A Bounded Real Lemma for Jump Systems

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Abstract

This paper presents a bounded real lemma for discrete-time Markovian jump linear systems (MJLS). We show that the linear matrix inequality in the bounded real lemma is both necessary and sufficient for this class of systems. For the case of one plant mode, this condition reduces to the standard necessary and sufficient condition for discrete-time systems. We envision this lemma being used to construct necessary and sufficient analysis and synthesis conditions for MJLS.

Index Terms

Markovian Jump Linear Systems, Bounded Real Lemma, Discrete-time, H_∞ norm

I. Introduction

THIS paper presents a bounded real lemma for discrete-time Markovian jump linear systems (MJLS). These systems have previously received significant attention and many theoretical results are available. One motivation for the continued research on these systems is the recent interest in networked control. Specifically, several authors have modeled the packet delivery characteristics of a network by a discrete-time jump system [5], [15], [20], [21], [22]. Empirical evidence suggests that a jump system is a reasonable model for the packet delivery characteristics of a wireless link [16].

Before proceeding, we briefly review some of the work on discrete-time jump systems that is relevant to the results in this paper. We will refer to the definitions of controllability and observability for MJLS given by Ji and Chizeck [12]. We note that these ideas were

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used to solve the Jump Linear Quadratic Gaussian control problem [7], [6], [13]. We will also apply stability results by Ji, et.al. [14] and Costa and Fragoso [8]. Finally, we note that several authors have developed bounded real lemmas for MJLS [2], [11], [4]. These results show that satisfying a matrix inequality condition is sufficient for the MJLS to have H_∞ norm less than a specified level. However, a proof of the necessity is lacking in the literature.

In this paper, we show that the linear matrix inequality (LMI) in the bounded real lemma is both a necessary and sufficient condition for a given class of stochastic inputs into the plant. The proof uses ideas from [18] which gives a dynamic game interpretation to the continuous time H_∞-control of jump linear systems. Reference [1] gives relevant information on generalized Riccati equations related to dynamic games.

II. Markov Jump Linear Systems (MJLS)

Consider the following stochastic system, denoted P:

$$\begin{bmatrix} x(k+1) \\ e(k) \end{bmatrix} = \begin{bmatrix} A_{\theta(k)} & B_{\theta(k)} \\ C_{\theta(k)} & D_{\theta(k)} \end{bmatrix} \begin{bmatrix} x(k) \\ d(k) \end{bmatrix} \tag{1}$$

where $x(k) \in \mathbb{R}^{n_x}$ is the state, $d(k) \in \mathbb{R}^{n_d}$ is the disturbance vector and $e(k) \in \mathbb{R}^{n_e}$ is the error vector. The state matrices are functions of a discrete-time Markov chain taking values in a finite set $\mathcal{N} = \{1, \ldots, N\}$. The Markov chain has transition probabilities $p_{ij} = \Pr(\theta(k+1) = j \mid \theta(k) = i)$ which are subject to the restrictions $p_{ij} \geq 0$ and $\sum_{j=1}^{N} p_{ij} = 1$ for any $i \in \mathcal{N}$. The plant initial conditions are given by specifying $\theta(0)$ and $x(0)$. When the plant is in mode $i \in \mathcal{N}$ (i.e. $\theta(k) = i$), we will use the following notation: $A_i := A_{\theta(k)}$, $B_i := B_{\theta(k)}$, $C_i := C_{\theta(k)}$, and $D_i := D_{\theta(k)}$. Plants of this form are called discrete-time Markovian jump linear systems.

We will work with sequences, $x := \{x(k)\}_{k=0}^{\infty}$, that depend on the sequence of Markov parameters, $\Theta = \{\theta(k)\}_{k=0}^{\infty}$. For notation, define $\Theta_k := \{\theta(1), \ldots, \theta(k)\}$. We define ℓ_2 as the space of square summable (stochastic) sequences:

$$\ell_2^n := \left\{ \{x(k)\}_{k=0}^{\infty} \ : \ \forall k \ x(k) \in \mathbb{R}^n \text{ is a random variable depending on } \Theta_k \text{ and } \|x\|_2 < \infty \right\}$$

where the ℓ_2-norm is defined by $\|x\|_2^2 := \sum_{k=0}^{\infty} \underset{\Theta_{k-1}}{E} \left[x(k)^T x(k) \right]$. Note that Θ_k does not contain $\theta(0)$ because this is assumed to be given as part of the plant initial conditions.

Several forms of stability exist for MJLS [14]. In the remainder of the paper, references to stability will be in the sense of second-moment stability.

Definition 1: For the system given by (1) with $d \equiv 0$, the equilibrium point at the origin is stochastically stable if for every initial state (x_0, θ_0), $\sum_{k=0}^{\infty} \underset{\Theta_{k-1}}{E} [\|x(k)\|^2 \mid x_0, \theta_0] < \infty$.

Ji, et.al. [14] showed that this definition of stability is equivalent to several common forms of stability for a MJLS. Furthermore, the following theorem gives a straightforward necessary and sufficient condition to check for second-moment stability. In the theorem, $\{G_i\} > 0$ means $G_i > 0 \ \forall i \in \mathcal{N}$. In the remainder of the paper, we will use similar notation whenever all matrices in the set satisfy a given condition.

Theorem 1 ([13], [8]) System (1) is SMS if and only if there exist matrices $\{G_i\} > 0$ that satisfy $A_i^T \tilde{G}_i A_i - G_i < 0 \ \forall i \in \mathcal{N}$ where $\tilde{G}_i := \sum_{j=1}^{N} p_{ij} G_j$.

III. Bounded Real Lemma

First we give the definition of the H_∞ norm [9] for discrete-time MJLS.

Definition 2: Assume that P is an SMS system. Let $x(0) = 0$ and define the H_∞ norm, denoted $\|P\|_\infty$, as:

$$\|P\|_\infty := \sup_{\theta(0) \in \mathcal{N}} \ \sup_{0 \neq d \in \ell_2^{n_d}} \frac{\|e\|_2}{\|d\|_2} \tag{2}$$

To derive the bounded real lemma, we need a definition of controllability for a MJLS.

Definition 3: The system, P, is weakly controllable if for every initial state/mode, (x_0, θ_0), and any final state/mode, (x_f, θ_f), there exists a finite time T_c and an input $d_c(k)$ such that

$Pr[x(T_c) = x_f \text{ and } \theta(T_c) = \theta_f] > 0$.

This version of weak controllability is motivated by the definition given by Ji and Chizeck [12]. The weak controllability assumption in the bounded real lemma ensures that the disturbance can affect the system state. If the system is not weakly controllable, the LMI condition is still sufficient, but it may not be necessary.

Theorem 2 (Bounded Real Lemma) Assume the system, P, is weakly controllable. P is SMS and satisfies $\|P\|_\infty < \gamma$ if and only if there exist matrices $\{G_i\} > 0$ that satisfy:

$$R_i := \begin{bmatrix} A_i & B_i \\ C_i & D_i \end{bmatrix}^T \begin{bmatrix} \tilde{G}_i & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A_i & B_i \\ C_i & D_i \end{bmatrix} - \begin{bmatrix} G_i & 0 \\ 0 & \gamma^2 I \end{bmatrix} < 0 \qquad i \in \mathcal{N} \tag{3}$$

where $\tilde{G}_i := \sum_{j=1}^N p_{ij} G_j$.

Proof: ($\Leftarrow$) Assume there exist $\{G_i\} > 0$ satisfying the matrix inequalities in Equation 3. These inequalities imply that the upper left blocks must also be negative definite: $A_i^T \tilde{G}_i A_i - G_i < C_i^T C_i \leq 0$ $\forall i \in \mathcal{N}$. By Theorem 1, we conclude that the system is SMS.

Next, define the function $V(x,i) := x^T G_i x$. Also, let e_M denote the sequence e truncated at time M: $e_M(k) = e(k)$ for $0 \leq k \leq M$ and $e_M(k) = 0$ for $k \geq M$. Given $x(0) = 0$, $V(x(0), \theta(0)) = 0$ for any initial mode $\theta(0) \in \mathcal{N}$ and hence:

$$\sum_{k=0}^{M} \underset{\Theta_{k+1}}{E} \left[V(x(k+1), \theta(k+1)) - V(x(k), \theta(k)) \right] = \underset{\Theta_{M+1}}{E} \left[\, V(x(M+1), \theta(M+1)) \, \right] \geq 0 \tag{4}$$

Inequality (a) below follows from Equation 4:

$$\begin{aligned} \|e_M\|_2^2 - \gamma^2 \|d_M\|_2^2 &\overset{(a)}{\leq} \sum_{k=0}^{M} \underset{\Theta_{k+1}}{E} \left[\|e(k)\|^2 - \gamma^2 \|d(k)\|^2 + V(x(k+1), \theta(k+1)) - V(x(k), \theta(k)) \right] \\ &\overset{(b)}{=} \sum_{k=0}^{M} \underset{\Theta_{k+1}}{E} \left[\begin{bmatrix} x(k) \\ d(k) \end{bmatrix}^T \left(\begin{bmatrix} A_{\theta(k)} & B_{\theta(k)} \\ C_{\theta(k)} & D_{\theta(k)} \end{bmatrix}^T \begin{bmatrix} G_{\theta(k+1)} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A_{\theta(k)} & B_{\theta(k)} \\ C_{\theta(k)} & D_{\theta(k)} \end{bmatrix} - \begin{bmatrix} G_{\theta(k)} & 0 \\ 0 & \gamma^2 I \end{bmatrix} \right) \begin{bmatrix} x(k) \\ d(k) \end{bmatrix} \right] \\ &\overset{(c)}{=} \sum_{k=0}^{M} \underset{\Theta_{k}}{E} \left[\begin{bmatrix} x(k) \\ d(k) \end{bmatrix}^T R_{\theta(k)} \begin{bmatrix} x(k) \\ d(k) \end{bmatrix} \right] \end{aligned}$$

Equality (b) follows by using the system dynamics to replace $e(k)$ and $x(k+1)$ in terms of $x(k)$ and $d(k)$. By taking the expectation over $\theta(k+1)$ we obtain equality (c) where $R_{\theta(k)}$ is defined in Equation 3. By assumption $R_{\theta(k)} < 0 \quad \forall \theta(k) \in \mathcal{N}$. Since $d(k) \neq 0$ for some k, there exists (for sufficiently large M) an $\epsilon > 0$ such that $\|e_M\|_2^2 < \gamma^2 \|d_M\|_2^2 - \epsilon$. Taking limits as $M \to \infty$ gives $\|e\|_2^2 \leq \gamma^2 \|d\|_2^2 - \epsilon$. Hence for any $\theta(0) \in \mathcal{N}$ and any $d \in \ell_2^{n_d}$, $\|e\|_2 < \gamma \|d\|_2$.
($\Rightarrow$) First we show that if $\|P\|_\infty < \gamma$ then there exist matrices, $\{G_i\} \geq 0$, that satisfy the following Generalized Riccati Equations:

$$G_i = A_i^T \tilde{G}_i A_i + C_i^T C_i + [B_i^T \tilde{G}_i A_i + D_i^T C_i]^T \; V_i^{-1} \; [B_i^T \tilde{G}_i A_i + D_i^T C_i] \qquad i \in \mathcal{N} \qquad (5)$$

where $V_i := \gamma^2 I - B_i^T \tilde{G}_i B_i - D_i^T D_i$. Consider the solution, $\{G_i(k)\}$, to the following Generalized Riccati Difference Equations (GRDE) with initial condition $\{G_i(0)\} = 0$:

$$G_i(k+1) = A_i^T \tilde{G}_i(k) A_i + C_i^T C_i + [B_i^T \tilde{G}_i(k) A_i + D_i^T C_i]^T \; V_i(k)^{-1} \; [B_i^T \tilde{G}_i(k) A_i + D_i^T C_i] \quad (6)$$

where $\tilde{G}_i(k) := \sum_{j=1}^N p_{ij} G_j(k)$ and $V_i(k) := \gamma^2 I - B_i^T \tilde{G}_i(k) B_i - D_i^T D_i$. Note that the GRDE will not be defined for $k > k_o$ if $V_i(k_o)$ is singular for some $i \in \mathcal{N}$. Next we show that $\|P\|_\infty < \gamma$ implies that this cannot occur and the solution GRDE exists for all k.

Suppose there does not exist $\alpha > 0$ such that $\{V_i(k)\} > \alpha I \; \forall k$. There are several ways that such an α may fail to exist. By Lemmas 3 and 4 in the Appendix, if $\exists T \geq 0$ such that $\{V_i(k)\} > 0$ for $0 \leq k \leq T-1$ and $V_{\theta_0}(T)$ has an eigenvalue $\lambda \leq 0$ for some $\theta_0 \in \mathcal{N}$ then $\|P\|_\infty \geq \gamma$. Similarly, suppose that $\{V_i(k)\} > 0 \; \forall k$ but there does not exist $\alpha > 0$ such that $\{V_i(k)\} > \alpha I \; \forall k$. In other words, for some $\theta_0 \in \mathcal{N}$, one eigenvalue of $V_{\theta_0}(k)$ tends to zero as $k \to \infty$. A proof similar to that given for Lemma 4 shows this implies $\|P\|_\infty \geq \gamma$.

By contraposition, $\|P\|_\infty < \gamma$ implies $\exists \alpha > 0$ such that $\{V_i(k)\} > \alpha I \; \forall k$. Also by contraposition, $\|P\|_\infty < \gamma$ implies that $\{G_i(k)\}$ are uniformly bounded (Lemma 5 in the Appendix). In summary, the GRDE is well-defined $\forall k$, its solutions $\{G_i(k)\}$ are uniformly bounded, and $\exists \alpha > 0$ such that $\{V_i(k)\} > \alpha I \; \forall k$.

We now show that the matrix sequences $\{G_i(k)\}$ are monotonically nondecreasing in k and thus boundedness of these sequences implies convergence. Since $\{G_i(0)\} = 0$ implies $\{\tilde{G}_i(0)\} = 0$, it is clear from the GRDE that $\{V_i(0)\} > 0$ implies $\{G_i(1)\} \geq \{\tilde{G}_i(0)\} = 0$. Now make the induction assumption that $G_i(k_0) \geq G_i(k_0 - 1)\ \forall i \in \mathcal{N}$. Define another solution to the GRDE, $\{G_i^0(k)\}$, on $k \geq k_0$ with initial condition $\{G_i^0(k_0)\} = \{G_i(k_0 - 1)\}$. Define the difference $\Delta G_i(k) = G_i(k) - G_i^0(k)$ for $i \in \mathcal{N}$ and $k \geq k_0$. Note that, by the induction assumption, $\{\Delta G_i(k_0)\} \geq 0$ and hence $\{\Delta\tilde{G}_i(k_0)\} \geq 0$. Apply Lemma 1 to show $\{\Delta G_i(k_0 + 1)\} \geq 0$ which implies $G_i(k_0 + 1) \geq G_i(k_0)\ \forall i \in \mathcal{N}$. Thus $\{G_i(k)\}$ are monotonic matrix sequences and, as stated above, they are uniformly bounded. Consequently these sequences must have a limit, $\{G_i\} \geq 0$, and this limit matrix satisfies the Generalized Riccati Equation (Equation 5). Since $V_i(k) := \gamma^2 I - B_i^T \tilde{G}_i(k) B_i - D_i^T D_i$, it also has a well defined limit as $k \to \infty$ which we denote by V_i. Finally, $\{V_i(k)\} > \alpha I > 0\ \forall k$ implies $\{V_i\} \geq \alpha I > 0$.

To conclude the proof of necessity, define the perturbed plant, P_ϵ:

$$\begin{bmatrix} x(k+1) \\ e(k) \end{bmatrix} = \begin{bmatrix} A_{\theta(k)} & B_{\theta(k)} \\ C^\epsilon_{\theta(k)} & D^\epsilon_{\theta(k)} \end{bmatrix} \begin{bmatrix} x(k) \\ d(k) \end{bmatrix} \tag{7}$$

The output equation matrices are given by $C^\epsilon_{\theta(k)} := \begin{bmatrix} C_{\theta(k)} \\ \epsilon I_{n_x \times n_x} \end{bmatrix}$ and $D^\epsilon_{\theta(k)} := \begin{bmatrix} c D_{\theta(k)} \\ 0_{n_x \times n_d} \end{bmatrix}$. For sufficiently small $\epsilon > 0$, P_ϵ is SMS and $\|P_\epsilon\|_\infty < \gamma$. By the argument above, there exist matrices, $\{G_i^\epsilon\} \geq 0$, that satisfy the following Generalized Riccati Equations:

$$G_i^\epsilon = A_i^T \tilde{G}_i^\epsilon A_i + (C_i^\epsilon)^T (C_i^\epsilon) + [B_i^T \tilde{G}_i^\epsilon A_i + (D_i^\epsilon)^T C_i^\epsilon]^T\ (V_i^\epsilon)^{-1}\ [B_i^T \tilde{G}_i^\epsilon A_i + (D_i^\epsilon)^T C_i^\epsilon] \quad i \in \mathcal{N} \tag{8}$$

where $V_i^\epsilon := \gamma^2 I - B_i^T \tilde{G}_i^\epsilon B_i - (D_i^\epsilon)^T D_i^\epsilon > 0$. After multiplying out all the matrices we obtain:

$$G_i^\epsilon = A_i^T \tilde{G}_i^\epsilon A_i + C_i^T C_i + [B_i^T \tilde{G}_i^\epsilon A_i + D_i^T C_i]^T\ (V_i^\epsilon)^{-1}\ [B_i^T \tilde{G}_i^\epsilon A_i + D_i^T C_i] = \epsilon^2 I > 0 \quad i \in \mathcal{N}$$

where $V_i^\epsilon := \gamma^2 I - B_i^T \tilde{G}_i^\epsilon B_i - D_i^T D_i > 0$. It follows from these inequalities that $\{G_i^\epsilon\} > 0$. Apply the Schur complement theorem [3] to show that $\{G_i^\epsilon\}$ is a solution of Equation 3. ■

The condition in Equation 3 reduces to the standard necessary and sufficient condition [17] for the case of one mode ($N = 1$). We also note that the 'worst-case' disturbances constructed to prove necessity (Lemmas 2 - 5) depend on the plant state and mode, $(x(k), \theta(k))$. As stated in the introduction, this result has interesting game theory interpretations.

IV. Conclusions

This paper presented a bounded real lemma for discrete-time Markovian jump linear systems. We showed that, given a class of stochastic inputs, the LMI in the bounded real lemma is both a necessary and sufficient condition. A stochastic Lyapunov function was used to prove sufficiency while stochastic disturbances were constructed (in the appendix) to prove the necessity of the lemma. We envision this lemma being used to derive necessary and sufficient LMI conditions for MJLS analysis and controller synthesis.

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Appendix

The following cost function is used in the lemmas:

$$J(d, x_0, \theta_0) := \|e\|_2^2 - \gamma^2 \|d\|_2^2 = \sum_{k=0}^{\infty} \underset{\Theta_k}{E} \left[e(k)^T e(k) - \gamma^2 d(k)^T d(k) \right] \tag{9}$$

where $d(k)$ and $e(k)$ are related by P (Equation 1) with $x(0) = x_0$ and $\theta(0) = \theta_0$. Note that $\|P\|_\infty > \gamma$ if and only if there exists $d \in \ell_2$ and an initial mode θ_0 such that $J(d, 0, \theta_0) > 0$.

Lemma 1: Let $\{G_i^{\gamma_1}(k)\}$ and $\{G_i^{\gamma_2}(k)\}$ be solutions of the GRDE with initial conditions $\{G_i^{\gamma_1}(0)\}$ and $\{G_i^{\gamma_2}(0)\}$, respectively. The superscript denotes the value of γ used in the iteration. Define $\Delta G_i(k) := G_i^{\gamma_2}(k) - G_i^{\gamma_1}(k)$. Then for $i \in \mathcal{N}$:

$$\Delta G_i(k+1) = \bar{A}_i(k)^T \Delta\tilde{G}_i(k) \bar{A}_i(k) + (\gamma_1 - \gamma_2) K_i(k)^T K_i(k) +$$
$$\left[(\gamma_1 - \gamma_2) K_i(k) + B_i^T \Delta\tilde{G}_i(k) \bar{A}_i(k) \right]^T \; (V_i^{\gamma_2}(k))^{-1} \left[(\gamma_1 - \gamma_2) K_i(k) + B_i^T \Delta\tilde{G}_i(k) \bar{A}_i(k) \right]$$

where $K_i(k) := (V_i^{\gamma_1}(k))^{-1}\left[B_i^T \Delta\tilde{G}_i^{\gamma_1}(k) A_i(k) + D_i^T C_i\right]$ and $\bar{A}_i(k) := A_i + B_i K_i(k)$.

Proof: The proof is a simple, albeit algebraically intensive, extension of a result by C. de Souza (Lemma 3.1 in [10]). A proof under the assumption that $p_{ij} = p_j \ \forall i,j \in \mathcal{N}$ is given in [19]. The proof of this lemma follows similarly. ■

Lemma 2: Let $\{G_i(k)\}$ be the solution to the GRDE with initial condition $\{G_i(0)\} = 0$. Fix T and assume $\{V_i(k)\} > 0$ for $0 \le k \le T-1$. Define the following disturbance:

$$\bar{d}(k) = \begin{cases} V_{\theta(k)}(T-k-1)^{-1}\left(B_{\theta(k)}^T \tilde{G}_{\theta(k)}(T-k-1) A_{\theta(k)} + D_{\theta(k)}^T C_{\theta(k)}\right) x(k) & 0 \le k \le T-1 \\ 0 & else \end{cases}$$

Then, $J(\bar{d}, x_0, \theta_0) \ge x_0^T G_{\theta_0}(T) x_0$.

Proof: The lemma is proved by the following string of equalities / inequalities:

$$\begin{aligned} J(\bar{d}, x_0, \theta_0) &\overset{(a)}{\ge} x_0^T G_{\theta_0}(T) x_0 + \sum_{k=0}^{T-1} \underset{\Theta_{k+1}}{E}\left[e(k)^T e(k) - \gamma^2 \bar{d}(k)^T \bar{d}(k) - x(k)^T G_{\theta(k)}(T-k) x(k)\right. \\ &\qquad \left. + x(k+1)^T G_{\theta(k+1)}(T-k-1) x(k+1)\right] \\ &\overset{(b)}{=} x_0^T G_{\theta_0}(T) x_0 + \sum_{k=0}^{T-1} \underset{\Theta_{k}}{E}\left[e(k)^T e(k) - \gamma^2 \bar{d}(k)^T \bar{d}(k) - x(k)^T G_{\theta(k)}(T-k) x(k)\right. \\ &\qquad \left. + x(k+1)^T \tilde{G}_{\theta(k)}(T-k-1) x(k+1)\right] \\ &\overset{(c)}{=} x_0^T G_{\theta_0}(T) x_0 \end{aligned}$$

Inequality (a) follows because $\bar{d}(k) = 0$ for $k \ge T$ and $\{G_i(0)\} = 0 \ \forall i \in \mathcal{N}$. Equality (b) is obtained after taking the expectation over $\theta(k+1)$. Next, substitute for $e(k)$, $x(k+1)$ using the system dynamics and for $G_{\theta(k)}(T-k)$ using the GRDE and then complete the square. Equality (c) follows because $\bar{d}(k)$ makes the resulting summation equal to zero. ■

Lemma 3: Let $\{G_i(k)\}$ be the solution to the GRDE with initial condition $\{G_i(0)\} = 0$. Assume there exists $T \ge 0$ such that $\{V_i(k)\} > 0$ for $0 \le k \le T-1$ and $V_{\theta_0}(T)$ has a negative eigenvalue for some $\theta_0 \in \mathcal{N}$. Then $\|P\|_\infty > \gamma$.

Proof: By assumption, $\exists r, \lambda$ such that $\lambda < 0$ and $V_{\theta_0}(T) r = \lambda r$. Define the disturbance:

$$\bar{d}(k) = \begin{cases} r & k = 0 \\ V_{\theta(k)}(T-k)^{-1} \left(B_{\theta(k)}^T \tilde{G}_{\theta(k)}(T-k) A_{\theta(k)} + D_{\theta(k)}^T C_{\theta(k)} \right) x(k) & 1 \le k \le T \\ 0 & else \end{cases}$$

Apply this disturbance to the system with $x(0) = 0$ and $\theta(0) = \theta_0$:

$$\begin{aligned} J(\bar{d}, 0, \theta_0) &\overset{(a)}{=} r^T (D_{\theta_0}^T D_{\theta_0} - \gamma^2 I) r + \sum_{k=1}^{\infty} \underset{\Theta_k}{E} \left[e(k)^T e(k) - \gamma^2 \bar{d}(k)^T \bar{d}(k) \right] \\ &\overset{(b)}{=} r^T (D_{\theta_0}^T D_{\theta_0} - \gamma^2 I) r + \underset{\theta(1)}{E} \left[J(\bar{d}, B_{\theta_0} r, \theta(1)) \right] \\ &\overset{(c)}{\ge} r^T (D_{\theta_0}^T D_{\theta_0} - \gamma^2 I) r + \underset{\theta(1)}{E} \left[r^T B_{\theta_0}^T G_{\theta(1)}(T) B_{\theta_0} r \right] \\ &\overset{(d)}{=} -r^T V_{\theta_0}(T) r = -\lambda \|r\|^2 > 0 \end{aligned}$$

Equality (a) follows from the choice of $\bar{d}(0)$. Equality (b) follows from a slight abuse of notation. $J(\bar{d}, B_{\theta_0} r, \theta(1))$ denotes the cost of applying $\bar{d}(k)$ for $k \ge 1$ to the system starting at $B_{\theta_0} r$. Inequality (c) then follows from Lemma 2. Equality (d) follows from the definition of $V_{\theta_0}(T)$ after taking the expectation over $\theta(1)$. $J(\bar{d}, 0, \theta_0) > 0$ implies that $\|P\|_\infty > \gamma$. ■

Lemma 4: Let $\{G_i(k)\}$ be the solution to the GRDE with initial condition $\{G_i(0)\} = 0$. Assume there exists $T \ge 0$ such that $\{V_i(k)\} > 0$ for $0 \le k \le T-1$ and $V_{\theta_0}(T)$ has an eigenvalue at zero for some $\theta_0 \in \mathcal{N}$. Then $\|P\|_\infty \ge \gamma$.

Proof: Using the notation of Lemma 1, let $\{G_i^\gamma(k)\}$ denote a solution of the GRDE with initial conditions $\{G_i^\gamma(0)\} = 0$. By assumption, $\{V_i^\gamma(k)\} > 0$ for $0 \le k \le T-1$ and $V_{\theta_0}^\gamma(T)$ has an eigenvalue at zero for some $\theta_0 \in \mathcal{N}$. Given $\epsilon > 0$, let $\{G_i^{\gamma-\epsilon}(k)\}$ denote a second solution of the GRDE with initial conditions $\{G_i^{\gamma-\epsilon}(0)\} = 0$. Thus $\{\Delta G_i(0)\} = 0$ which implies $\{\Delta \tilde{G}_i(0)\} = 0$. If $\epsilon > 0$ is sufficiently small, then $\{V_i^{\gamma-\epsilon}(k)\} > 0$ for $1 \le k \le T$. We can then use Lemma 1 and induction to show that $\{\Delta G_i(0)\} \ge 0$ for $0 \le k \le T$. It follows that $V_{\theta_0}^{\gamma-\epsilon}(T) < V_{\theta_0}^\gamma(T)$ and hence $V_{\theta_0}^{\gamma-\epsilon}(T)$ has a negative eigenvalue. By Lemma 3,

$\|P\|_\infty > \gamma - \epsilon$. Since this holds for all sufficiently small $\epsilon > 0$, $\|P\|_\infty \geq \gamma$. ■

Lemma 5: Assume the plant is weakly controllable. Let $\{G_i(k)\}$ be the solution to the GRDE with initial condition $\{G_i(0)\} = 0$ and assume that $\forall k \geq 0$, $\{V_i(k)\} > 0$. If, for some $\theta_f \in \mathcal{N}$, the sequence $\lambda_{\max}(G_{\theta_f}(k))$ is unbounded, then $\|P\|_\infty > \gamma$.

Proof: By assumption, $\exists \theta_f \in \mathcal{N}$ and a sequence $\{T_j\}_{j=0}^{\infty}$ such that $\lambda_{\max}(G_{\theta_f}(T_j)) \to \infty$ as $j \to \infty$. For each j, let r_j be the eigenvector associated with $\lambda_{\max}(G_{\theta_f}(T_j))$ normalized to $\|r_j\| = 1$. Then $\exists r^* \in \mathbb{R}^{n_x}$ and a subsequence j_l such that $\lim_{j_l \to \infty} r_{j_l} = r^*$. Furthermore, $\lim_{j_l \to \infty} (r^*)^T G_{\theta_f}(T_{j_l}) r^* = \infty$. To ease some of the notation below, we refer to this subsequence as $G_{\theta_f}(T_l)$ with $(r^*)^T G_{\theta_f}(T_l) r^* \to \infty$.

Now apply the assumption of weak controllability: Given $x(0) = 0$ and any initial mode, $\theta(0)$, $\exists T_c$, and an input, $d_c(k)$, such that $p_c := \Pr[x(T_c) = r^* \text{ and } \theta(T_c) = \theta_f] > 0$. Define a second input on $T_c \leq k \leq T_c + T_l - 1$:

$$d_l(k) = V_{\theta(k)}(T_l - T_c - k - 1)^{-1} \left(B_{\theta(k)}^T \tilde{G}_{\theta(k)}(T_l - T_c - k - 1) A_{\theta(k)} + D_{\theta(k)}^T C_{\theta(k)} \right) x(k)$$

We now construct a disturbance which can make the cost function arbitrarily large:

$$\bar{d}_l(k) = \begin{cases} d_c(k) & 0 \leq k \leq T_c - 1 \\ d_l(k) & \text{if } (x(T_c), \theta(T_c)) = (r^*, \theta_f) \text{ and } T_c \leq k \leq T_c + T_l - 1 \\ 0 & else \end{cases}$$

The first portion of the disturbance attempts to move the system from $(0, \theta_0)$ to (r^*, θ_f). Note that $d_l(k)$ is only applied if the system is moved to (r^*, θ_f). By construction of $d_c(k)$, this occurs with some positive probability. The cost can now be lower bounded as follows:

$$J(\bar{d}_l, 0, \theta_0) \overset{(a)}{\geq} \underset{\Theta_{T_c - 1}}{E} \left[\sum_{k=0}^{T_c - 1} e(k)^T e(k) - \gamma^2 \bar{d}_l(k)^T \bar{d}_l(k) \right] + p_c \cdot (r^*)^T G_{\theta_f}(T_l) r^*$$

Inequality (a) follows by the construction of $\bar{d}_l(k)$ and by Lemma 2. The summation is a fixed cost for all l while the second term can be made arbitrarily large as $l \to \infty$. Thus $\exists l$ such that $J(\bar{d}_l, 0, \theta_0) > 0$. Hence $\|P\|_\infty > \gamma$. ■