquad m_1\rightleftharpoons\sqrt{m_2^2+m_3^2-1},$$

$$V\rightleftharpoons\begin{pmatrix}v_1^2&0&0\\0&v_2^2&0\\0&0&v_3^2\end{pmatrix},\qquad R\rightleftharpoons\begin{pmatrix}-2m_1^2-1&2m_1m_2&2m_1m_3\\-2m_1m_2&2m_2^2-1&2m_2m_3\\-2m_1m_3&2m_2m_3&2m_3^2-1\end{pmatrix}$$

with V and R written in some orthonormal basis of signature $-++$.

We require that

$$\operatorname{Re}(u_2w)<\operatorname{Re}u_2^3,\qquad \operatorname{Re}(v_2w)<\operatorname{Re}v_2^3,\qquad \operatorname{Re}(v_3v_1^2)\geq\operatorname{Re}(v_3w),\qquad \operatorname{Re}(u_2^2\overline{w})>1,\qquad \operatorname{Re}(v_2^2\overline{w})>1,$$

$$m_2^2\frac{\operatorname{Re}\big(u_1^{-2}(v_2^2-v_1^2)\big)}{1+\operatorname{Re}(u_1^{-2}v_2^2)}+m_3^2\frac{\operatorname{Re}\big(u_1^{-2}(v_3^2-v_1^2)\big)}{1+\operatorname{Re}(u_1^{-2}v_3^2)}\geq 1,$$

$$\Big(|w|-\frac{\operatorname{Re}(u_2^2\overline{w})}{|w|}\Big)^2\operatorname{Re}\big((u_1^2-w)u_2\big)<\big(|w|^2-1\big)\operatorname{Re}\big((u_2^2-w)u_2\big),$$

$$\Big(|w|-\frac{\operatorname{Re}(v_2^2\overline{w})}{|w|}\Big)^2\operatorname{Re}\big((v_1^2-w)v_2\big)<\big(|w|^2-1\big)\operatorname{Re}\big((v_2^2-w)v_2\big),$$

$$\operatorname{Re}(u_1^{-1}v_1)\cdot\operatorname{Re}(u_1^{-1}v_2)\cdot\operatorname{Re}(u_1^{-1}v_3)\cdot\operatorname{Im}\Big(u_1^{-1}v_2^2\big((2m_2^2-1)(u_1^2+v_1^2-v_2^2+v_3^2)+u_2^2+u_3^2\big)\Big)\geq 0.$$

Then the group F_n generated by V and R is discrete in $\mathrm{PU}(2,1)$ and has the defining relations $V^n=R^2=(VR)^n=1$. For odd n, a suitable subgroup $T_n\subset F_n$ of index $4n$ (for even n, a suitable subgroup $G_n\subset F_n$ of index $2n$) defines a complex hyperbolic manifold $M(n,l,k,p)\rightleftharpoons \mathrm{B}\,V/T_n$ (for even n, $N(n,l,k,p)\rightleftharpoons\mathrm{B}\,V/G_n$) which is homeomorphic to a disc bundle over a closed orientable surface of Euler characteristic $\chi=8-2n$ (for even n and for $N(n,l,k,p)$, $\chi=4-n$). For this bundle, the relation $2(\chi+e)=3\tau \mod 8n$ holds (for even n, $2(\chi+e)=3\tau \mod 4n$ holds), where e and τ stand for the Euler number of the bundle and for the Toledo invariant of the representation defined by the manifold. Moreover, $\tau=\frac{8}{3}t-4n$ (for even n and for $N(n,l,k,p)$, we have $\tau=\frac{4}{3}t-2n$), where $0\leq t<3n$ is defined by $t\equiv 2np-k-l \mod 3n$.

Proof. The inequalities $n\geq 3$ and $0\leq k,l\leq n-3$ imply both (1) and the inequalities $\operatorname{Re}u_2^3<\operatorname{Re}(u_2u_1^2)$, $\operatorname{Re}u_3^3<\operatorname{Re}(u_3u_1^2)$, $\operatorname{Re}v_2^3<\operatorname{Re}(v_2v_1^2)$, and $\operatorname{Re}v_3^3<\operatorname{Re}(v_3v_1^2)$. Since $4np+3n-2k+4l+6\not\equiv 0 \mod 6n$, we obtain $u_1^2+v_3^2\neq 0$. It is easy to see that (8) holds. So, all conditions (1–9) are valid.

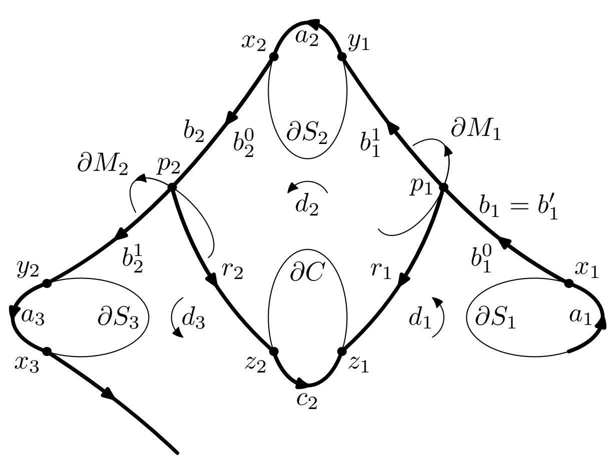
By the above considerations, $R_1 = R$ is the reflection in the projective line M_1 of signature $+-$ such that the rotations $U \risingdotseq VR$ and V satisfy $U^n = V^n = 1$ in $\mathrm{PU}(2,1)$. There are a U-invariant projective line S_2 and a V-invariant projective line C, both of signature $+-$. Denoting by $s' \in S_2$ and $q \in C$ the fixed points of U and of V, respectively, the transversal triangle $\Delta(S_2, M_2, M_1)$ oriented in counterclockwise sense suits the transversal triangle $\Delta(C, M_1, M_2)$ oriented in counterclockwise sense and both the angle of $\Delta(S_2, M_2, M_1)$ at $s' \in S_2$ and the angle of $\Delta(C, M_1, M_2)$ at $q \in C$ equal $\dfrac{2\pi}{n}$, where $M_2 \risingdotseq VM_1$.

For $j = 1, 2, \dots, n$, we put $R_j \risingdotseq R^{V^{j-1}}$, $M_j \risingdotseq V^{j-1}M_1$, and $S_j \risingdotseq V^{j-2}S_2$ (we remember that indices are considered modulo n). Since $RV^{-1} = U^{-1}$ and $U^{-1}S_2 = S_2$, we obtain $R_jS_j = R^{V^{j-1}}V^{j-2}S_2 = V^{j-1}RV^{-1}S_2 = V^{j-1}S_2 = S_{j+1}$ and $R_n \dots R_1 = V^n(V^{-1}R)^n = 1$ in $\mathrm{PU}(2,1)$ (in $\mathrm{SU}(2,1)$, we have $R_n \dots R_1 = \delta$, where $\delta \risingdotseq \exp\dfrac{2(k+l+np)\pi i}{3}$). Clearly, R_j is the reflection in the projective line M_j of signature $+-$. Since S_2 and M_2 are ultraparallel, S_j and M_j are ultraparallel. Thus, we have a cycle of bisectors with positive foci. Applying V^{j-2}, we see that the triangle $V^{j-2}\Delta(S_2, M_2, M_1) = \Delta(S_j, M_j, M_{j-1})$ suits the triangle $V^{j-2}\Delta(C, M_1, M_2) = \Delta(C, M_{j-1}, M_j)$ for all j and that C is a centre of the cycle with central angle 2π. Hence, the cycle is transversal and simple by Criterion 6.3.2. We put $s'_1 \risingdotseq Rs' \in \check S_1$ and $s'_{j+1} \risingdotseq R_js'_j \in \check S_{j+1}$ for $j = 1, 2, \dots, n$ (so, $s'_2 = s'$ and $s'_{n+1} = \delta s'_1$). Obviously, s'_j's are the vertices of some meridian of the cycle. Therefore, the total angle of the cycle is 2π. By Remark 6.4.3, the polyhedron related to the cycle is fibred. By Proposition 6.4.5, we arrive at the desired manifold $M(n, l, k, p)$ for odd n and at $N(n, l, k, p)$ for even n.

In order to calculate the Toledo invariant, we apply the formula $\tau = \dfrac{8}{\pi}\displaystyle\sum_{i=1}^{n}\Big(\frac{1}{2}\operatorname{Arg}\frac{\langle u, s'_{i+1}\rangle}{\langle u, s'_i\rangle} - \frac{\pi}{2}\Big)$ obtained in the proof of Proposition 6.1.1, taking $u \risingdotseq q$. Since $RV^{-1} = U^{-1}$, $Us' = u_1^2 s'$, and $Vq = v_1^2 q$, for $j = 1, \dots, n+1$, we have $s'_j = V^{j-2}(RV^{-1})^{j-2}s' = V^{j-2}U^{2-j}s' = u_1^{4-2j}V^{j-2}s'$, $\langle q, s'_j\rangle = u_1^{2j-4}\langle q, V^{j-2}s'\rangle = u_1^{2j-4}\langle V^{2-j}q, s'\rangle = u_1^{2j-4}v_1^{4-2j}\langle q, s'\rangle = (u_1^2v_1^{-2})^{j-2}\langle q, s'\rangle$, and $\dfrac{\langle q, s'_{j+1}\rangle}{\langle q, s'_j\rangle} = u_1^2v_1^{-2}$. It follows that $\tau = 4n\Big(\dfrac{\operatorname{Arg}(u_1^2v_1^{-2})}{\pi} - 1\Big) = \dfrac{4n\operatorname{Arg}\exp\frac{2(2np-k-l)\pi i}{3n}}{\pi} - 4n = \dfrac{8}{3}t - 4n$ for odd n and $\tau = \dfrac{4}{3}t - 2n$ for even n.

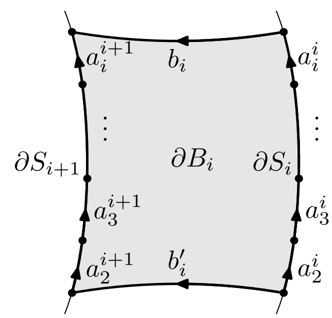
Until the end of the article, all points will 'live' and 'die' in $\overline{\mathrm{B}}V$ and all isometries, in $\mathrm{PU}(2,1)$.

Let $p_1 \in \partial M_1$. We draw an ideal ℓ-meridian r_1 of the bisector $B(C, M_1)$ that begins with $p_1 \in \partial M_1$ and ends with $z_1 \in \partial C$. We put $z_j \risingdotseq V^{j-1}z_1 \in \partial C$, $p_j \risingdotseq V^{j-1}p_1 \in \partial M_j$, and $r_j \risingdotseq V^{j-1}r_1$. Clearly, r_j is the ideal ℓ-meridian of the bisector $B(C, M_j)$ that begins with p_j and ends with z_j. Since $p_j \in \partial M_j$ and M_j is the middle slice of the bisector $B_j \risingdotseq B(S_j, S_{j+1})$, we can draw an ideal meridian $b_j = b_j^0 \cup b_j^1$ of B_j passing through p_j so that b_j^0 begins with $x_j \in \partial S_j$ and ends with $p_j \in \partial M_j$, b_j^1 begins with $p_j \in \partial M_j$ and ends with $y_j \in \partial S_{j+1}$, and b_j begins with $x_j \in \partial S_j$ and ends with $y_j \in \partial S_{j+1}$. Since $B_j = V^{j-1}B_1$ and $p_j = V^{j-1}p_1$, we obtain $b_j^0 = V^{j-1}b_1^0$, $b_j^1 = V^{j-1}b_1^1$, $b_j = V^{j-1}b_1$, $x_j =$



$V^{j-1}x_1$, and $y_j = V^{j-1}y_1$. Following the natural orientation of ∂C, we draw an arc $c_j \subset \partial C$ that begins with z_j and ends with z_{j-1}. Following the natural orientation of ∂S_j, we draw an arc $a_j \subset \partial S_j$ that begins with y_{j-1} and ends with x_j. We put $b \rightleftharpoons b_1 \cup a_2 \cup b_2 \cup \dots \cup a_n \cup b_n \cup a_1$, $d_j \rightleftharpoons r_{j-1}^{-1} \cup b_{j-1}^1 \cup a_j \cup b_j^0 \cup r_j \cup c_j$, and $U_j \rightleftharpoons U^{V^{j-2}}$ and denote by $b' = b'_1 \cup b'_2 \cup \dots \cup b'_n \subset T \rightleftharpoons \partial\partial_1 P$ the closed meridian that begins with x_1 and whose edges are b'_j's (so, $U_2 = U$ and $b'_1 = b_1$). It is obvious that $b, b' \subset T$ are closed curves that begin with x_1, that $d_j \subset \partial_1 P$ is a closed curve that begins with z_{j-1}, that $d_j = V^{j-1}d_1$, and that $R_j U_j^i = U_{j+1}^i R_j$ (since $RV^{-1} = U^{-1}$). Identifying $\check C$ with the unitary disc of complex numbers so that q corresponds to 0, we can easily see that $V|_C : z \mapsto \beta^{-1}z$, where $\beta \rightleftharpoons v_1^2 v_3^{-2} = \exp\dfrac{2(l+1)\pi i}{n}$. The same is valid for $U_2|_{S_2} = U|_{S_2}$: when identifying $\check S_2$ with the unitary disc so that $s'_2 = s'$ corresponds to 0, we obtain $U|_{S_2} : y \mapsto \alpha y$, where $\alpha \rightleftharpoons u_3^2 u_1^{-2} = \exp\dfrac{2(k+1)\pi i}{n}$. By induction on j, we show that s'_j is a fixed point for U_j : $s'_{j+1} = R_j s'_j = R_j U_j s'_j = (VRV^{-1}U)^{V^{j-2}} s'_j = V s'_j$ and $U_{j+1} = U_j^V$. So, $U_j|_{S_j} : y \mapsto \alpha y$, when we identify $\check S_j$ with the unitary disc with s'_j corresponding to 0. Since $Up_1 = VRp_1 = Vp_1 = p_2$, $UM_1 = VRM_1 = VM_1 = M_2$, $US_2 = S_2$, and $UB(S_2, M_1) = B(S_2, M_2)$, we obtain $Ub_1^1 = (b_2^0)^{-1}$. In particular, $Uy_1 = x_2$. Thus, we conclude that $U_j y_{j-1} = x_j$.

We know that $\partial_1 P$ is a solid torus and that $\pi_1\partial_1 P$ is generated by $[c]$, $\pi_1\partial_1 P = \mathbb{Z}[c]$, where c stands for ∂C naturally oriented. Hence, $[d_2] = f[c]$ in $\pi_1\partial_1 P$ for some $f \in \mathbb{Z}$. Since $d_{j+1} = Vd_j$ and $c = Vc$, we obtain $[d_j] = f[c]$ in $\pi_1\partial_1 P$. Consequently, $[d_1] + \dots + [d_n] = nf[c]$ in $\pi_1\partial_1 P$. It is easy to verify that $[d_2 \cup c_2^{-1} \cup d_3 \cup c_3^{-1} \cup \dots \cup d_{n+1} \cup c_{n+1}^{-1}] = [b]$. Hence, $[d_1] + \dots + [d_n] = [b] + [c_{n+1} \cup c_n \cup \dots \cup c_2]$. Since $z_{j+1} = Vz_j$ and, when identifying $\check C$ with the unitary disc of complex numbers, $Vz_j = \beta^{-1}z_j$ for $\beta = \exp\dfrac{2(l+1)\pi i}{n}$, we arrive at $[c_{n+1} \cup c_n \cup \dots \cup c_2] = (l+1)[c]$ in $\pi_1\partial_1 P$. Therefore, $[b] = \big(nf - (l+1)\big)[c]$ in $\pi_1\partial_1 P$.



Let us prove that $[b] = [b'] + (k+1)[c]$ in $\pi_1\partial_1 P$. We denote $a_j^j \rightleftharpoons a_j \subset \partial S_j$ and put $a_j^{i+1} \rightleftharpoons R_i a_j^i$. Clearly, $a_j^i \subset \partial S_i$ is an arc following the natural orientation of ∂S_i and a_j^i ends with the same point as a_{j+1}^i begins with. Moreover, it follows from the relations $R_j U_j = U_{j+1} R_j$ and $U_j y_{j-1} = x_j$ that, when identifying $\check S_i$ with the unitary disc of complex numbers, we have $x = \alpha y$, where y and x stand respectively for the initial point and for the final point of a_j^i. It is easy to see that $a_2^i \cup a_3^i \cup \dots \cup a_i^i \cup b_i \sim b'_i \cup a_2^{i+1} \cup a_3^{i+1} \cup \dots \cup a_i^{i+1}$, where $\sim$ stands for 'is homotopic to' in T. This implies that $b = b_1 \cup a_2^2 \cup b_2 \cup a_3^3 \cup b_3 \cup a_4^4 \cup \dots \cup a_n^n \cup b_n \cup a_1^1 \sim b'_1 \cup a_2^2 \cup b_2 \cup a_3^3 \cup b_3 \cup a_4^4 \cup \dots \cup a_n^n \cup b_n \cup a_1^1 \sim b'_1 \cup b'_2 \cup a_2^3 \cup a_3^3 \cup b_3 \cup a_4^4 \cup \dots \cup a_n^n \cup b_n \cup a_1^1 \sim \dots \sim b'_1 \cup b'_2 \cup \dots \cup b'_n \cup a_2^{n+1} \cup \dots \cup a_n^{n+1} \cup a_1^1 = b'_1 \cup b'_2 \cup \dots \cup b'_n \cup a_2^1 \cup \dots \cup a_n^1 \cup a_1^1$.

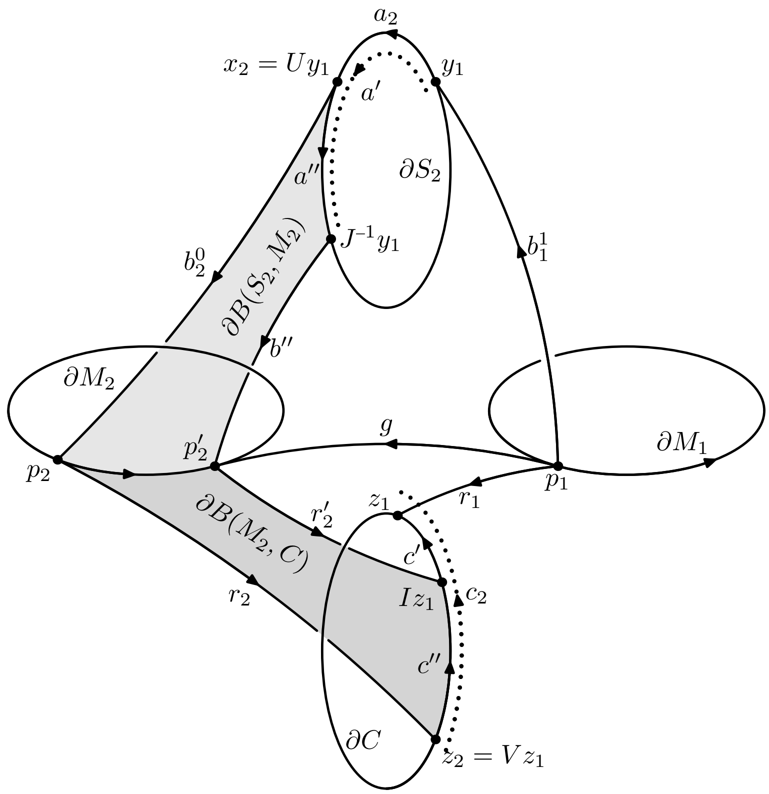
It follows from $\alpha = \exp\dfrac{2(k+1)\pi i}{n}$ that $[a_2^1 \cup \dots \cup a_n^1 \cup a_1^1] = (k+1)[\partial S_1]$. Since $[\partial S_1] = [c]$ in $\pi_1\partial_1 P$, the result follows.

Finally, by Remark 6.4.4, $eP = nf - (k+l+2)$ and $e = 4\big(nf - (k+l+2)\big)$ (for even n and $N(n,l,k,p)$, $e = 2\big(nf - (k+l+2)\big)$) ■

In order to explicitly find f, we need to study the mutual position of triangles $\Delta(C, M_1, M_2)$ and $\Delta(S_2, M_2, M_1)$. For $t_1, t_2, t_3 \in S$, where S is some oriented circle, we put $o(t_1, t_2, t_3) = 0$ if t_1, t_2, t_3 are in the cyclic order of the circle and $o(t_1, t_2, t_3) = 1$, otherwise. For an isometry K of a slice D of a transversal triangle and for $d \in \partial D$, we define $\lambda(d, K) = 0$ if p belongs to the L-part of K and $\lambda(d, K) = 1$, otherwise. We denote by I the isometry of C in the triangle $\Delta(C, M_1, M_2)$ and by J, the isometry of S_2 in the triangle $\Delta(S_2, M_2, M_1)$.

We draw, in the bisector $B(S_2, M_2)$, an ideal ℓ-meridian b'' that begins with $J^{-1}y_1 \in \partial S_2$. Let $p'_2 \in \partial M_2$ denote its final point. There exists an ideal ℓ-meridian g of $B(M_1, M_2)$ that begins with p_1

and ends with p_2'. Following the natural orientation of ∂S_2, we draw an arc a' from y_1 to $J^{-1}y_1$. By Theorem 6.5.1, the closed curve $\gamma \rightleftharpoons b''\cup g^{-1}\cup b_1^1\cup a'$ is trivializing for the triangle $\Delta(S_2,M_2,M_1)$ if $J^{-1}y_1$ belongs to the L-part of ∂S_2, and $[\gamma]=[\partial S_2]$, otherwise. This fact can be written as $[\gamma]=\lambda(y_1,J)[c]$ since $[\partial S_2]=[c]$.



Let r_2' be an ideal ℓ-meridian of $B(M_2,C)$ that begins with $p_2'\in\partial M_2$. Clearly, $Iz_1\in\partial C$ is the final point of r_2'. Following the natural orientation of ∂C, we draw an arc c' from Iz_1 to z_1. As above, by Theorem 6.5.1, $[\vartheta]=\lambda(z_1,I)[c]$, where $\vartheta\rightleftharpoons r_1^{-1}\cup g\cup r_2'\cup c'$.

Following the natural orientation of ∂S_2 and of ∂C, we draw arcs $a''\subset\partial S_2$ from $x_2=Uy_1$ to $J^{-1}y_1$ and $c''\subset\partial C$ from $z_2=Vz_1$ to Iz_1. It is easy to see that $b_2^0\cup r_2\sim a''\cup b''\cup r_2'\cup c''^{-1}$, that $[a_2\cup a''\cup a'^{-1}]=o(J^{-1}y_1,y_1,Uy_1)[\partial S_2]=o(J^{-1}y_1,y_1,Uy_1)[c]$, and that $[c_2\cup c'^{-1}\cup c''^{-1}]=-o(Iz_1,z_1,Vz_1)[c]$. In terms of $H_1(\partial_1P,\mathbb{Z})=\pi_1\partial_1P$, we have $[d_2]=[-r_1+b_1^1+a_2+b_2^0+r_2+c_2]=[-r_1+b_1^1+a_2+a''+b''+r_2'-c''+c_2]=[-r_1+g+r_2'+c']+[-g+b_1^1+a'+b'']+[a_2+a''-a']+[c_2-c'-c'']=\lambda(z_1,I)[c]+\lambda(y_1,J)[c]+o(J^{-1}y_1,y_1,Uy_1)[c]-o(Iz_1,z_1,Vz_1)[c]$. We arrive at $f=\lambda(y_1,J)+o(J^{-1}y_1,y_1,Uy_1)+\lambda(z_1,I)-o(Iz_1,z_1,Vz_1)$. Clearly, the terms $\lambda(y_1,J)$ and $\lambda(z_1,I)$ vanish if both triangles $\Delta(C,M_1,M_2)$ and $\Delta(S_2,M_2,M_1)$ are elliptic.

7.2. Some Interesting Examples. In this subsection, we present some explicit examples of complex hyperbolic bundles obtained with straightforward computer calculations (a simple program that verifies our computational claims is available upon request: all we need is to find parameters satisfying the conditions in Theorem 7.1.2).

We have tested all $n\le 1001$ and obtained the following. For any n satisfying the inequalities $9\le n\le 1001$ and $n\ne 11,12$, there is at least one example of the type described in Theorem 7.1.2. The total number of such examples is 308359, and only 89546 of them have the Toledo invariant integer. Each example satisfies the equality $2(\chi+e)=3\tau$ and the inequalities $\tau<0$ and $\frac12\chi<e<0$. For every such example, both triangles $\Delta(C, M_1, M_2)$ and $\Delta(S_2, M_2, M_1)$ are elliptic and $o(Iz_1, z_1, Vz_1) = 0$. For $p = 1$, we have $f = 1$ with $o(J^{-1}y_1, y_1, Uy_1) = 1$, and, for $p = 2$, we have $f = 0$ with $o(J^{-1}y_1, y_1, Uy_1) = 0$.

The following table contain all the examples with extreme values of g, e, and e/χ :

Manifold	g	χ	e	τ	Comment
$N(10,6,3,1)$	4	-6	-2	$-5\frac{1}{3}$	minimal g, maximal e
$M(9,4,4,1)$	6	-10	-4	$-9\frac{1}{3}$	next to minimal g, next to maximal e
$M(9,5,3,1)$	6	-10	-4	$-9\frac{1}{3}$	next to minimal g, next to maximal e
$M(9,6,2,1)$	6	-10	-4	$-9\frac{1}{3}$	next to minimal g, next to maximal e
$N(14,7,7,1)$	6	-10	-4	$-9\frac{1}{3}$	next to minimal g, next to maximal e
$N(14,8,6,1)$	6	-10	-4	$-9\frac{1}{3}$	next to minimal g, next to maximal e
$N(14,9,5,1)$	6	-10	-4	$-9\frac{1}{3}$	next to minimal g, next to maximal e
$N(14,10,4,1)$	6	-10	-4	$-9\frac{1}{3}$	next to minimal g, next to maximal e
$N(14,11,3,1)$	6	-10	-4	$-9\frac{1}{3}$	next to minimal g, next to maximal e
$N(14,0,0,2)$	6	-10	-4	$-9\frac{1}{3}$	next to minimal g, next to maximal e
$N(16,0,0,2)$	7	-12	-4	$-10\frac{2}{3}$	next to maximal e
$N(18,0,0,2)$	8	-14	-4	-12	next to maximal e
$N(20,0,0,2)$	9	-16	-4	$-13\frac{1}{3}$	next to maximal e
$N(22,0,0,2)$	10	-18	-4	$-14\frac{2}{3}$	next to maximal e
$N(24,0,0,2)$	11	-20	-4	-16	next to maximal e
$N(26,0,0,2)$	12	-22	-4	$-17\frac{1}{3}$	next to maximal e
$N(28,0,0,2)$	13	-24	-4	$-18\frac{2}{3}$	minimal $e/\chi = \frac{1}{6}$, next to maximal e

The four tables below expose all 55 examples that satisfy the inequality $\frac{1}{3}\chi \leq e$ and, therefore, due to [Kui], admit a real hyperbolic structure:

Manifold	g	χ	e	τ
$N(10,6,3,1)$	4	-6	-2	$-5\frac{1}{3}$
$N(16,0,0,2)$	7	-12	-4	$-10\frac{2}{3}$
$N(18,0,0,2)$	8	-14	-4	-12
$N(20,0,0,2)$	9	-16	-4	$-13\frac{1}{3}$
$N(22,0,0,2)$	10	-18	-4	$-14\frac{2}{3}$
$N(22,1,0,2)$	10	-18	-6	-16
$N(24,0,0,2)$	11	-20	-4	-16
$N(24,1,0,2)$	11	-20	-6	$-17\frac{1}{3}$
$N(26,0,0,2)$	12	-22	-4	$-17\frac{1}{3}$
$N(26,1,0,2)$	12	-22	-6	$-18\frac{2}{3}$
$N(28,0,0,2)$	13	-24	-4	$-18\frac{2}{3}$
$N(28,1,0,2)$	13	-24	-6	-20
$N(28,1,1,2)$	13	-24	-8	$-21\frac{1}{3}$
$N(28,2,0,2)$	13	-24	-8	$-21\frac{1}{3}$
$M(17,0,0,2)$	14	-26	-8	$-22\frac{2}{3}$
$N(30,1,1,2)$	14	-26	-8	$-22\frac{2}{3}$

Manifold	g	χ	e	τ
$N(30,2,0,2)$	14	-26	-8	$-22\frac{2}{3}$
$N(32,1,1,2)$	15	-28	-8	-24
$N(32,2,0,2)$	15	-28	-8	-24
$M(19,0,0,2)$	16	-30	-8	$-25\frac{1}{3}$
$N(34,1,1,2)$	16	-30	-8	$-25\frac{1}{3}$
$N(34,2,1,2)$	16	-30	-10	$-26\frac{2}{3}$
$N(34,3,0,2)$	16	-30	-10	$-26\frac{2}{3}$
$N(36,1,1,2)$	17	-32	-8	$-26\frac{2}{3}$
$N(36,2,1,2)$	17	-32	-10	-28
$M(21,0,0,2)$	18	-34	-8	-28
$N(38,2,1,2)$	18	-34	-10	$-29\frac{1}{3}$
$N(40,2,2,2)$	19	-36	-12	-32
$N(40,3,1,2)$	19	-36	-12	-32
$M(23,0,0,2)$	20	-38	-8	$-30\frac{2}{3}$
$M(23,1,0,2)$	20	-38	-12	$-33\frac{1}{3}$
$N(42,2,2,2)$	20	-38	-12	$-33\frac{1}{3}$

Manifold	g	χ	e	τ
$N(44,2,2,2)$	21	-40	-12	$-34\frac{2}{3}$
$M(25,0,0,2)$	22	-42	-8	$-33\frac{1}{3}$
$M(25,1,0,2)$	22	-42	-12	-36
$N(46,3,2,2)$	22	-42	-14	$-37\frac{1}{3}$
$M(27,0,0,2)$	24	-46	-8	-36
$M(27,1,0,2)$	24	-46	-12	$-38\frac{2}{3}$
$N(52,3,3,2)$	25	-48	-16	$-42\frac{2}{3}$
$M(29,1,0,2)$	26	-50	-12	$-41\frac{1}{3}$
$M(29,1,1,2)$	26	-50	-16	-44
$M(29,2,0,2)$	26	-50	-16	-44
$M(31,1,1,2)$	28	-54	-16	$-46\frac{2}{3}$
$M(31,2,0,2)$	28	-54	-16	$-46\frac{2}{3}$

Manifold	g	χ	e	τ
$M(33,1,1,2)$	30	-58	-16	$-49\frac{1}{3}$
$M(35,1,1,2)$	32	-62	-16	-52
$M(35,2,1,2)$	32	-62	-20	$-54\frac{2}{3}$
$M(37,1,1,2)$	34	-66	-16	$-54\frac{2}{3}$
$M(37,2,1,2)$	34	-66	-20	$-57\frac{1}{3}$
$M(41,2,2,2)$	38	-74	-24	$-65\frac{1}{3}$
$M(41,3,1,2)$	38	-74	-24	$-65\frac{1}{3}$
$M(43,2,2,2)$	40	-78	-24	-68
$M(45,2,2,2)$	42	-82	-24	$-70\frac{2}{3}$
$M(47,3,2,2)$	44	-86	-28	-76
$M(53,3,3,2)$	50	-98	-32	$-86\frac{2}{3}$

The following tables show all the examples for $n=101$ ($g=98$, $\chi=-194$). This illustrates a typical behaviour of l, k, p, and the Euler number for some fixed parameter n :

$e=-96$, $\tau=-193\frac{1}{3}$
$M(101,62,61,1)$
$M(101,63,60,1)$
$M(101,64,59,1)$
$M(101,65,58,1)$
$M(101,66,57,1)$
$M(101,67,56,1)$
$M(101,68,55,1)$
$M(101,69,54,1)$
$M(101,70,53,1)$
$M(101,71,52,1)$
$M(101,72,51,1)$
$M(101,73,50,1)$
$M(101,74,49,1)$
$M(101,75,48,1)$
$M(101,76,47,1)$
$M(101,77,46,1)$
$M(101,78,45,1)$

$e=-96$, $\tau=-193\frac{1}{3}$
$M(101,79,44,1)$
$M(101,80,43,1)$
$M(101,81,42,1)$
$M(101,82,41,1)$
$M(101,83,40,1)$
$M(101,84,39,1)$
$M(101,85,38,1)$
$M(101,86,37,1)$
$M(101,87,36,1)$
$M(101,88,35,1)$
$M(101,89,34,1)$
$M(101,90,33,1)$
$M(101,91,32,1)$
$M(101,92,31,1)$
$M(101,93,30,1)$
$M(101,94,29,1)$
$M(101,95,28,1)$

$e=-96$, $\tau=-193\frac{1}{3}$
$M(101,96,27,1)$
$M(101,97,26,1)$
$M(101,98,25,1)$
$M(101,11,11,2)$
$M(101,12,10,2)$
$M(101,13,9,2)$
$M(101,14,8,2)$
$M(101,15,7,2)$
$M(101,16,6,2)$
$M(101,17,5,2)$
$M(101,18,4,2)$
$M(101,19,3,2)$
$M(101,20,2,2)$
$M(101,21,1,2)$
$M(101,22,0,2)$

$e=-92$, $\tau=-190\frac{2}{3}$
$M(101,96,26,1)$
$M(101,97,25,1)$
$M(101,98,24,1)$
$M(101,11,10,2)$
$M(101,12,9,2)$
$M(101,13,8,2)$
$M(101,14,7,2)$
$M(101,15,6,2)$
$M(101,16,5,2)$
$M(101,17,4,2)$
$M(101,18,3,2)$
$M(101,19,2,2)$
$M(101,20,1,2)$
$M(101,21,0,2)$

$e=-88$, $\tau=-188$
$M(101,10,10,2)$
$M(101,11,9,2)$
$M(101,12,8,2)$
$M(101,13,7,2)$
$M(101,14,6,2)$
$M(101,15,5,2)$
$M(101,16,4,2)$

$e=-84$, $\tau=-185\frac{1}{3}$
$M(101,10,9,2)$
$M(101,11,8,2)$
$M(101,12,7,2)$

$e=-80$, $\tau=-182\frac{2}{3}$
$M(101,9,9,2)$

Some of the examples in the above tables could represent the same complex hyperbolic bundle, that is, the corresponding groups would be conjugated in $\mathrm{PU}(2,1)$. For instance, examples 2–10 in the first table. (In general, it makes sense to treat as representing the 'same example' two discrete groups of isometries having some subgroups of finite index conjugated in the group of all isometries.) Nevertheless, it can be proven that these complex hyperbolic bundles define different points in the space of representations.

In subsequent articles (for instance, in [Ana]), we will study the connected components of the Teichmüller space (the space of discrete, faithful, and type-preserving representations $\varrho:\pi_1\Sigma\to\mathrm{PU}(2,1)$) and will distinguish some of our examples having the same g, e, and τ with new discrete invariants of complex hyperbolic structure on the bundles in question.

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