Exact maximum singular value calculation of an interval matrix

Hyo-Sung Ahn† and YangQuan Chen‡

Abstract— In this note, we present a method for calculating the maximum singular value of an interval matrix. First, we provide an algorithm for calculating the maximum singular value of a square interval matrix. Then, based on this algorithm, we extend the result to non-square interval matrix case and to the case of computing the minimum singular value. Through numerical examples, the validity of the suggested methods is illustrated. Particularly, we compare the newly-proposed method with an existing method to show that the new method finds the correct bound of the maximum singular value with no exception.

Index Terms— Maximum singular value, interval matrix.

I. INTRODUCTION

A great amount of literature is available for interval uncertain matrix and its stability conditions [1], [2], [3], [4]. In particular, the Hurwitz stability [5], [6], the Schur stability [7], [8], and the eigenvalue boundary problem with perturbation [3], [4], [9], [10] have been well studied and formulated. However, some fundamental interval computational problems such as “power of an interval matrix,” “analytical stability condition of an interval polynomial matrix,” and “maximum singular value bound of an interval matrix” have not been well solved yet. For more detailed discussions about these problems and for some initial results, refer to [11], [12]. This note provides a solution for determining the bound of the maximum singular value of an interval matrix for the first time based on authors’ best knowledge.

From literature search, in fact, boundaries of singular values of an interval matrix were studied in [13]. However, the sign of eigenvectors was limited to be unchanged with interval perturbation. Thus, the algorithm presented in [13] was developed based on some restrictive assumptions. In $H_{\infty}$ robust control, the singular value of an uncertain plant is popularly used under the name of structured singular value (SSV) [14] for designing a robust controller. However, in this traditional robust control, an inequality of the maximum singular value is estimated under the condition that the uncertainty is structured. In this note, we provide a generalized method, which can be used for finding the maximum singular value of a general (unstructured) interval matrix. As possible applications of the results, the monotonic convergence condition of an uncertain discrete-time linear system can be effectively checked and possibly it can be used for $\mu$ synthesis [14] of a robust control system.

This paper is structured as follows. In Section II, we provide the main result for square matrices. Then, in Section III we extend the result to non-square matrices, and in Section IV we consider the minimum singular value of an interval matrix. In Section V, numerical examples are offered for demonstration purpose and conclusion is given in Section VI.

II. MAIN RESULTS

For our main results, we make use of Hertz’s idea for finding extreme eigenvalues of a symmetric interval matrix [2]. In this paper, let us consider a real square non-symmetric interval matrix such as:

$$A^f = [a^f_{ij}], \quad a^f_{ij} := [a_{ij}, \bar{a}_{ij}], i,j = 1, \ldots, n$$

where $a^f_{ij}$ is an element of interval matrix $A^f$, $a_{ij}$ is the lower boundary of an interval $a^f_{ij}$, and $\bar{a}_{ij}$ is the upper boundary of an interval $a^f_{ij}$. If we define the lower boundary matrix and the upper boundary matrix as $A = [a_{ij}]$ and $\bar{A} = [\bar{a}_{ij}]$ respectively, the interval matrix can then be written as $A^f := [A^0 - \Delta, A^0 + \Delta]$, where the center matrix ($A^0$) and the radius matrix ($\Delta$) are defined as

$$A^0 = \frac{1}{2}(A + \bar{A}); \quad \Delta = \frac{1}{2}(A - \bar{A}).$$

In fact, the upper boundary of singular values of an interval matrix can be estimated using the relationship $\sigma_i(A^f) = \sqrt{\lambda_i([A^f]^T \otimes A^f)}$ where $\otimes$ represents multiplication of interval matrices, $\sigma_i$ is the singular value, $\lambda_i$ is the eigenvalue, and $i = 1, 2, \ldots, \text{rank}(A)$. However, as commented in [13], the upper boundary of singular values of an interval matrix cannot be obtained. Instead, in this paper, we suggest using the following relationship between singular values and eigenvalues:

$$\sigma_i(A) = \text{Positive}(\lambda_i \left[ \begin{array}{c}
0 \\
A
\end{array} \right])$$

where Positive($\cdot$) considers only the positive part of ($\cdot$). Obviously, $H$ is a symmetric matrix and it is a member of the symmetric interval matrix

$$H^f = \begin{bmatrix}
0 & (A^f)^T \\
A^f & 0
\end{bmatrix}.$$ 

Hence, if we make use of the results of [2], there will be a way to find maximum singular value (denoted as $\sigma$) of $A^f$. In the sequel, we briefly summarize our main idea and results. Based on [2], since $H$ and $H^f$ are symmetric matrices, we
Therefore, one of the corresponding-vertex matrices for the sign pattern \( + + - \) of \( y \) and for the sign pattern \( + - - \) of \( z \), the sign of the corresponding-vertex matrix is defined by \( y z \) such as:

\[
\begin{bmatrix}
+ \\
- \\
+
\end{bmatrix} \begin{bmatrix}
+ & + & - \\
+ & - & - \\
+ & - & +
\end{bmatrix} = \begin{bmatrix}
+ & + & - \\
- & + & + \\
+ & - & -
\end{bmatrix}.
\] (5)

Therefore, one of the corresponding-vertex matrices for the calculation of maximum singular value is found as:

\[
\begin{bmatrix}
\sigma_{ij} & \sigma_{ij} & \sigma_{ij} \\
\sigma_{ij} & \sigma_{ij} & \sigma_{ij} \\
\sigma_{ij} & \sigma_{ij} & \sigma_{ij}
\end{bmatrix}.
\] (6)

Based on the above discussion, for a general size \( n \) square interval matrix, we can develop an algorithm such as:

- **Step-1**: Produce the set of \( \pm 1 \) vectors with \( y_1 = 1 \) of length \( n \) such as
  \[
  Y = \{ y \in \mathbb{R}^n : y_1 = 1, \ |y_j| = 1, \text{ for } j = 2, \ldots, n \}.
  \]

- **Step-2**: Produce the set of \( \pm 1 \) vectors of length \( n \) such as
  \[
  Z = \{ z \in \mathbb{R}^n : |z_j| = 1, \text{ for } j = 1, \ldots, n \}.
  \]

- **Step-3**: Make \( n \times n \) diagonal matrix \( T_y \) defined by \( (T_y)_{ii} = y_i \) and \( (T_y)_{ij} = 0 \) for \( i \neq j, i, j = 1, \ldots, n \) where \( y \in Y \).

- **Step-4**: Make \( n \times n \) diagonal matrix \( T_z \) defined by \( (T_z)_{ii} = z_i \) and \( (T_z)_{ij} = 0 \) for \( i \neq j, i, j = 1, \ldots, n \) where \( z \in Z \).

- **Step-5**: Produce a matrix set \( S^n : A_{yz} = A^n + T_y T_z \forall y \in Y \text{ and } \forall z \in Z \).

- **Step-6**: Find maximum singular values of all element of the finite set \( S^n \) and select the largest one as the maximum singular value of the interval matrix \( A' \).

Now, to summarize our idea in a compact form and to improve the clarity of the presentation, we make the following theorem.

*Theorem 2.1*: Given a square interval matrix \( A' \), the maximum singular value can be found from the reduced-vertex matrices set \( S^n \), i.e., \( \max\{\sigma(A), A \in A'\} = \max\{\sigma(S), S \in S^n\} \).

*Proof*: Since \( H' \) is a symmetric interval matrix, from
\[ \lambda = 2 \left( \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij}y_{i}z_{j} \right) \] where \( a_{ij} \in A_{ij} \), \( \max \{ a_{ij}y_{i}z_{j} \} \) occurs at one of vertex points of \( A_{ij} \) depending on sign of \( y_{i}z_{j} \). Furthermore, since \( \text{sign}(zy^{T}) = \text{sign}((-z)(-y)^{T}) \) as illustrated from Table I and Table II, the proof is direct by the relationship (2).

### III. Maximum Singular Value of Non-square Interval Matrix

Results of the preceding section can be extended to the general non-square interval matrix case easily. Let us consider \( m \times n \) interval matrix \( A^{T} \). Then, \( H^{T} \) is \( (m + n) \times (m + n) \) interval matrix. Now, introducing length-\( n \) vector \( y \) and length-\( m \) vector \( z \), using the same procedure as in the square matrix case, we have \( \lambda = 2 \left( \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij}y_{i}z_{j} \right) \). Then, we can find that there are a total number of \( 2^{m+n-1} \) possible combinations of vertex matrices that can be used for the calculation of the maximum singular value of a non-square interval matrix. For example, for \( 3 \times 2 \) matrix, we have a total number of \( 2^{3 \times 2} \) combinations as shown in Table III.

#### Table III

16 Sign Patterns for \( 3 \times 2 \) Non-square Matrix.

<table>
<thead>
<tr>
<th>y</th>
<th>z</th>
<th>y</th>
<th>z</th>
</tr>
</thead>
<tbody>
<tr>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>+</td>
<td>-</td>
<td>+</td>
<td>-</td>
</tr>
<tr>
<td>-</td>
<td>+</td>
<td>-</td>
<td>+</td>
</tr>
<tr>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
</tbody>
</table>

In Table III, for example, for the sign pattern \(+ -\) of \( y \) and for the sign pattern \(+ +\) of \( z \), the sign of vertex matrix is defined by \( zy^{T} \) such as:

\[
\begin{bmatrix}
+ \\
- \\
+
\end{bmatrix}
\begin{bmatrix}
+ & - \\
- & +
\end{bmatrix}
= \begin{bmatrix}
+ & - \\
- & +
\end{bmatrix},
\]

which provides the corresponding vertex matrix:

\[
\begin{pmatrix}
a_{ij} & a_{ij} \\
\frac{a_{ij}}{a_{ij}} & \frac{a_{ij}}{a_{ij}}
\end{pmatrix}.
\]

A similar algorithm and a similar theorem as given in Section II can be developed for a non-square case. However, due to the simplicity, detailed discussions are omitted.

### IV. Minimum Singular Value of an Interval Matrix

It is clear that the minimum singular value (denoted as \( \sigma \)) of an interval matrix does not occur at one of the vertex matrices. For example, for the following interval matrix

\[
A^{T} = \begin{bmatrix}
-3.3 & -2.24 & 0.06 \\
1.03 & -0.34 & 1.09 \\
-2.02 & -1.02 & 2.27
\end{bmatrix},
\]

\[
\Delta = \begin{bmatrix}
1.32 & 0.68 & 4.38 \\
0.84 & 2.97 & 1.42 \\
1.61 & 3.06 & 0.55
\end{bmatrix},
\]

Using the suggested method, we found that the maximum singular value of \( A^{T} \) as 9.8549, but from Deif’s method, we have 9.7408. For demonstration purpose, we performed random tests. Fig. 1 shows the Monte-Carlo type random tests. In this figure, the dashed-dot line is the calculated maximum singular value from the suggested method (9.8549) and the solid line is the maximum singular value from Deif’s method. Clearly there exist exceptions in the case of Deif’s method, while the suggested method provides the exact boundary without an exception.

### C. Example-3: Minimum Singular Value

For the minimum singular value, let us consider an interval matrix with the following center matrix and radius matrix

\[
A^{o} = \begin{bmatrix}
1.5 & -0.01 & 3.4 \\
7.1 & -3.4 & -1.3 \\
2.1 & 0.01 & -7
\end{bmatrix},
\]

\[
\Delta = \begin{bmatrix}
0.75 & 0.005 & 1.7 \\
3.55 & 1.7 & 0.65 \\
1.05 & 0.005 & 3.5
\end{bmatrix}.
\]

The following remark is provided for the calculation of the minimum singular value in some cases.

**Remark 4.1:** When an interval matrix is square and regular (i.e., nonsingular of an interval matrix [8]), the inverse of an interval matrix can be found using the method suggested in [16]. Then, from the relationship \( \sigma(A) = 1/\sigma(A^{-1}) \), we can find the minimum singular value of an interval matrix.
where property than the asymptotical stability \[17\], \[18\]. Let us use \[\|\cdot\|\]\text{cally convergent (no-overshoot) to zero (i.e.,} \[\|\cdot\|\text{transient. Whereas, from the following relationship:}\
\[\|x_k\| = \|Ax_k\| \leq \|A\|\|x_k\|,\]
if \[\|A\|<1\] for all \[A \in A^I\], then the state \[x_k\] will be monotonically convergent (no-overshoot) to zero (i.e., \[\|x_{k+1}\| < \|x_k\|\]) in a 2-norm topology, which is a more desirable convergence property than the asymptotical stability \[17\], \[18\]. Let us use the following example for a demonstration purpose:
\[x_{k+1} = Ax_k,\]
where
\[A \in A^I = \begin{bmatrix} [0.7650, 0.9350] & [0.0, 0.0] \\ [-6.0500, -4.9500] & [0.6700, 0.8250] \end{bmatrix} \].

Since \[A^I\] is a lower triangular matrix, the maximum spectral radius is 0.9350. Hence, the system is considered robustly asymptotical stable. However, using the suggested algorithm in this paper, the maximum singular value is calculated as 6.1759. Hence, the monotonic convergent is not guaranteed. Fig. 3 shows the transients of the randomly-selected plants \[A \in A^I\]. As shown in these plots, it is clearly observed that there are overshoots during the transient responses although all plants converge to zero eventually as \[k\] increases.

![Fig. 3. Transient responses of randomly-selected plants \(A \in A^I\).](image)

Let us consider another example:
\[A^I = \begin{bmatrix} [0.8075, 0.8925] & [0.0, 0.0] \\ [-0.2625, -0.2375] & [0.7125, 0.7875] \end{bmatrix} \].

Since \[A^I\] is a lower triangular matrix, the maximum spectral radius is found as 0.8925. Using the suggested algorithm in this paper, the maximum singular value is found as 0.9916. Thus, the system will be monotonic convergent (no-overshoot). Hence, we expect no overshoot during the transient response. Fig. 4 shows the transient responses of the randomly-selected plants. From this figure, we conclude that when the maximum singular value of an interval uncertain system is less than 1, the robustly monotonic convergence of the system is guaranteed.

![Fig. 4. Transient responses of randomly-selected plants \(A \in A^I\).](image)

**Remark 5.1:** In control system, a big overshoot during the transient is not acceptable because the actuator may not provide enough control force to the control system. Thus,
if possible, it is desirable to guarantee the monotonic convergence; hence finding the maximum singular value of an uncertain system matrix is practically important.

VI. CONCLUSIONS

In this paper, an algorithm for calculating the maximum singular value of a general square/non-square interval matrix was suggested. Using the existing result [13], which was developed based on perturbation under some restrictive assumptions, we verified that the proposed method can calculate the maximum singular value accurately. Furthermore, from a created example, we have shown that the existing method does not find the maximum singular value in some cases while the suggested method finds the maximum singular value without an exception. Practical importance of the maximum singular value of an uncertain system was also illustrated through an example. In authors’ best knowledge, this paper presented a solution for the maximum singular value of an interval matrix, which is a fundamental question in the robust control system, for the first time.

REFERENCES


