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More Thwarts in Transversal Designs

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Using thwarts, new transversal designs are determined for orders 201, 336, 360, 365, 393, 429, 501, 749, 845, 1080, 1120, 1324, 1400, 1632, 1760, 1824, 1904, and for numerous larger orders. Incomplete transversal designs with block size 8, and PBDs having three consecutive block sizes, are also constructed from thwarts.

1. THWARTS

A *transversal design* of *order n* and *block size k*, or $TD(k; n)$, is a triple $(X, \mathcal{G}, \mathcal{B})$, where X is a set of kn elements. $\mathcal{G} = \{G_1, \ldots, G_k\}$ is a partition of X into k sets each of size n; each class of the partition is a *group*. $\mathcal{B}$ is a set of k-subsets of X, with the property that each $B \in \mathcal{B}$ satisfies $|B \cap G_i| = 1$ for each $1 \le i \le k$; sets in $\mathcal{B}$ are *blocks*. Finally, each unordered pair of elements in X occurs together either in a group or in a single block, but not both.

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A $TD(k, n)$ is equivalent to $k - 2$ mutually orthogonal latin squares (MOLS) of order n. It is well known that a $TD(q + 1, q)$ exists when q is a prime or prime power; the desarguesian TD is constructed from the finite field $GF(q)$. Moreover, $q + 1$ is the largest possible block size for a TD of order q.

Bounds on the block size have great importance in the construction of combinatorial designs (see [1], for example). Many of these bounds arise directly or indirectly from the $TD(q + 1, q)$ arising from the field; we develop some further bounds of this type in this paper.

In the construction of transversal designs, a main ingredient is a certain type of partial transversal design. An *incomplete transversal design* of *order* n and *block size* k with *holes* of sizes $h_1, \ldots, h_l$, or $TD(k; n) - \sum_{i=1}^{l} TD(k; h_i)$, is a quadruple $(X, \mathscr{H}, \mathscr{G}, \mathscr{B})$. X and $\mathscr{G}$ are as before. $\mathscr{H} = \{H_1, \ldots, H_l\}$ is a set of pairwise disjoint subsets of X, with the property that $|H_j \cap G_i| = h_j$ for $1 \leq j \leq l$ and $1 \leq i \leq k$; each H_i is a *hole*. Then $\mathscr{B}$ is a set of k-subsets of X as before, with the property that every unordered pair of elements from X is either in a hole or group together, or in exactly one block of $\mathscr{B}$.

Colbourn *et al.* [8] explore the use of a restricted form of Wilson's theorem [13]. Let x be a nonnegative integer, and let $\mathscr{I} = \{i_1, \ldots, i_s\}$ with $0 \leq i_1 < i_2 < \cdots < i_s \leq x$. Further suppose that $0 \leq s_1 \leq s_2 \leq \cdots \leq s_x \leq n$. Let $(X, \mathscr{G}, \mathscr{B})$ be a $TD(k + x; n)$ with $\mathscr{G} = \{G_1, \ldots, G_k, H_1, \ldots, H_x\}$. Then an $(x, \mathscr{I}, s_1, s_2, \ldots, s_x)$-*thwart* is a set $S = \bigcup_{j=1}^{x} S_j$, where $S_j \subseteq H_j$ with $|S_j| = s_j$ for each $1 \leq j \leq x$, such that for every $B \in \mathscr{B}$, $|B \cap S| \in \mathscr{I}$. We use the notation $(x, \mathscr{I} \cup \{a^\star\}, s_1, s_2, \ldots, s_x)$-*thwart* for a set S in which one block intersects S in a entries, and every other block B satisfies $|B \cap S| \in \mathscr{I}$.

$N(n)$ denotes the largest value of $k - 2$ for which a $TD(k, n)$ exists, and $N^\star(n)$ is the largest value of $k - 2$ for which a $TD(k, n)$ having n disjoint blocks exists.

Then a restriction of Wilson's theorem (see [3]) is:

THEOREM 1.1. *If a* $TD(k + x; n)$ *exists having an* $(x, \mathscr{I}, s_1, s_2, \ldots, s_x)$-*thwart, and if for every* $i \in \mathscr{I}$ *there exists a* $TD(k; m + i) - iTD(k; 1)$, *then there exists a*

$$TD\left(k; mn + \sum_{j=1}^{x} s_j\right) - \sum_{j=1}^{x} TD(k; s_j).$$

If, in addition, a $TD(k; s_j)$ *exists for each* $1 \leq j \leq x$, *then a* $TD(k; mn + \sum_{j=1}^{x} s_j)$ *exists.*

If we impose the stronger condition that $N^\star(m + i) \geq k - 2$ for every $i \in \mathscr{I}$, we obtain the stronger conclusion that a

$$TD\left(k; mn + \sum_{j=1}^{x} s_j\right) - mTD(k, n) - \sum_{j=1}^{x} TD(k; s_j)$$

exists. Numerous variations of Wilson's basic theorem are possible; see [3, 7].

Colbourn *et al.* [8] employ elementary combinatorial methods along with the structure of the $TD(q + 1, q)$ from the field in order to establish the presence of certain thwarts. In particular, they examine the presence of thwarts arising from $TD(3, m)$ in $TD(k, mn)$. We generalize this here to explore the consequences of the presence of $TD(l, m)$, in particular those arising from a generalization of Baer subplanes.

Colbourn *et al.* also examine $(3, \{1, 2, 3\}, a, b, c)$-thwarts and observe that in the $TD(p + 1, p)$ arising from fields of prime order, a simple condition on $a + b + c$ ensures the presence of the thwart. (The proof of Lemma 5.1 in [8] does not indicate how to treat the case when $n = p^\alpha$ is not prime; however, a simple inductive argument on α suffices in that case.) We investigate a similar situation here, but consider thwarts that are subsets of two groups ("levels") and one block ("spike").

Interesting thwarts also arise from geometric structures in planes such as ovals, hyperovals, and Denniston arcs [7]. For example Denniston arcs give rise to the following thwarts.

THEOREM 1.2. *For any integers r and s such that* $0 < r \leq s$, *there exists a* $TD(2^s + 1, 2^s)$ *containing a* $(2^s + 1 - 2^{s-r}, \{0, 2^r\}, 2^r, \ldots, 2^r) -$ *thwart.*

2. Baer Configurations

A (p, α, β)-*Baer configuration* is a set X of $p^\alpha + p^\beta + 1$ points in a projective plane of order p^α, with the properties that, restricting the lines of the plane to the ponts of X,

1. one point of X, the *focus,* lies on $p^{\alpha-\beta} + 1$ lines each of size $p^\beta + 1$; and
2. the remaining points each lie on one line of size $p^\beta + 1$, and on p^β lines each of size $p^{\alpha-\beta} + 1$.

We adopt the name "Baer" because the special case when $\alpha = 2\beta$ is a Baer subplane.

It is easy to verify that if a projective plane of order p^α contains a (p, α, β)-Baer configuration, then every line of the plane meets the configuration in 1, $p^{\alpha-\beta} + 1$, or $p^\beta + 1$ points; all lines meet the configuration at least as a tangent. Thus a Baer configuration, if present, forms a special type of blocking set. It is in this context that they have been studied before.

THEOREM 2.1. *The desarguesian projective plane of order p^α contains a (p, α, β)-Baer configuration whenever $(\alpha - \beta)$ is a divisor of α.*

Theorem 2.1 appears (in different language) in Bruen [5], who credits Ostrom with the result; a short proof appears in [4]. A Baer configuration is essentially a sub-$TD(p^{\alpha-\beta} + 1, p^\beta)$ in the $TD(p^\alpha + 1, p^\alpha)$ arising from the plane. Indeed, deleting the focus of a Baer configuration yields:

COROLLARY 2.2. *Let p be a prime, and $\alpha \geq \beta \geq 1$ satisfy $\alpha - \beta \mid \alpha$. Then there is a $TD(p^\alpha + 1, p^\alpha)$ containing a $(p^{\alpha-\beta} + 1, \{1, p^{\alpha-\beta} + 1\}, p^\beta, \ldots, p^\beta)$-thwart, and a $(p^{\alpha-\beta} + 1, \{0, p^{\alpha-\beta}\}, p^\alpha - p^\beta, \ldots, p^\alpha - p^\beta)$-thwart.*

Applying Wilson's theorem, we obtain:

THEOREM 2.3. 1. *Let $k \leq \min(p^\alpha - p^{\alpha-\beta} - 2, p^\beta - 1, N(m + 1), N^\star(m + p^{\alpha-\beta} + 1))$. Then there are k MOLS of order $(m + 1)p^\alpha + p^\beta$.*

2. *Let $k \leq \min(p^\alpha - p^{\alpha-\beta} - 2, N(p^\alpha + 1), N(p^\beta + 1), N^\star(m + 1), N^\star(m + p^{\alpha-\beta} + 1))$. Then there are k MOLS of order $(m + 1)p^\alpha + p^\beta + 1$.*

THEOREM 2.4. 1. *Let $k \leq \min(p^\alpha - p^{\alpha-\beta} - 2, N(p^\alpha - p^\beta), N(m), N^\star(m + p^{\alpha-\beta}))$. Then there are k MOLS of order $mp^\alpha + (p^{\alpha-\beta} + 1)(p^\alpha - p^\beta)$.*

2. *Let $k \leq \min(p^\alpha - p^{\alpha-\beta} - 2, N(p^\alpha + 1), N(p^\alpha - p^\beta + 1), N^\star(m), N^\star(m + p^{\alpha-\beta}))$. Then there are k MOLS of order $mp^\alpha + (p^{\alpha-\beta} + 1)(p^\alpha - p^\beta) + 1$.*

Table I reports on the consequences for the existence of MOLS up to order 10,000, using Baer configurations. The column "old bound" reports the previously available bound on $N(n)$ from a 1993 update of Brouwer's table [2].

By retaining l levels $(1 \leq l \leq p^{\alpha-\beta})$, we obtain $(l, \{0, 1, l\}, p^\beta, \ldots, p^\beta)$-thwarts. Using $p = 17$, $\alpha = 2$, $\beta = 1$, $m = 31$, and $l \in \{6, 10, 12, 18\}$, we obtain $N(n) \geq 16$ for $n \in \{9061, 9129, 9163, 9265\}$. In one case, we can also truncate one of the levels in the thwart to obtain a $(16, \{0, 1, 15, 16\}, 19, \ldots, 19, 13)$-thwart in a $TD(362, 361)$. Using $m = 16$ establishes that $N(6074) \geq 12$.

We can also employ fewer levels in the complementary thwart; in each of the applications here, an extra point is added as in Theorem 2.4(2).

TABLE I

Order n	Old bound	$N(n) \geq$	p	α	β	m	Authority
201	7	8	2	4	3	11	Theorem 2.4(2)
336	7	8	2	5	4	9	Theorem 2.4(1)
360	7	8	3	3	2	12	Theorem 2.3(1)
365	7	10	2	4	2	19	Theorem 2.4(2)
393	7	8	2	4	3	23	Theorem 2.4(2)
429	7	10	2	4	2	23	Theorem 2.4(2)
749	7	10	2	4	2	43	Theorem 2.4(2)
845	7	10	2	4	2	49	Theorem 2.4(2)
1080	7	12	3	4	3	12	Theorem 2.3(1)
1120	15	16	2	6	5	16	Theorem 2.3(1)
1400	8	10	5	3	2	10	Theorem 2.3(1)
1632	15	24	2	6	5	24	Theorem 2.3(1)
1760	15	26	2	6	5	26	Theorem 2.3(1)
1824	15	26	2	6	5	27	Theorem 2.4(1)
1904	12	15	2	5	4	58	Theorem 2.3(1)
2240	15	16	2	7	6	16	Theorem 2.3(1)
2376	8	26	3	4	3	28	Theorem 2.3(1)
2720	15	31	2	6	5	41	Theorem 2.4(1)
3040	30	31	2	6	5	46	Theorem 2.3(1)
3472	11	15	2	5	4	107	Theorem 2.4(1)
3520	16	26	2	7	6	26	Theorem 2.3(1)
3648	15	26	2	7	6	27	Theorem 2.4(1)
3808	11	31	2	6	5	58	Theorem 2.3(1)
4480	15	16	2	8	7	16	Theorem 2.3(1)
4576	15	31	2	6	5	70	Theorem 2.3(1)
4640	15	31	2	6	5	71	Theorem 2.4(1)
5088	15	31	2	6	5	78	Theorem 2.3(1)
5152	15	31	2	6	5	79	Theorem 2.4(1)
5280	30	31	2	6	5	81	Theorem 2.4(1)
5440	15	40	2	7	6	41	Theorem 2.4(1)
6080	30	46	2	7	6	46	Theorem 2.3(1)
6496	15	31	2	6	5	100	Theorem 2.3(1)
6528	15	24	2	8	7	24	Theorem 2.3(1)
6560	18	31	2	6	5	101	Theorem 2.4(1)
6849	15	21	2	8	6	23	Theorem 2.4(2)
6880	15	31	2	6	5	106	Theorem 2.3(1)
6944	30	31	2	6	5	107	Theorem 2.4(1)
7400	13	24	5	3	2	58	Theorem 2.3(1)
7616	30	58	2	7	6	58	Theorem 2.3(1)
8096	30	31	2	6	5	125	Theorem 2.4(1)
8800	30	31	2	6	5	136	Theorem 2.3(1)
9152	30	63	2	7	6	70	Theorem 2.3(1)
9265	15	16	17	2	1	31	Theorem 2.3(1)
9280	30	63	2	7	6	71	Theorem 2.4(1)
9568	15	31	2	6	5	148	Theorem 2.3(1)
9632	15	31	2	6	5	149	Theorem 2.4(1)

n	$N(n)\geq$	p	α	β	l	m
501	12	2	4	2	3	29
1324	8	3	4	2	6	11
8113	25	2	8	4	5	27
8145	27	2	8	4	3	29

Next we consider the structure of the tangent lines to a Baer configuration. Easy counting shows that $p^\alpha - p^{\alpha-\beta}$ lines are tangent at the focus; at every other point of the configuration, $p^\alpha - p^\beta$ lines are tangent. Points not in the configuration come in two types. Those lying on a line that meets the configuration in $p^\beta + 1$ points lie on p^α tangent lines, while the remainder lie on $p^\alpha - p^{2\beta-\alpha} + 1$ tangent lines (one of them tangent at the focus). Considering the structure induced by the tangent lines and the exterior points of the configuration, and dualizing, the dual plane contains a certain group divisible design:

THEOREM 2.5. *Let p be a prime, $\alpha > \beta \geq 1$, and $\alpha - \beta \mid \alpha$. Then the (dual of the) desarguesian plane contains a group divisible design with block sizes $\{p^\alpha, p^\alpha - p^{2\beta-\alpha} + 1\}$, and one group of size $p^\alpha - p^{\alpha-\beta}$, and $p^\alpha + p^\beta$ groups of size $p^\alpha - p^\beta$.*

By way of example, take $p = 2$, $\alpha = 6$, and $\beta = 4$. Then there is a plane of order 64 containing a $\{64, 61\}$-GDD of type $60^1 48^{80}$. Using the standard GDD construction for MOLS, we obtain $N(3901) \geq 48$.

Baer configurations form a natural generalization of Baer subplanes; in that restricted case, one can partition a plane of order q^2 into Baer subplanes of order q. The arithmetic does not in general support such a partition except in the case of subplanes; however, it may be of interest to determine possible intersections of Baer configurations in the plane.

3. TWO LEVELS AND A SPIKE

In this section, we establish the presence of a further useful thwart in transversal designs of prime and prime power orders, extending an idea in [9]. We first give some definitions.

Let A and B be subsets of Z_n. Then define

$$A -_n B = \{a - b \bmod n : a \in A, b \in B\}.$$

Now define

$$m(n, a, b) = \min\{|A -_n B| : A, B \subseteq Z_n, |A| = a, |B| = b\}.$$

We use the fact that the multiplicative group of the finite field of order q (a prime power) is cyclic in order to establish the presence of certain thwarts:

THEOREM 3.1. *For q a prime or prime power, there exists a $TD(q+1, q)$ containing the thwarts*:

1. *an* $(l+2, \{0, 1, 2, (l+\alpha+\beta)^\star\}, a+\alpha, b+\beta, 1, \ldots, 1)$*-thwart for all* $0 \le l \le q-1-m(q-1, a, b)$, *and* $\alpha, \beta \in \{0, 1\}$.
2. *an* $(l+2, \{1, 2, 3, (l+2)^\star\}, q-a, q-b, 1, \ldots, 1)$*-thwart for all* $m(q-1, a, b) \le l \le q-1$.

Proof. Consider the $TD(q+1, q)$ arising from the finite field $GF(q)$ (i.e., the transversal design arising from the desarguesian projective plane). This transversal design can be represented as ordered pairs from

$$(GF(q)) \times (GF(q)^\star \cup \{R, C\}),$$

with groups $GF(q) \times \{x\}$ for $x \in GF(q)^\star \cup \{R, C\}$, and blocks defined by

$$\{(i, R), (j, C)\} \cup \{(i + \lambda j, \lambda) : \lambda \in GF(q)^\star\},$$

for $i, j \in GF(q)$. Evidently $Z = \{(0, x) : x \in GF(q)^\star \cup \{R, C\}\}$ is a block.

Using this transversal design, we prove the first statement above. Choose $A, B \subseteq Z_{q-1}$ with $|A| = a$ and $|B| = b$, so that $A -_{q-1} B$ has cardinality $m(q-1, a, b)$. Let $D \subseteq GF(q)^\star \setminus (A -_{q-1} B)$ satisfy $|D| = l$. Now let ω be a primitive root of $GF(q)$, and define

$$\begin{array}{lll}
X_1 = \{(\omega^i, R) : i \in A\} & & \\
X_2 = \{(\omega^j, C) : j \in B\} & & \\
X_3 = \{(0, \omega^k) : k \in D\} & & \text{if } q \text{ even} \\
\quad\;\;\; \{(0, \omega^{k+(q-1)/2}) : k \in D\} & & \text{if } q \text{ odd} \\
X_4 = \{(0, R)\} & & \text{if } \alpha = 1, \{\,\} \text{ otherwise,} \\
X_5 = \{(0, C)\} & & \text{if } \beta = 1, \{\,\} \text{ otherwise.}
\end{array}$$

Now $T = \bigcup_{i=1}^{5} X_i$ is the desired thwart. Evidently $T \cap Z$ contains $l + \alpha + \beta$ points (those of $X_3 \cup X_4 \cup X_5$); that all other blocks have intersection of size at most two with the thwart is verified as follows. Suppose to the contrary that some block intersects the thwart in (ω^a, R), (ω^b, C), and $(0, \omega^d)$. Then $\omega^a + \omega^d\omega^b = 0$, so $\omega^{a-b} = -\omega^d$. If q is even, $-1 = \omega^0$, so $a - b \equiv d \pmod{q-1}$; but $d \in A -_{q-1} B$ so $d \notin D$, a contradiction. If instead q is odd, $-1 = \omega^{(q-1)/2}$ so that $a - b \equiv d + (q-1)/2 \pmod{q-1}$; but

$d + (q-1)/2) \in A -_{q-1} B$, and the same contradiction is obtained. Hence in either case, no block intersects the thwart in X_1, X_2, and X_3.

The presence of the second thwarts is proved similarly, complementing the sets A, B, and $A -_{q-1} B$ with respect to $GF(q)^\star$. In this thwart, $(0, R)$ and $(0, C)$ must be present to avoid having a block exterior to the thwart. ∎

Theorem 3.1(2) gives an example of a blocking set in the corresponding projective plane, in which all points are contained in the union of three lines. Cameron [6] gives an excellent discussion of such blocking sets.

Thwarts of this type were used in [9] to construct GDDs with three consecutive block sizes. Here is an example of a class of GDDs that can be obtained in this way.

COROLLARY 3.2. *Suppose q is a prime or prime power and γ, a, and b are integers such that $0 \le \gamma \le q - 1 - m(q-1, a, b)$. Then there exists a group-divisible design having group type $(q-1)^\gamma(q-a-1)^1(q-b-1)^1$, with all block sizes in the set $\{\gamma - 1, \gamma, \gamma + 1\}$.*

Proof. Take $\alpha = \beta = 1$ and $l = q - 1$ in Theorem 3.1(1). Then delete the $q - 1 - \gamma$ groups of the TD disjoint from the thwart, as well as all the points in the thwart. ∎

In applying Theorem 3.1, the essential question is to determine $m(n, a, b)$. The Cauchy–Davenport theorem [11] provides the classical result in this area:

THEOREM 3.3. *For all $n \ge 1$ and $0 \le a, b \le n$, $m(n, a, b) \le \min(n, a + b - 1)$. Moreover, when n is prime, $m(n, a, b) = \min(n, a + b - 1)$.*

When n is composite, the situation is somewhat more complex:

THEOREM 3.4. *Let $d \ge 1$. Then $m(dn, da + s, db + t) \le d\cdot(a + b) + m(d, s, t)$ for $1 \le s, t \le d$.*

Kemperman [12] shows that the bound obtained is best possible.

We develop another application of Theorem 3.1, to the existence of incomplete transversal designs with block size 8; see [10] for more information on this problem. We use an application of Wilson's theorem based on Theorem 3.1, using the facts that $TD(8, 7)$, $TD(8, 8)$, $TD(8, 9)$, and $TD(q + 1, q)$ all exist; indeed a $TD(8, 7) - TD(8, 1)$, $TD(8, 8) - 2TD(8, 1)$, and $TD(8, 9) - 3TD(8, 1)$ exist.

THEOREM 3.5. *Let q be a prime or prime power. Let $1 \le s, t \le q - 1$, and $\alpha, \beta \in \{0, 1\}$.*

A. *Choose an integer u so that $1 \le u \min(q - 9, q - 3 - m(q - 1, s, t))$.*

(a) *If a $TD(8, 7 + \alpha + \beta + u)$ and a $TD(8, s + \alpha)$ both exist, then a $TD(8, 7q + u + s + t + \alpha + \beta) - TD(8, t + \beta)$ exists.*

(b) *If a* $TD(8, 7 + \alpha + \beta + u)$ *and a* $TD(8, t + \beta)$ *both exist, then a* $TD(8, 7q + u + s + t + \alpha + \beta) - TD(8, s + \alpha)$ *exists.*

(c) *If a* $TD(8, s + \alpha)$ *and a* $TD(8, t + \beta)$ *both exist, then a* $TD(8, 7q + u + s + t + \alpha + \beta) - TD(8, 7 + \alpha + \beta + u)$ *exists.*

B. *Choose an integer* u *so that* $m(q - 1, s, t) \le u \le q - 9$. *Further suppose that* $\alpha = \beta = 1$.

(a) *If a* $TD(8, 6 + \alpha + \beta + u)$ *and a* $TD(8, q - 1 - s + \alpha)$ *both exist, then a* $TD(8, 8q + u - 2 - s - t + \alpha + \beta) - TD(8, q - 1 - t + \beta)$ *exists.*

(b) *If a* $TD(8, 6 + \alpha + \beta + u)$ *and a* $TD(8, q - 1 - t + \beta)$ *both exist, then a* $TD(8, 8q + u - 2 - s - t + \alpha + \beta) - TD(8, q - 1 - s + \alpha)$ *exists.*

(c) *If a* $TD(8, q - 1 - s + \alpha)$ *and a* $TD(8, q - 1 - t + \beta)$ *both exist, then a* $TD(8, 8q + u - 2 - s - t + \alpha + \beta) - TD(8, 6 + \alpha + \beta + u)$ *exists.*

Applications of Theorem 3.5 are given in Table II. In each case, a $TD(8, n) - TD(8, h)$ is constructed using the statement of the Theorem given in column T. The column x gives the value of $m(q - 1, s, t)$.

TABLE II
Applications of Theorem 3.5

n	h	T	q	x	s	t	u	α	β	n	h	T	q	x	s	t	u	α	β
100	12	B	13	4	4	4	4	1	1	106	5	A	13	8	4	7	2	1	1
107	6	A	13	6	6	6	3	0	1	123	13	B	16	5	5	5	5	1	1
124	14	B	16	5	5	5	6	1	1	125	15	B	16	5	5	5	7	1	1
148	18	B	19	9	6	8	10	1	1	155	10	A	19	9	7	9	5	0	1
181	22	B	23	11	7	10	14	1	1	178	22	B	23	11	10	10	14	1	1
196	24	B	25	16	6	14	16	1	1	194	24	B	25	16	6	16	16	1	1
195	24	B	25	12	9	12	16	1	1	213	26	B	27	13	10	11	18	1	1
212	26	B	27	13	11	11	18	1	1	228	28	B	29	20	4	20	20	1	1
229	28	B	29	14	10	13	20	1	1	227	28	B	29	14	12	13	20	1	1
226	28	B	29	14	13	13	20	1	1	244	30	B	31	20	6	20	22	1	1
242	30	B	31	20	8	20	22	1	1	243	30	B	31	15	12	15	22	1	1
293	36	B	37	27	5	26	28	1	1	292	36	B	37	27	6	26	28	1	1
290	36	B	37	27	8	26	28	1	1	291	36	B	37	24	12	21	28	1	1
324	40	B	41	32	4	32	32	1	1	323	40	B	41	30	9	28	32	1	1
325	40	B	41	30	10	25	32	1	1	322	40	B	41	30	10	28	32	1	1
341	42	B	43	28	11	26	34	1	1	340	42	B	43	28	11	27	34	1	1
339	42	B	43	28	12	27	34	1	1	371	45	B	47	23	20	22	37	1	1
372	46	B	47	23	20	22	38	1	1	370	46	B	47	23	22	22	38	1	1
388	48	B	49	40	6	38	40	1	1	387	48	B	49	36	9	36	40	1	1

4. Concluding Remarks

We have improved the lower bounds on the number of MOLS of several orders by using thwarts. Also, we have given a brief description of known constructions of thwarts and some applications to the construction of other types of designs.

In closing, it is important to remark that the presence of the thwarts examined here rests on relatively simple structure of the desarguesian plane, inherited naturally from the arithmetic of the field. It is reasonable to expect that other useful configurations can be found using more subtle properties of finite fields.

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