A New Trigonometric Spline Approach to Numerical Solution of Generalized Nonlinear Klien-Gordon Equation

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Abstract

The generalized nonlinear Klien-Gordon equation plays an important role in quantum mechanics. In this paper, a new three-time level implicit approach based on cubic trigonometric B-spline is presented for the approximate solution of this equation with Dirichlet boundary conditions. The usual finite difference approach is used to discretize the time derivative while cubic trigonometric B-spline is applied as an interpolating function in the space dimension. Several examples are discussed to exhibit the feasibility and capability of the approach. The absolute errors and L_∞ error norms are also computed at different times to assess the performance of the proposed approach and the results were found to be in good agreement with known solutions and with existing schemes in literature.

Citation: Mat Zin S, Abbas M, Abd Majid A, Md Ismail AI (2014) A New Trigonometric Spline Approach to Numerical Solution of Generalized Nonlinear Klien-Gordon Equation. PLoS ONE 9(5): e95774. doi:10.1371/journal.pone.0095774

Editor: Vladimir E. Bondarenko, Georgia State University, United States of America

Received February 22, 2014; **Accepted** March 28, 2014; **Published** May 5, 2014

Funding: This study was fully supported by FRGS Grant of No. 203/PMATHS/6711324 from the School of Mathematical Sciences, Universiti Sains Malaysia, 11800 USM Penang, Malaysia. The funders had no role in study design, data collection and analysis, decision to publish, or preparation of the manuscript.

Competing Interests: The authors have declared that no competing interests exist.

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Introduction

The generalized nonlinear Klien-Gordon (KG) equation arises in various problems in science and engineering. This paper focuses on the analysis and numerical solution of the generalized nonlinear KG equation, which is given in the following form [14]:

$$\frac{\partial^2}{\partial t^2}u(x,t)+\alpha\frac{\partial^2}{\partial x^2}u(x,t)+\beta u(x,t)+G(u(x,t))=f(x,t), \quad a\leq x\leq b,\ 0\leq t\leq T \tag{1}$$

subject to initial conditions

$$\begin{cases} u(x,0)=\omega_1(x), \\ \dfrac{\partial}{\partial t}u(x,0)=\omega_2(x), \end{cases} \quad a\leq x\leq b \tag{2}$$

and with Dirichlet boundary conditions

$$\begin{cases} u(a,t)=\phi_1(t), \\ u(b,t)=\phi_2(t), \end{cases} \quad 0\leq t\leq T \tag{3}$$

where $u=u(x,t)$ denotes the wave displacement at position and time (x,t), α and β are real constants, $G(u)$ is a nonlinear function of u and $f(x,t)$, $\omega_1(x)$, $\omega_2(x)$, $\phi_1(t)$ and $\phi_2(t)$ are known functions.

In particular, the KG equation is important in mathematical physics especially in quantum mechanics and it is well known as a soliton equation. A study on the interaction of soliton in collisionless plasma, the recurrence of initial states and examination of the nonlinear wave equations was in [1].

Several methods, in addition to several finite difference schemes, have been developed to solve the nonlinear KG equation. Jiminez and Vazquez [2] introduced four numerical schemes for solving the nonlinear KG equation. Ming and Guo utilized a Fourier collocation method for solving the nonlinear KG equation [3]. The KG equation was approximated using decomposition scheme by Deeba and Khuri [4] and using the Legendre spectral method by Guo et al [5]. Wong et. al. solved an initial value problem involving the nonlinear KG equation using fully implicit and discrete energy conserving finite difference scheme [6]. Wazwaz introduced the tanh and sine-cosine method to obtain compact and noncompact solutions for the nonlinear KG equation [7]. Sirendaoreji solved the nonlinear KG equation using the auxiliary equation method to construct new exact traveling wave solutions with quadratic and cubic nonlinerity [8]. Yucel solved the nonlinear KG equation using homotopy analysis method [9] and Chowdhury and Hashim solved the equation using homotopy-pertubation method [10].

B-spline functions can be used to solve numerically linear and non-linear differential equations. Caglar et. al. [11] has introduced a cubic B-spline interpolation method to solve the two-point boundary value problem. Hamid et al. [12] has introduced an alternative cubic trigonometric B-spline interpolation method to solve the same problem. Dehghan and Shokri [13] have obtained a numerical solution of the nonlinear KG equation using Thin Plate Splines radial basis functions. Khuri and Sayfy [14] have solved the generalized nonlinear KG equation using a finite

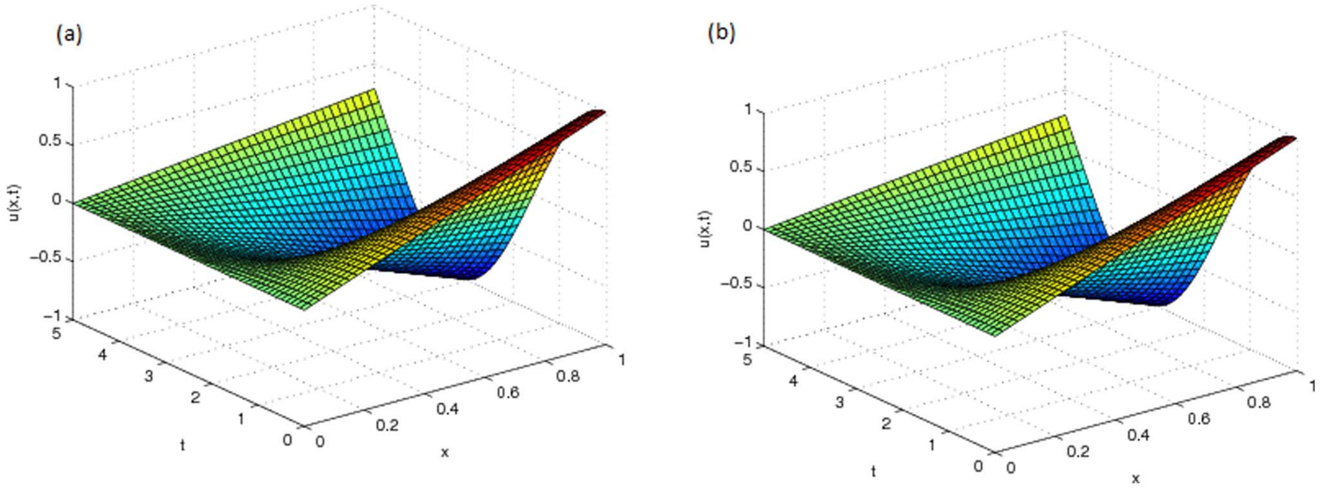

Figure 1. (a) Analytical solution, $\bar{u}(x,t)$ of Problem 1 (b) Approximate solution, $u(x,t)$ of Problem 1 with $n=40$ and $\Delta t = 0.1$.
doi:10.1371/journal.pone.0095774.g001

element collocation approach based on third degree B-spline polynomials.

In this work, a new three-time level implicit approach which combines a finite difference approach and cubic trigonometric B-spline collocation method (CTBCM) is proposed to solve generalized nonlinear KG equation. The finite difference approach is proposed to discretize time derivative and cubic trigonometric collocation method is applied to interpolate the solutions at time$t$. Two numerical experiments are carried out to calculate the numerical solutions, absolute errors, L_∞ error norms and order of convergence for each problem in order to show the accuracy of method.

Temporal Discretization

Consider a uniform mesh Ω with grid points (x_j,t_k) to discretize the grid region $\Delta=[a,b]\times[0,T]$ with $x_j=a+jh,\ j=0,1,2,...,n$ and $t_k=k\Delta t,\ k=0,1,2,3,...,N,\ T=N\Delta t$. h and Δt denote mesh space size and time step size respectively. The time derivative is approximated using the central finite difference formula

$$\frac{\partial^2 u^k}{\partial t^2}=\frac{u_j^{k+1}-2u_j^k+u_j^{k-1}}{(\Delta t)^2} \tag{4}$$

Using the approximation of equation (4), equation (1) becomes

$$\frac{u_j^{k+1}-2u_j^k+u_j^{k-1}}{(\Delta t)^2}+\alpha\frac{\partial^2 u^k}{\partial x^2}+\beta u_j^k=f(x_j,t_k)-G(u(x_j,t_k)) \tag{5}$$

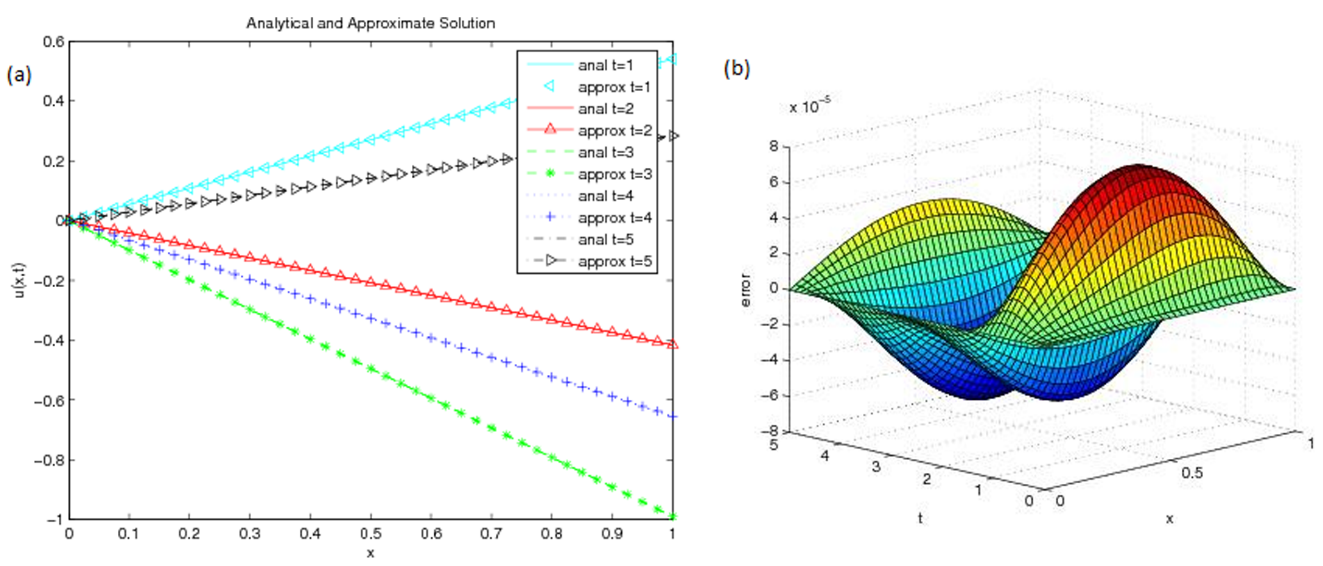

Figure 2. (a) Analytical and approximate solution at different time (b) Error of Problem 1 with $n=40$ and $\Delta t = 0.1$.
doi:10.1371/journal.pone.0095774.g002

Table 1. Numerical solution of Problem 1 at $t=5$.

x/n	0	0.2	0.4	0.6	0.8	1.0
10	0	0.0567	0.1134	0.1702	0.2269	0.2837
20	0	0.0567	0.1135	0.1702	0.2269	0.2837
40	0	0.0567	0.1135	0.1702	0.2269	0.2837
80	0	0.0567	0.1135	0.1702	0.2269	0.2837
$\bar{u}(x,t)$	0	0.0567	0.1135	0.1702	0.2269	0.2837

doi:10.1371/journal.pone.0095774.t001

Using the $\theta-$weighted technique, the space derivatives of equation (5) becomes

$$\frac{u_j^{k+1}-2u_j^k+u_j^{k-1}}{(\Delta t)^2}+(1-\theta)g_j^k+\theta g_j^{k+1}=f_j^k-G\left(u_j^k\right) \quad (6)$$

where $g_j^k=\alpha\frac{\partial^2 u^k}{\partial x^2}+\beta(u)_j^k,\ g_j^{k+1}=\alpha\frac{\partial^2 u^{k+1}}{\partial x^2}+\beta(u)_j^{k+1},\ 0\leq\theta\leq 1$ and the subscripts k and $k+1$ are successive time levels.

After simplification, equation (6) leads to

$$\begin{aligned} u_j^{k+1}+\theta(\Delta t)^2 g_j^{k+1} &= 2u_j^k+(\Delta t)^2\left[f_j^k-G\left(u_j^k\right)\right] \\ &-(1-\theta)(\Delta t)^2 g_j^k-u_j^{k-1} \end{aligned} \quad (7)$$

Trigonometric B-Spline Collocation method

In this section, CTBCM is used to solve Klien-Gordon equation. Cubic trigonometric B-spline are used to approximate the space derivatives. To construct the numerical solution, nodal points (x_j,t_k) defined in the region $[a,b]\times[0,T]$ where $a=x_0<x_1<\cdots<x_n=b,\ \ x_{j+1}-x_j=h$.

The approximate solution $u(x,t)$ to the exact solution $\bar{u}(x,t)$ is defined as [18]:

$$u(x,t)=\sum_{j=-3}^{n-1}C_j(t)T_{4,j}(x) \quad (8)$$

where $C_j(t)$ are time dependent unknowns to be determined and $T_{4,j}(x)$ are cubic trigonometric B-spline basis function given as:

$$T_{4,j}(x)=\frac{1}{\omega}\begin{cases} p^3(x_j), & x\in[x_j,x_{j+1}] \\ p(x_j)\left(p(x_j)q(x_{j+2})+q(x_{j+3})p(x_{j+1})\right)+q(x_{j+4})p^2(x_{j+1}), & x\in[x_{j+1},x_{j+2}] \\ q(x_{j+4})\left(p(x_{j+1})q(x_{j+3})+q(x_{j+4})p(x_{j+2})\right)+p(x_j)q^2(x_{j+3}), & x\in[x_{j+2},x_{j+3}] \\ q^3(x_{j+4}), & x\in[x_{j+3},x_{j+4}] \end{cases} \quad (9)$$

where

$$p(x_j)=\sin\left(\frac{x-x_j}{2}\right),$$

$$q(x_j)=\sin\left(\frac{x_j-x}{2}\right),$$

$$\kappa=\sin\left(\frac{h}{2}\right)\sin(h)\sin\left(\frac{3h}{2}\right).$$

Due to local support properties of B-spline basis function, there are only three non-zero basis functions $T_{4,j-3}(x_j)$, $T_{4,j-2}(x_j)$ and $T_{4,j-1}(x_j)$ are included over subinterval $[x_j,x_{j+1}]$. Thus, the approximation u_j^k and its derivatives with respect to x can be simplified as:

$$\begin{cases} u_j^k &= \eta_1 C_{j-3}(t)+\eta_2 C_{j-2}(t)+\eta_1 C_{j-1}(t) \\ (u_x)_j^k &= -\eta_3 C_{j-3}(t)+\eta_3 C_{j-1}(t) \\ (u_{xx})_j^k &= \eta_4 C_{j-3}(t)+\eta_5 C_{j-2}(t)+\eta_4 C_{j-1}(t) \end{cases} \quad (10)$$

where

$$\eta_1=\csc(h)\csc\left(\frac{3h}{2}\right)\sin^2\left(\frac{h}{2}\right),$$

Table 2. Absolute error of Problem 1 at $t=5$.

x/n	0	0.2	0.4	0.6	0.8	1.0
10	0	2.473×10^{-5}	3.810×10^{-5}	3.559×10^{-5}	2.152×10^{-5}	0
20	0	6.486×10^{-6}	1.496×10^{-6}	8.892×10^{-6}	5.485×10^{-6}	0
40	0	1.709×10^{-6}	2.490×10^{-6}	2.337×10^{-6}	1.442×10^{-6}	0
80	0	5.119×10^{-7}	7.460×10^{-7}	7.003×10^{-7}	4.320×10^{-7}	0

doi:10.1371/journal.pone.0095774.t002

Table 3. Comparison of L_∞ errors norms with Khuri & Sayfy [14] using $h=0.1$, $\Delta t=0.005$ and Dehghan and Shokri [13] using $h=0.02$, $\Delta t=0.0001$.

t	1	2	3	4	5
Dehghan& Shokri [13]	1.254×10^{-5}	--------------	1.555×10^{-5}	-------------	3.379×10^{-5}
Khuri& Sayfy [14]	2.838×10^{-4}	3.299×10^{-4}	7.005×10^{-5}	3.018×10^{-4}	3.249×10^{-4}
Present method	4.552×10^{-5}	4.358×10^{-5}	1.359×10^{-5}	5.353×10^{-5}	3.868×10^{-5}

doi:10.1371/journal.pone.0095774.t003

$$\eta_2=\frac{2}{1+2\cos(h)},$$

$$\eta_3=\frac{3}{4}\csc\left(\frac{3h}{2}\right),$$

$$\eta_4=\frac{3+9\cos(h)}{4\cos(h/2)-4\cos(5h/2)},$$

$$\eta_5=-\frac{3\cot^2(h/2)}{2+4\cos(h)}.$$

The solution to problem (1) is obtained by substituting equation (10) into equation (7). Initially, time dependent unknowns $\mathbf{C}^0$ are calculated and shown in section 3.1. Next, the following initial condition is substituted into the last term of equation (7) for computing $\mathbf{C}^1$

$$u_j^{-1}=u_j^1-2\Delta t\omega_2(x) \tag{10a}$$

Subsequently, the time dependent unknowns, $\mathbf{C}^k$ for $k\geq1$ should be generated. After simplification, equation (7) leads to

$$\rho_5u_j^{k+1}+\rho_2(u_{xx})_j^{k+1}=\rho_3u_j^k-\rho_4(u_{xx})_j^k-u_j^{k-1}+D_j^k \tag{11}$$

where,

$$\rho_5=1+\theta\gamma_1,$$

$$\rho_2=\theta\gamma_2,$$

$$\rho_3=2-(1-\theta)\gamma_1,$$

$$\rho_4=(1-\theta)\gamma_2,$$

$$D_j^k=D\left(x_j,t_k\right)=(\Delta t)^2\left(F_j^k-G\left(u_j^k\right)\right)$$

with

$$\gamma_1=\beta(\Delta t)^2,$$

$$\gamma_2=\alpha(\Delta t)^2.$$

The system obtained on simplifying (11) consists of $n+3$ unknowns $\mathbf{C^k}=(C_{-3}^k,C_{-2}^k,C_{-1}^k,\cdots,C_{n-1}^k)$ in $n+1$ linear equations at the time level $t=t_{k+1}$. In order to obtain a unique solution, the equation (8) is applied to the boundary conditions given in equation (3) for two additional linear equations.

$$u(a,t_{k+1})=\eta_1C_{j-3}(t)+\eta_2C_{j-2}(t)+\eta_1C_{j-1}(t)=\phi_1(t_{k+1}) \tag{12}$$

$$u(b,t_{k+1})=\eta_1C_{j-3}(t)+\eta_2C_{j-2}(t)+\eta_1C_{j-1}(t)=\phi_2(t_{k+1}) \tag{13}$$

From equations (11), (12) and (13), the system can be written as

$$\mathbf{MC}^{k+1}=\mathbf{NC}^k-\mathbf{PC}^{k-1}+\mathbf{E} \tag{14}$$

where,

Table 4. The maximum L_∞ errors norms and order of convergence, *p* for Problem 1.

n	$t=2$		$t=5$	
	L_∞	p	L_∞	p
10	4.358×10^{-5}	--------	3.868×10^{-5}	---------
20	1.094×10^{-5}	1.9942	9.643×10^{-6}	2.0039
40	2.886×10^{-6}	1.9927	2.532×10^{-6}	1.9290
80	6.648×10^{-7}	1.7384	7.592×10^{-7}	1.7379

doi:10.1371/journal.pone.0095774.t004

Table 5. Numerical solution of Problem 1 at $t=5$.

x/n	0	0.2	0.4	0.6	0.8	1.0
25	0	0.0567	0.1134	0.1702	0.2269	0.2837
50	0	0.0567	0.1135	0.1702	0.2269	0.2837
100	0	0.0567	0.1135	0.1702	0.2269	0.2837
200	0	0.0567	0.1135	0.1702	0.2269	0.2837
$\bar{u}(x,t)$	0	0.0567	0.1135	0.1702	0.2269	0.2837

doi:10.1371/journal.pone.0095774.t005

$$M\begin{pmatrix} \eta_1 & \eta_2 & \eta_1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ M_1 & M_2 & M_3 & 0 & & & & & 0 \\ 0 & M_1 & M_2 & M_3 & & & & & 0 \\ \vdots & & & & \ddots & & & & \vdots \\ 0 & & & & & M_1 & M_2 & M_3 & 0 \\ 0 & & & & & 0 & M_1 & M_2 & M_3 \\ 0 & 0 & 0 & 0 & \cdots & 0 & \eta_1 & \eta_2 & \eta_1 \end{pmatrix}$$

$$\mathbf{E}=\begin{pmatrix} \phi_1(t_{k+1}) \\ (\Delta t)^2\left[F_0^k-G\left(u_0^k\right)\right] \\ (\Delta t)^2\left[F_1^k-G\left(u_1^k\right)\right] \\ \vdots \\ (\Delta t)^2\left[F_n^k-G\left(u_n^k\right)\right] \\ \phi_2(t_{k+1}) \end{pmatrix}$$

$$\mathbf{N}=\begin{pmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ N_1 & N_2 & N_3 & 0 & & & & & 0 \\ 0 & N_1 & N_2 & N_3 & & & & & 0 \\ \vdots & & & & \ddots & & & & \vdots \\ 0 & & & & & N_1 & N_2 & N_3 & 0 \\ 0 & & & & & 0 & N_1 & N_2 & N_3 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\mathbf{P}=\begin{pmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \eta_1 & \eta_2 & \eta_1 & 0 & \cdots & & & & 0 \\ 0 & \eta_1 & \eta_2 & \eta_1 & & & & & 0 \\ \vdots & & & & \ddots & & & & \vdots \\ 0 & & & & & \eta_1 & \eta_2 & \eta_1 & 0 \\ 0 & & & & & 0 & \eta_1 & \eta_2 & \eta_1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \end{pmatrix}$$

With

$$M_1=\rho_5\eta_1+\rho_2(-\eta_3),$$

$$M_2=\rho_5\eta_2,$$

$$M_3=\rho_5\eta_1+\rho_2(\eta_3),$$

$$N_1=\rho_3\eta_1-\rho_4(-\eta_3),$$

$$N_2=\rho_3\eta_2,$$

$$N_3=\rho_3\eta_1-\rho_4(\eta_3).$$

Half explicit and half implicit scheme is produced by choosing θ to be 0.5. This scheme is known as Crank-Nicolson scheme. System (14) becomes a tri-diagonal matrix system of dimension $(n+3)\times(n+3)$ that can be solved using the Thomas Algorithm [17].

Table 6. Absolute errors of Problem 1 at $t=5$.

x/n	0	0.2	0.4	0.6	0.8	1.0
25	0	4.404×10^{-6}	6.172×10^{-6}	5.745×10^{-6}	3.524×10^{-6}	0
50	0	1.101×10^{-6}	1.550×10^{-6}	1.439×10^{-6}	8.827×10^{-7}	0
100	0	2.783×10^{-7}	3.926×10^{-7}	3.644×10^{-7}	2.236×10^{-7}	0
200	0	7.320×10^{-8}	1.033×10^{-7}	9.587×10^{-8}	5.882×10^{-8}	0

doi:10.1371/journal.pone.0095774.t006

Table 7. Comparison of L_∞ errors norms with Khuri & Sayfy [14] using $h=0.04$, $\Delta t=0.001$ and Dehghan and Shokri [13] using $h=0.02$, $\Delta t=0.0001$.

t	1	2	3	4	5
Dehghan& Shokri [13]	1.254×10^{-5}	------------	1.555×10^{-5}	------------	3.379×10^{-5}
Khuri& Sayfy [14]	4.599×10^{-5}	8.053×10^{-5}	1.276×10^{-5}	7.292×10^{-5}	5.128×10^{-5}
Present method	7.316×10^{-6}	6.986×10^{-6}	2.089×10^{-6}	8.596×10^{-6}	6.245×10^{-6}

doi:10.1371/journal.pone.0095774.t007

Initial vector $\mathbf{C}^0$

The initial vectors $\mathbf{C}^0$can be obtained from the initial condition as well as boundary values of the derivatives of the initial condition [11,15]:

$$\begin{cases} (u_j^0)_x = -\eta_3 C_{j-3}^0 + \eta_3 C_{j-1}^0 = w_1'(x_i),\ j=0 \\ u_j^0 = \eta_1 C_{j-3}^0 + \eta_2 C_{j-2}^0 + \eta_1 C_{j-1}^0 = w_1(x_j),\ j=0,1,2,...,n \\ (u_j^0)_x = -\eta_3 C_{j-3}^0 + \eta_3 C_{j-1}^0 = w_1'(x_j),\ j=n \end{cases} \tag{15}$$

This yields a $(n+3)\times(n+3)$matrix system:

$$\mathbf{AC}^0 = \mathbf{B}$$

$$\begin{pmatrix} -\eta_3 & 0 & \eta_3 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \eta_1 & \eta_2 & \eta_1 & 0 & & & & & 0 \\ 0 & \eta_1 & \eta_2 & \eta_1 & & & & & 0 \\ \vdots & & & & \ddots & & & & \vdots \\ 0 & & & & & \eta_1 & \eta_2 & \eta_1 & 0 \\ 0 & & & & & 0 & \eta_1 & \eta_2 & \eta_1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & -\eta_3 & 0 & \eta_3 \end{pmatrix} \begin{pmatrix} C_{-3}^0 \\ C_{-2}^0 \\ C_{-1}^0 \\ \vdots \\ C_{n-3}^0 \\ C_{n-2}^0 \\ C_{n-1}^0 \end{pmatrix} = \begin{pmatrix} \omega_1'(x_0) \\ \omega_1(x_0) \\ \vdots \\ \omega_1(x_n) \\ \omega_1'(x_n) \end{pmatrix} \tag{16}$$

The solution can be obtained by using the Thomas Algorithm [17].

Stability analysis using von Neumann method

The von Neumann analysis of stability considers the growth of error in a single Fourier mode [19–20]

$$C_j^k = \delta^k \exp(i\eta jh) \tag{17}$$

where $i=\sqrt{-1}$ and η is the mode number. It is known that this method can be used to analyze the stability of linear scheme. All the nonlinear term in (11) are assumed to be zero [19]. Thus, equation (11) becomes

$$\begin{aligned}[1+\theta\gamma_1]u_j^{k+1} + \theta\gamma_2(u_{xx})_j^{k+1} = [2-(1-\theta)\gamma_1]u_j^k \\ -(1-\theta)\gamma_2(u_{xx})_j^k - u_j^{k-1}\end{aligned} \tag{18}$$

where $\gamma_1=(\Delta t)^2\beta$ and $\gamma_2=(\Delta t)^2\alpha$. Substituting the approximate solution (8) in equation (18) leads to

$$\begin{aligned} p_1C_{j-3}^{k+1} + p_2C_{j-2}^{k+1} + p_1C_{j-1}^{k+1} = p_3C_{j-3}^k + p_4C_{j-2}^k + p_3C_{j-1}^k \\ -\eta_1C_{j-3}^{k-1} - \eta_2C_{j-2}^{k-1} - \eta_1C_{j-1}^{k-1}\end{aligned} \tag{19}$$

where

$$p_1 = \eta_1 + \theta(\rho_1),$$

$$p_2 = \eta_2 + \theta(\rho_2),$$

$$p_3 = 2\eta_1 + (1-\theta)(\rho_1),$$

$$p_4 = 2\eta_2 + (1-\theta)(\rho_2),$$

$$\rho_1 = \eta_4\gamma_1 + \eta_1\gamma_2,$$

$$\rho_2 = \eta_5\gamma_1 + \eta_2\gamma_2.$$

Table 8. The maximum L_∞ error norms and order of convergence, *p* for Problem 1.

n	$t=2$		$t=5$	
	L_∞	p	L_∞	p
25	6.9860×10^{-6}	---------	6.2453×10^{-6}	--------
50	1.7540×10^{-6}	1.9938	1.5704×10^{-6}	1.9916
100	4.4432×10^{-7}	1.9810	3.9775×10^{-7}	1.9812
200	1.1690×10^{-7}	1.9263	1.0466×10^{-7}	1.9262

doi:10.1371/journal.pone.0095774.t008

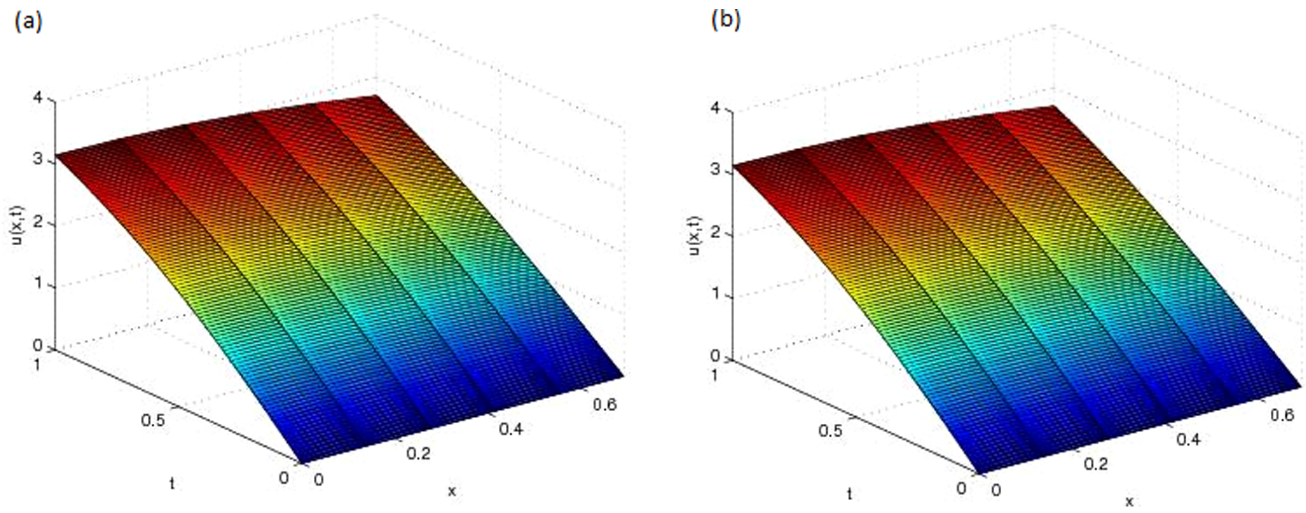

Figure 3. (a) Analytical solution, $\bar{u}(x,t)$ of Problem 2 (b) Approximate solution $u(x,t)$ of Problem 2 with $n=5$, $\Delta t=0.01$.
doi:10.1371/journal.pone.0095774.g003

In order to obtain the amplification factor $|\delta|$, the trial solution (17) is substituted in (19) and after some simplification, we obtain,

$$\delta^2 A + \delta B + C = 0 \tag{20}$$

where $A = p_1(2\cos\eta h) + p_2$, $B = p_3(2\cos\eta h) + p_4$ and $C = \eta_1(2\cos\eta h) + \eta_2$. Since $A,B,C \geq 0$ and $0 \leq \theta \leq 1$, the amplification factor of this scheme is

$$|\delta| = \sqrt{\frac{M}{M+\theta N}} \leq 1 \tag{21}$$

where $M = \eta_1(2\cos\eta h) + \eta_2$ and $N = \rho_1(2\cos\eta h) + \rho_2$. Hence, this scheme is unconditionally stable.

Numerical Results and Discussions

In this section, the CTBCM is applied on two numerical problems. In order to measure the accuracy of the method, absolute errors and L_∞error norms are calculated using the following formulas [16]

$$\text{Absoluteerror} = |\bar{u}_i - u_i| \tag{22}$$

$$L_\infty = \max_i |\bar{u}_i - u_i| \tag{23}$$

where $\bar{u}_i$ and u_i are analytical solution and approximate solution of proposed problem (1), respectively. The numerical order of convergence, p is obtained by using following formula [14]

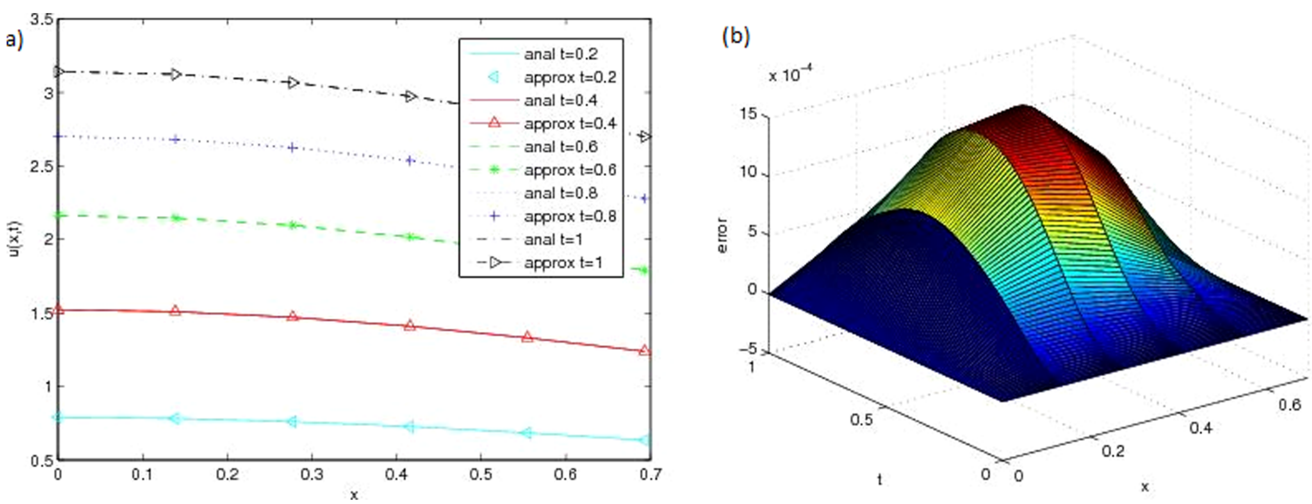

Figure 4. (a) Analytical and approximate solution at different time (b) Error of Problem 2 with $n=5$, $\Delta t=0.01$.
doi:10.1371/journal.pone.0095774.g004

Table 9. Numerical solution of Problem 2 at $t=1$.

x/n	0	$\ln 2/5$	$2\ln 2/5$	$3\ln 2/5$	$4\ln 2/5$	$\ln 2$
5	3.1416	3.1222	3.0654	2.9733	2.8495	2.6990
10	3.1416	3.1223	3.0655	2.9734	2.8496	2.6990
20	3.1416	3.1223	3.0655	2.9734	2.8496	2.6990
40	3.1416	3.1223	3.0655	2.9734	2.8496	2.6990
$\bar{u}(x,t)$	3.1416	3.1224	3.0657	2.9736	2.8497	2.6990

doi:10.1371/journal.pone.0095774.t009

$$p=\frac{Log(L_\infty(n))-Log(L_\infty(2n))}{Log\left(\frac{T}{n}\right)-Log\left(\frac{T}{2n}\right)} \tag{24}$$

where $L_\infty(n)$ and $L_\infty(2n)$ are the L_∞ at number of partition n and $2n$ respectively.

Problem.1

Consider the following nonlinear Klien-Gordon equation [13,14]

$$u_{tt}-u_{xx}+u^2=-x\cos t+x^2\cos^2 t \tag{25}$$

subject to the following boundary and initial conditions

$$u(x,0)=x,\quad u_t(x,0)=0;\qquad 0\le x\le 1$$

$$u(0,t)=0,\quad u(1,t)=\cos t;\qquad 0\le t\le 5$$

The analytical solution of problem (25) is known to be $\bar{u}(x,t)=x\cos t$ and graphically shown in Fig.1 (a). This problem is tested using different values of h and Δt to show the capability of the present method. The final time is taken to be $T=5.0$. Fig.1 (b) and Fig.2 (a) show the approximate solution and Fig.2 (b) shows the error of this problem with $n=40$ and $\Delta t=0.1$. Two cases of this problem are discussed where Case 1 and Case 2 consider $\Delta t=0.001$ and $\Delta t=0.005$,respectively. Numerical solutions, absolute errors and order of convergence of each case are tabulated in Tables1–4 and Tables5–8. The L_∞error norms are compared to Dehghan and Shokri [13] and Khuri and Sayfy [14] in Table3 and Table7. The comparison indicates that the present method is more accurate. The order of convergence of the present problem is calculated by the use of the formula given in (24) and is tabulated in Table4 and Table8. An examination of these tables indicates the method has a nearly second order of convergence.

Case.1. Numerical solutions, absolute errors, L_∞error norms and order of convergence using time step size $\Delta t=0.005$

Case.2. Numerical solutions, absolute errors, L_∞ error norms and order of convergence using time step size $\Delta t=0.001$

Problem.2

The following nonlinear Klien-Gordon equation which is also known as Sine-Gordon equation is considered [14]

$$u_{tt}-u_{xx}+\sin u=0 \tag{26}$$

It is subjected to initial conditions and boundary conditions as

$$u(x,0)=0,\quad u_t(x,0)=4\,\mathrm{sech}\,x;\qquad 0\le x\le \ln 2$$

$$u(0,t)=4\tan^{-1}t,\quad u(\ln 2,t)=4\tan^{-1}\left(\frac{4t}{5}\right);\qquad 0\le t\le 1$$

The analytical solution of this problem is $\bar{u}(x,t)=4\tan^{-1}(t\,\mathrm{sech}\,x)$. Fig.3 (a) depicts a graph of this analytical solution. The final time is taken as$T=5.0$. The approximate solutions are calculated at time step size $\Delta t=0.01$with different mesh space size, h. Numerical solutions are recorded in Table9 and graphical solutions are plotted in Fig.3 (b) and Fig.4 (a). Absolute errors are calculated and shown in Table10 while the 3D error plot is depicted in Fig.4 (b). Table11 shows the comparison of L_∞ error norms between the present method with Khuri and Sayfy [14] method. This comparison shows that the present method gives better results.

Table 10. Absolute error of Problem 2 at $t=1$.

x/n	0	$\ln 2/5$	$2\ln 2/5$	$3\ln 2/5$	$4\ln 2/5$	$\ln 2$
5	0	2.686×10^{-4}	3.534×10^{-4}	3.072×10^{-4}	1.942×10^{-4}	0
10	0	1.744×10^{-4}	2.525×10^{-4}	1.994×10^{-4}	1.080×10^{-4}	0
20	0	1.492×10^{-4}	2.266×10^{-4}	1.737×10^{-4}	8.401×10^{-5}	0
40	0	1.435×10^{-4}	2.193×10^{-4}	1.678×10^{-4}	7.786×10^{-5}	0

doi:10.1371/journal.pone.0095774.t010

Table 11. Comparison of L_∞ errors norms with Khuri and Sayfy [14] using $h=(\ln 2)/5$, $\Delta t=0.01$.

t	0.01	0.02	0.10	0.50	1.00
Khuri& Sayfy [14]	1.3×10^{-6}	6.6×10^{-6}	3.8×10^{-4}	4.8×10^{-3}	2.2×10^{-3}
Present method	5.566×10^{-7}	1.618×10^{-6}	9.851×10^{-5}	1.323×10^{-3}	3.534×10^{-4}

doi:10.1371/journal.pone.0095774.t011

Conclusions

In this work, Klien-Gordon equation has been successfully solved using CTBCM incorporating a finite difference scheme. Specifically, the central difference approach is used to discretize the time derivatives and cubic trigonometric B-spline is used to interpolate the solutions at displacement x. Well-known two test problem were solved using the proposed method and the solution obtained were in good agreement with the known solution. Accurate solutions at intermediate points can be easily obtained.

Acknowledgments

The authors are indebted to the anonymous reviewers for their helpful, valuable comments and suggestions in the improvement of this manuscript.

Author Contributions

Conceived and designed the experiments: SMZ MA AAM AIMI. Performed the experiments: SMZ MA. Analyzed the data: SMZ MA AAM AIMI. Contributed reagents/materials/analysis tools: SMZ. Wrote the paper: SMZ MA AAM AIMI. Calculations and employing B-spline method: SMZ MA. Solved the problem: SMZ MA AAM AIMI. Obtained the results: SMZ.

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