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Chiral Symmetry Breaking in $2+1$ dimensions due to Sphalerons

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Abstract

In $2+1$ dimensional gauge thories with $SU(N_c)$ color and $2N_f$ flavors of quarks in the fundamental representation, sphalerons, if present may lead to quark condensation and chiral symmetry breaking. The effect is similar to instanton induced chiral symmetry breaking in $3+1$ dimensions and is due to the interaction of quark propagators with sphalerons. The existence of sphalerons requires that color symmetries be broken by a Higgs in the fundamental representation. We show that the sphaleron effect may persist for arbitrarily large N_f but vanishes along with the mass of the gauge fields corresponding to broken generators of $SU(N_c)$. The effect is inherently non-Abelian and absent for QED.

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1 Introduction

Understanding the nature of chiral symmetry breaking in quantum chromo-dynamics (QCD) is one of the more difficult problems in non-Abelian gauge theories. There are analogous but simpler problems in $2 + 1$ dimensions [1] where usually one writes a gap equation for the quark propagator in the presence of gluon exchange or four-fermi interactions. A shift in the pole of the quark propagator to a non-zero value signifies chiral symmetry breaking. In $3+1$ dimensions a completely different non-perturbative approach may also be taken, namely, semiclassical expansion about instantons, which may give effects comparable to gluon exchange [2, 3]. Instantons, technically, are not present in $2+1$ dimensional gauge theories. Nevertheless one can ask if there is a semiclassical effect paralleling the instanton effect of $3 + 1$ dimensions.

In this paper we show that in $2 + 1$ dimensions, a new non-perturbative mechanism can lead to chiral symmetry breaking. The new mechanism is an interaction between quarks and instanton-like field configurations that, in the context of $3 + 1$ dimensional theories, would be called sphalerons. The relevance of instantons to the problem originates from their being the least action gauge field configurations in 4- dimensions that suport a normalizable quark zero mode. Sphalerons, though being different from instantons in many respects, share this vital property.

The existence of the sphaleron requires that the color symmetries be broken. Upon integrating out the massive gauge and Higgs fields one obtains a quark action with four-fermi (vector-vector or Thirring type) effective interactions. Recent studies indicate that if this interaction is strong enough and the number of flavors is small, quarks can condense. However the sphaleron effects are distinguished from gluon exchange effects in two respects. Firstly, we can consider a case where the mass of the Higgs $M_H \sim \sqrt{\lambda v^2}$ is large (λ is a dimensionful coupling and the vacuum expectation value of the Higgs is $v/\sqrt{2}$) and the gauge field is heavy ($M_W \sim gv$ where g is the gauge coupling). For small g/v, the low energy four-fermi theory is weakly coupled. Four-fermi interactions can not consistently break flavor symmetries in this theory, but sphaleron effects can. Secondly, we find that sphaleron effects may persist to arbitrarily large values of flavor (N_f).

In section 2 we explain the notation and review the sphaleron solution. In section 3 we solve a gap equation for sphaleron- induced quark mass. In section 4 we explain the significance of the sphaleron in 3- dimensional physics and discuss some generalizations.

2 The $SU(2)$ Model

Let us consider the case $N_c = 2$. The Euclidean action is

$$
\begin{aligned}
S &= \int d^3x \left[\mathcal{L}_A + \mathcal{L}_F + \mathcal{L}_S \right] \\
\mathcal{L}_A &= \frac{1}{4}(F^a_{\mu\nu})^2 \\
\mathcal{L}_F &= \bar{\psi}_{\alpha a} i\gamma_\mu D_\mu \psi_{\alpha a} \\
\mathcal{L}_S &= |D_\mu \phi|^2 + U(|\phi|)
\end{aligned}
\tag{2.1}
$$

where $a = 1, 2$ is the color index and $\alpha = \pm 1, ... \pm N_f$ is a flavor index. The scalar field ϕ is a doublet under the color $SU(2)$ and U is a potential, $F^a_{\mu\nu} = \partial_\mu W^a_\nu - \partial_\nu W^a_\mu + g\epsilon^{abc} W^b_\mu W^c_\mu$ is the $SU(2)$ field strength, $W_\mu = W^a_\mu \tau_a$ are the $SU(2)$ gauge fields defined using the Pauli matrices τ_a and the covariant derivative is $D_\mu = \partial_\mu - \frac{1}{2} ig W_\mu$. The Euclidean γ matrices are 2×2 and anti-Hermitian. Gauge and global symmetries prevent Yukawa couplings from being generated perturbatively.

Let us summarize what is known about the chiral symmetries in this model. If the scalar sector was absent then a $1/N_f$ expansion would show that the $U(2N_f)$ flavor symmetry is broken to $U(N_f) \times U(N_f)$ and quarks $\psi_{\pm\alpha a}$ acquire masses $\pm m$ when N_f is smaller than a critical value N_{fc} [1]. (The term chiral symmetry is appropriate precisely because of this pairwise generation of masses of opposite signs for the quarks). This pattern of masses preserves the action under a parity transformation described by: $\psi_\alpha(x) \to \sigma_2 \psi_{-\alpha}(x')$; $W_\mu(x) \to P_\mu W_\mu(x')$; $x'_\mu = P_\mu x_\mu$; $P_1 = -P_2 = P_3 = 1$. For the color group G, $N_{fc} = [\frac{128}{3} C_2(R)]/[T(R)\pi^2]$, where $C_2(R)$ is the quadratic Casimir invariant in representation R and $T(R) = C_2(R)d(R)/r$, with $d(R)$ being the dimension of the representation R and r the dimension of the group G [1]. For $SU(N_c)$ in the fundamental represantation, $C_2(R) = (N_c^2 - 1)/N_c$ and $T(R) = 1/2$. The scale of the quark masses is g^2, although the masses are exponentially small for $N_f \to N_{fc}-$. For $N_f > N_{fc}$, there is no flavor symmetry breaking and the quarks are massless. For $SU(2)$ color $N_{fc} = 64/\pi^2$.

The summary above is likely to be qualitatively unchanged upon including the scalar field if the scalar field is heavy, and the color symmetries are not broken. On the other hand, if U has a non-trivial minimum, color symmetries are broken and some of the gauge fields acquire mass $\sim gv$, where g is the gauge coupling and v is the vacuum expectation value (VEV) of the Higgs. Perturbatively, the low energy theory then consists of the light quarks which can condense due to the Thirring type four-fermi interactions that are obtained by integrating out the massive vector bosons. Several approaches to Thirring

models of this kind have been discussed whose results often contradict each other [4]. However, the model we are interested in is solidly grounded on an underlying gauge theory, where the Dyson-Schwinger equations for the quark two point function can be easily written down in a $1/N_f$ expansion. For instance, the Dyson-Schwinger equation for the quark self energy $\Sigma(p)$ can be expanded in $1/N_f$ for our case exactly as in ref. [1] with a massive gauge field propagator.

$$\Sigma(p) = \frac{2C_2(R)\alpha}{N_f}\text{Tr} \int \frac{d^3k}{(2\pi)^3} \frac{\gamma^\mu D_{\mu\nu}(p-k)\Sigma(k)\gamma^\nu}{k^2 + \Sigma^2(k)} \quad , \tag{2.2}$$

where the leading order (in $1/N_f$) gauge field propagator is

$$D_{\mu\nu}(p-k) = \frac{g_{\mu\nu} - [1-\xi]\left[(p-k)_\mu(p-k)_\nu\right]/[k(p-k)^2 + \frac{M_w^2\xi}{(1+\Pi(p-k))}]}{(p-k)^2[1+\Pi(p-k)] + M_w^2} \quad , \tag{2.3}$$

and where the non-local gauge ξ must be chosen to make the quark wave function renormalization zero [5, 6]. The the lowest order approximation in $1/N_f$ gives $\Pi(p-k) = T(R)\alpha/|p-k|$ and α is defined as $N_f g^2/8$. The gauge field mass M_w is given by $M_w^2 = 8v^2\alpha/N_f$. In the $1/N_f$ expansion, α and M_w are to be kept fixed as N_f is made large. In solving the gap equation one makes the approximation that the relevant physics comes from small momenta and $\alpha/|p-k| >> 1$. In this approximation the gauge fixing parameter ξ (found by equating the quark wave function renormalization to zero) is known to approach 2/3 when $M_w \to 0$ [5]. Using the same approximation one can show that $\xi \to 1$ when $M_w^2/(\alpha|p-k|) >> 1$. Thus ξ is usually a momentum dependent function that interpolates between the constant values 2/3 and 1 as the mass M_w increase from 0 to α. We will make the approximation that $\xi = 2/3$ for small M_w and $\xi = 1$ for $M_w \sim \alpha$. The angular integration in (2.2) can be exactly performed to get

$$\Sigma(p) = \frac{2C_2(R)}{T(R)\pi^2 N_f p} \int dk \frac{\Sigma(k)k}{(k^2 + \Sigma^2(k))} I_0 \tag{2.4}$$

where

$$I_0 = \left[\frac{16}{3} + \frac{20M_w^4}{27\alpha^2(p^2 - k^2)}\right]\text{Min}(p,k) - \left[2\frac{M_w^2}{\alpha}\log\left(\frac{|p+k|}{|p-k|}\right)\right] \tag{2.5}$$

The gap equation is difficult to solve for arbitrary M_w. But In the limit $M_w^2/\alpha p \to 0$ one can make the further approximation that only values of k lying in the interval $[\Sigma(k), \alpha]$ contribute in the gap equation. Then for for $p << \Sigma(0)$, I_0 can be replaced by the expression

$$I_0' = \frac{16}{3}\left(1 - \frac{5}{36}\frac{M_w^4}{\alpha^2(k^2)}\right)\text{Min}(p,k) \, . \tag{2.6}$$

3

In this approximation equation (2.4) reduces to the gap equation in [5] with a small perturbation proportional to the fourth power of M_w. The corresponding critical number of flavors is less than $64/\pi^2$ by a number of order $M_w^4/(\alpha^2 \Sigma(0)^2)$. Qualitatively, this result agrees with some recent studies of the Thirring model where the four- fermi- coupling G is analogous to $\frac{\alpha}{M_w^2}$ in our case. As αG decreases, N_{fc} appears to decrease to zero [7].

Keeping in mind that in the $1/N_f$ expansions mentioned above, the condensation takes place due to gluon exchange forces or forces arising from four-fermi contact interactions, we now turn to a different effect, namely the effect of sphalerons, which is non-perturbative in both M_w/α and $1/N_f$. The phenomenon is interesting because if a dynamical mass is generated by sphalerons, a residual mass may be present even when the number of flavors exceeds the critical value N_{fc} or the effective four-fermi coupling strength α/M_w^2 is smaller than the critical value (for a given N_f) obtained from previous studies.

Let us briefly review the salient features of the sphaleron. To make contact with existing literature we choose $U = \lambda(|\phi|^2 - v^2/2)^2$. A symmetric *ansatz* for the sphaleron is the following [8].

$$
\begin{aligned}
W_\mu &= -\frac{2i}{g} f(R)(\partial_\mu U^\infty)(U^\infty)^{-1} \\
\phi &= \frac{v}{\sqrt{2}} h(R) U^\infty \begin{pmatrix} 0 \\ 1 \end{pmatrix}
\end{aligned}
\tag{2.7}
$$

where the $SU(2)$ elements U^∞ are defined as $U^\infty = i\vec{\tau} \cdot \hat{\mathbf{x}}$ and all fields are functions of the dimensionless radial coordinate $R \equiv vgr$ where $r^2 \equiv x_1^2 + x_2^2 + x_3^2$. There exist non-trivial solutions for $h(R)$ and $f(R)$ (not known in closed form) that satisfy the equations of motion. Note that this *ansatz* is in a manifestly $SO(3)$ symmetric gauge.

Historically the sphaleron originates from being a saddle point of the energy functional in a 3+1 dimensional gauge theory. In that context it is a 3- dimensional field configuration at the "top" (the height measures energy in 3+1 dimensions and action in 2+1 dimensions) of a non-contractible loop (NCL) in the configuration space (where the "bottom" of the loop is the vaccum configuration). Its existence therefore is due to a topological reason resembling that of the instantons in gauge theories. Yet, being at the "top" of the NCL, it is, in the present context, a saddle point of the action and unlike the usually encountered instantons, is not a local minimum of the action. Nevertheless, the sphaleron is believed to have only one direction of instability (*i.e.* the energy/action decreases in only one direction). For our purposes the most important property of the sphaleron is that it has quark zero modes.

4

Non-perturbative contributions to quantum amplitudes due to the sphaleron can be calculated by a saddle point expansion around the sphaleron. Multiple sphaleron effects are most easily taken into account by considering a dilute gas of non-interacting sphalerons. For instance the contribution of sphalerons to the vacuum to vacuum transition amplitude is:

$$< 0|0 >= \sum_n < 0|0 >_n= \sum_n \frac{\left[J \exp[-S_S] d^{N_f} [\frac{m_0}{(vg)}]^{2N_f} [\int d^3x (vg)^3] \right]^n}{n!} \qquad (2.8)$$

The r.h.s is the sum of contributions from n non-interacting sphalerons. The contribution of a single sphaleron is raised to the n-th power and divided by the symmetry factor $n!$. Various terms in equation (2.8) can be understood as follows. S_s is the action of the sphaleron and d is the determinant coming from (properly renormalized) Gaussian integrals over a pair of quarks of opposite chirality (ψ_α and $\psi_{-\alpha}$) in the sphaleron background. In computing this determinant the zero-modes of the operator $i\gamma D$ are not included. The contribution of these zero modes is contained in the factor $m_0^{2N_f}$, where m_0 is the expectation value of the quark mass operator in the zero eigenmode ψ_0 (summed over the color index a)

$$m_0 =< \psi_{0a}|m|\psi_{0a} > . \qquad (2.9)$$

Note that if no dynamical mass is generated for the quarks ($m_0 = 0$), the whole amplitude collapses to zero. The factor J contains properly renormalized determinants coming from Gaussian integrals over bosonic and ghost fields and Jacobian factors obtained after converting the final integral over translational zero modes of fields to the collective coordinate x denoting the location of the sphaleron.

$$J = \text{Det}'[-D^2 - U'']^{-1/2}\text{Det}'[-D^2 - 2F]^{-1/2}\text{Det}[-D^2]M \qquad (2.10)$$

where the sphaleron background fields are to be used in computing D and U. Each determinant is understood to be renormalized by division with corresponding determinants with D and U computed with trivial background fields. The prime on some of the determinants implies omission of zero eigenvalues from translation and gauge rotation and omission of the negative eigenvalue associated with the bosonic determinants. The negative eigenvalue appears because the sphaleron is not a local minimum of the action but a saddle point. Such a negative eigenvalue appears also in connection with saddle points (called bounces) that are associated with vacuum tunneling. There, one usually completes the Gaussian integration over the negative eigenmode by analytic continuation, obtaining an imaginary part in the vacuum energy signalling vacuum instability. In the present

case, there is no vacuum instability and the non-convergent integral over the negative eigenmode is simply cut-off at a finite point to yield the unknown but finite factor M, which is then abosrbed in the definition of J. Note that appropriate factors of vg have been inserted in (2.8) to make J and d dimensionless.

3   The Gap Equation

The interaction of a propagating quark with a single background sphaleron can be thought of as a mass insertion $m(p)/p^2$ [3]. Summing the effect of a dilute gas of sphalerons amounts to modifying the massless propagator to $i/[A(p)\gamma_\mu p^\mu - m(p)]$. This gives a self consistent gap equation whose kernel is the mass insertion due to the single sphaleron:

$$\frac{m(p)}{p^2} = J \exp[-S_s] d^{N_f} \left[\frac{m_0}{(vg)}\right]^{2N_f} \frac{\psi_0(p)\psi_0^\dagger(p)}{m_0} \,. \tag{3.11}$$

The r.h.s comes from path integral evaluation of the quark two point function in the zero mode approximation (averaged over color indices) [3]. The product of the Fourier transform of the zero modes $\psi_0(p)\psi_0^\dagger(p)$ has been summed over the color index a and determines the momentum dependence of the mass. Note that once the integral over the collective degrees of freedom of the sphaleron are performed the equation in momentum space is a purely algebraic one. In contrast, the gap equation from the instanton in the $3+1$ dimensional case is an integral equation involving an integration over instantons of all sizes.

The form of the zero modes of $i\gamma D$ can be found in ref. [9]. In the gauge where the sphaleron solution is given by expression [8], the zero mode is:

$$\psi_{0ai}(R) = \epsilon_{ai} N \exp\left[-2\int_0^r \frac{f(R)}{xdx}\right] \tag{3.12}$$

where i and a are spin and color indices respectively and N is a normalization (unknown, since $f(R)$ is not known in closed form). The Fourier transform of the zero mode is:

$$\psi_0(p) = (vg)^3 \int d^3x \psi_0(x) \exp(-ip \cdot x) \,. \tag{3.13}$$

Then the expression for m_0 becomes

$$m_0 = \frac{1}{(vg)^3} \int dp\, m(p) p^2 \psi_0(p) \psi_0^\dagger(p) \,. \tag{3.14}$$

Writing $m(p) = K_{N_f}(p^2/vg)\psi_0(p)\psi_0^\dagger(p)$, one gets from equation (3.11)

$$K_{N_f} = J \exp[-S_s] d^{N_f} \left[\int dp \frac{(\psi_0(p)\psi_0^\dagger(p))^4 p^4}{(vg)^5} \right]^{2N_f - 1} K_{N_f}^{2N_f - 1}. \qquad (3.15)$$

To simplify notation we define

$$C = \left[\int dp \frac{(\psi_0(p)\psi_0^\dagger(p))^2}{(vg)^5} p^4 \right], \qquad (3.16)$$

$$L = J \exp[-S_s]. \qquad (3.17)$$

Then the gap equation has, apart from the trivial solution, the non-trivial solution

$$K_{N_f} = \frac{[Ld^{N_f}C]^{\frac{1}{2(1-N_f)}}}{C}. \qquad (3.18)$$

Equation (3.18) is also valid for half-integral values of N_f for which a parity breaking mass term is expected. Indeed the case $N_f = 1/2$ is interesting because the gap equation reduces to a single sphaleron approximation. (Theories with half integral N_f are ill defined due to the existence of global anomalies [10]. One can however add very heavy "spectator quarks" to cancel anomalies). The dynamical mass is found from the relation:

$$K_{1/2} = Ld^{\frac{1}{2}} \qquad (3.19)$$

$N_f = 2$ is the next interesting case to consider. One finds that no non-trivial solution exists for this case unless $F \equiv [Ld^{N_f}C] = 1$. But the quantities d, C and L depend on the parameters v, g and λ and so, one expects, F should depend on the dimensionless quantities v/g and λ/vg.

Using the relation (3.19) one otains in the arbitrary N_f case:

$$K_{N_f} = K_{1/2} \left[\frac{K_{1/2}^2 C}{L} \right]^{\frac{2N_f - 1}{2(1-N_f)}} \qquad (3.20)$$

from where the large N_f limit is obtained to be:

$$K_\infty = K_{1/2} \left[\frac{L}{K_{1/2}^2 C} \right]. \qquad (3.21)$$

It is necessary to know for what range of these parameters is the semiclassical formula derived above a good approximation. The average sphaleron density must be small for the

dilute gas approximation to hold. In reference [8], several *ansatze* for the sphaleron were used to compute its action. The results there indicate that the action of the sphaleron is $O(v/g)$ with a relatively weak dependence on $\frac{\lambda}{vg}$. The average sphaleron density is found by maximizing $< 0|0 >_n$ with respect to n. This gives the most likely value $\bar{n}$ of n to be:

$$\bar{n} = \left[J\exp[-S_S]d^{N_f} \left[\frac{m_0}{(vg)} \right]^{2N_f} [(vg)^3 \int d^3x] \right] \tag{3.22}$$

whence the mean sphaleron density ρ is obtained to be

$$\rho = \frac{\bar{n}}{\int d^3x} = J\exp[-S_S]d^{N_f} \left[\frac{m_0}{(vg)} \right]^{2N_f} (vg)^3 . \tag{3.23}$$

The dilute gas approximation holds if the mean sphaleron density multiplied by the sphaleron size $(\sim (vg)^{-3})$ is small. Substituting the expression for m_0 we get

$$\rho(vg)^{-3} = \left[L(Cd)^{N_f} \right]^{\frac{1}{1-N_f}} << 1 . \tag{3.24}$$

Computing any term on the r.h.s of equation (3.24) requires exact knowledge of the sphaleron solution. Lacking that, we can still argue that the condition can be satisfied for particular ranges of parameters. Consider the large N_f limit. In this limit, $\rho \sim (Cd)^{-1}$. Note that C depends only on the coupling $\frac{\lambda}{vg}$ which we hold fixed. Because d is the (normalized) product of the squares of the eigenvalues of the operator $i\gamma D$ in the sphaleron background and the normalization depends only on g, for fixed g, d is a function of v/g. The natural scale of the eigenvalues associated with the sphaleron background is vg, hence the eigenvalues increase if the charge g is kept fixed and v is increased, which is to say that their product d is an increasing function of v/g. But d is normalized so that $d(v = 0) = 1$. Therefore $d \geq 1$. It is not possible, without a more detailed calculation to say if ρ can be indefinitely large, or to say how large must v/g be for the dilute gas approximation to be valid. However it is interesting to note that the dilute gas approximation may be valid in both the regimes $0 < v/g \leq 1$ and $1 < v/g$. In the first case, it is known by $1/N_f$ expansions that the low energy theory (after integrating out the massive gauge fields) may break flavor symmetries due to strong four-fermi interactions. The sphaleron method is more trustworthy in the second case, where the four-fermi interaction is weak, and one does not find chiral symmetry breaking in $1/N_f$ expansions. Note that the dynamical mass obtained in (3.20) has the peculiar feature of becoming a constant for large N_f.

4 The Sphaleron and Quark Zero Modes

We have shown that in the $SU(2)$ broken-color theory, sphalerons induce parity breaking mass in the $N_f = 1/2$ theory and may break chiral symmetries for arbitrary N_f. Some other questions one may ask at this point are: (i) Is the result valid for other color groups? (ii) Is it valid if color is unbroken? (iii) Are there objects other than the sphaleron which can induce chiral symmetry breaking?

The first question can be immediately answered. Suppose we have an $SU(N_c)$ gauge theory ($N_c > 2$) which is broken to $SU(N_c - 1)$ by a Higgs in the fundamental representation. Then one can take three of the $2N_c - 1$ broken generators to form an $SU(2)$ in N_c different ways and get N_c distinct sphalerons that suffice to give mass to all the quarks. Note that there is an unbroken $SU(N_c - 1)$ whose massless gauge bosons also give a quark condensation effect that vanishes for $N_f > N_{fc}$.

Let us now consider the second question. First we summarize what we know about different limiting cases:
(i) When $v/g >> 1$, sphaleron gas is a good approximation and seems to give an order vg mass to quarks for large N_f.
(ii) When $v/g \sim 1$, the sphaleron effect may or may not be trustworthy depending on the validity of the dilute gas approximation. However, upon integrating out the massive gauge bosons, one obtains a strongly coupled theory of four-fermi interactions (interaction strength $G \sim g^2/(gv)^2$). This leads to an order vg mass for quarks when $N_f < N_{fc}$.
(iii) When $v/g << 1$, the four-fermi theory is no longer a good low energy theory because the gauge fields are too light. The dilute gas approximation for the sphaleron also breaks down. Since $g^2 >> vg$, the gauge fields can be treated as massless. In this case, an order g^2 quark mass is dynamically generated by gluon exchange. Like case (ii), this mass vanishes as N_f approaches N_{fc} from below.

There is one picture that is consistent with each of the above limiting cases. One can think of the quark mass as made of two parts. There is a part that comes from light gluon exchange and is order g^2. This mass vanishes for large N_f. The second part is an order vg part that comes from sphaleron effects and approaches a non-zero constant value for large N_f. The picture is heuristic since there is no region of parameters where controlled calculation of all three dynamical effects described above can be performed.

The $N_f = 1/2$ case is interesting because it does not require a gap equation or the dilute gas approximation. In this case the quark mass is order vg and goes to zero as

$v \to 0$. Also, for $v = 0$ no sphaleron exists, indicating that the answer to question (ii) may be "no" unless the sphaleron method can be generalized to work when $v = 0$. This brings us to a discussion of question (iii).

Consider a parity invariant theory with two quarks. Recall that the trace of the quark two point function with the background gauge field W can be written as

$$\int d^3x \, \mathrm{Tr}[S(x,x)]_W = \left[\sum_\lambda \frac{1}{\lambda + im} - \frac{1}{\lambda - im}\right] = \sum_{\lambda \geq 0} \left[\frac{2im}{\lambda^2 + m^2}\right] \qquad (4.25)$$

where λ are the eigenvalues of the Dirac operator and ψ_λ are the corresponding eigenfuctions. Note that we have taken a parity invariant trace over two quarks with positive and negative mass terms. In the chiral limit $m \to 0$, the sum over λ will be zero unless there was an infrared divergence. Hence it is the eigenvalues close to zero which lead to quark condensation [11].

Because a sphaleron has a normalizable zero mode, a dilute gas of sphalerons has a large number of almost zero modes, with the eigenvalues vanishing for infinite dilution. Gauge field configurations that have special, normalizable zero modes like the sphaleron are most suitable for writing a gap equation. Let us denote their space to be S_0. What do we know about S_0? Recall the level crossing picture of the chiral anomaly of 4- dimensions [2, 12]. Euclidean gauge field configurations with winding number n contribute to fermion number violating processes by n units. Consider the static energy eigenvalue equation:

$$\gamma_i D_i (1 - \gamma_5) \psi(x_i) = E\psi(x_i) \quad (i = 1, 2, 3) \qquad (4.26)$$

where the γ matrices are four dimensional and there is a background gauge field $A_i(x_\mu)$, $(\mu = 1, 2, 3, 4)$ of winding number n. The level crossing picture tells us that as x_4 is changed from $-\infty$ to $+\infty$, n negative energy levels cross the zero energy level and become positive energy levels (implying a creation of n quarks). Thus in the instanton background $(n = 1)$ there is a value of x_4 $(x_4 = 0)$ for which a normalizable zero eigenmode of the three dimensional Dirac operator exists.

In the $2 + 1$ dimensional gauge theory at hand take a one parameter family of field configurations $A_i(x_\mu)$ that begin $(x_4 \to \infty)$ and end $(x_4 \to -\infty)$ at the vacuum. This is a loop in the space of field configurations which is non-contractible (an NCL) if $\int d^4x \, F \wedge F = n \neq 0$. The winding number of this loop is just like the winding number of an n instanton. Therefore there is level crossing as described above, and for $n \neq 0$ the loop must intersect the surface S_0.
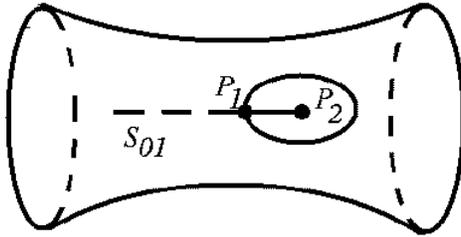
Fig. 1. The schematic picture shows a torus-like space that is not simply connected. The co-dimension 1 surface S_{01} is a line here with a point-like boundary at P_2. The circle centered at P_2 and passing through P_1 intersects S_{01} only once.

Since every NCL must intersect S_0, S_0 has a part which is a co-dimension 1 surface in the space of all field configurations of finite action. This surface S_{01} must have no boundaries which is proven as follows. Suppose P_2 is a point on the boundary (Fig. 1). Then there can be a contractible circle with center at P_2 that intersects S_{01} only at a single point P_1 which is not possible because level crossing does not occur around contractible loops. Thus S_{01} is an unbounded surface that intersects every NCL. How many such unbounded surfaces are there? Suppose their number is s. Since every NCL must cross each of these surfaces, one finds s by considering a particular NCL. The instanton defines a special NCL with a highly symmetric form for the gauge fields. It is known that only the $x_4 = 0$ cross section of the instanton has a zero mode (and exactly one zero mode) [2, 13]. Hence $s = 1$. The justification for a saddle point expansion around the sphaleron is that it is likely to be the least action point on S_{01} at least for small Higgs mass. Indeed for $M_H < 12 M_W$ the sphaleron is the least action spherically symmetric saddle point of the action [14].

We can now also answer question (iii). Firstly, for large Higgs mass the minimum action point on S_0 is not the sphaleron but some other non-trivial field configuration [14, 15]. Furthermore, if the Higgs potential U is minimized at $\phi = 0$ (or if the Higgs does not exist in the theory), the sphaleron is no longer a saddle point of the action. For a given configuration of the gauge field W, the action is minimized by taking $\phi \equiv 0$, and the action can be lowered indefinitely by scaling $x \to \alpha x$, $W \to 1/\alpha W$ for $\alpha > 1$. Thus the minimum action point on S_0 does not exist. It may still be possible to extract the contribution of S_0 by the constrained instanton approach [16]. Namely, one can fix a scale for field configuration W, minimize the action keeping the scale fixed, find the contribution

of this point to the quark condensate and then integrate over the scale. Formally this amounts to finding the contribution of a line in the space S_0. This is an interesting idea that can be the topic of future research in this area.

The sphaleron-instanton parallel may seem a coincidence, since the topological resons for their existence are somewhat different. However, there are objects in $3+1$ dimensional gauge theories that are closer parallels of sphalerons. In an $SU(2)$ gauge theory these are called the I^* instantons [17]. In reference [18], sphaleron like extrema of the Euclidean action were dubbed boomerons and a list of theories were they may exist was presented. The present results indicate a possible role of boomerons in chiral symmetry breaking. Whether I^* plays a similar role in $3+1$ dimensions is yet to be investigated.

5   Conclusions

In conclusion, we have shown that quark mass generation by instanton effects may be possible in $2 + 1$ dimensions. The role of the instantons is played by sphalerons. The unifying idea behind the instanton effects is the existence of a space of gauge field configurations with a normalizable quark zero mode. If the minimum action point of this space is well defined, a semiclassical expansion about that point can lead to a gap equation. Non-trivial solutions of the gap equation signal quark condensation and chiral symmetry breaking. The most interesting aspect of the new dynamical mass is that it vanishes with the color symmetry breaking breaking parameter v/g but approaches a constant value for large N_f. The effect is inherently non-Abelian and is absent in massive QED.

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