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Martingale Property and Capacity under G-Framework

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Abstract

The main purpose of this article is to study the symmetric martingale property and capacity defined by G-expectation introduced by Peng (cf. http://arxiv.org/PS_cache/math/pdf/0601/0601035v2.pdf) in 2006. We show that the G-capacity can not be dynamic, and also demonstrate the relationship between symmetric G-martingale and the martingale under linear expectation. Based on these results and path-wise analysis, we obtain the martingale characterization theorem for G Brownian motion without Markovian assumption. This theorem covers the Lèvy's martingale characterization theorem for Brownian motion, and it also gives a different method to prove Lèvy's theorem.

Key words: G-Brownian motion, G-expectation, Martingale characterization, Capacity.

AMS 2000 Subject Classification: Primary 60H05, 60H10.

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1 INTRODUCTION

In 1969, Robert C. Merton introduced stochastic calculus to finance, see [20], and indeed to the broader field of economics, beginning an amazing decade of developments. The most famous pricing formula for the European call option was given by Black and Scholes in 1973, see [2]. As these developments unfolded, Cox and Ross [6] notice that, without loss of generality, to price some derivative security as an option, one would get the correct result by assuming that all of the securities have the same expected rate of return. This is the "risk neutral" pricing method. This notion is first given by Harrison and Kreps [14], who formalize (under conditions) the near equivalence of the absence arbitrage with the existence of some new "risk neutral" probability measure under which all expected rates of return are indeed equal to the current risk free rate. To get this "risk-neutral" probability measure, Harrison and Kreps apply Girsanov's theorem to change the measure.

Black & Scholes's result and Cox's result have been widely used since it appeared. But their work deeply depends on certain assumptions, e.g., the interest rate and the volatility of the stock price remain constant and known. In fact, the interest rate and the volatility of the stock price are not always constant and known, which are called mean uncertainty and volatility uncertainty. As for the mean uncertainty, Girsanov's theorem or Peng's g-expectation is a powerful tool to solve the problem, see [3] and [11]. How to deal with the volatility uncertainty is a big problem, see [1], [15], [19], [27]. As in [9], the main difficulty is that we have to deal with a series of probability measures which are not absolutely continuous with respect to one single probability measure. It shows that this problem can not be solved in a given probability space.

In 2006, Peng made a change to the heat equation that Brownian motion satisfies, see [23], [24], and constructed the G-normal distribution via the modified heat equation. With this G-normal distribution, a nonlinear expectation is given which is called G-expectation and the related conditional expectation is constructed, which is a kind of dynamic coherent risk measure introduced by Delbaen in [7]. Under this framework, the canonical process is a G-Brownian motion. The stochastic calculus of Itô's type with respect to the G-Brownian motion and the related Itô's formula are also derived. G-Brownian motion, different from Brownian motion in the classical case, is not defined on a given probability space.

It is interesting to get a pricing formula (Black-Scholes formula) in G-framework. To do this, it is important to give the Girsanov theorem in this framework. Its proof depends heavily on the martingale characterization of Brownian motion, due to Lèvy. This theorem enables us to recognize a Brownian motion just with one or two martingale properties of a process.

In order to get the Girsanov theorem, we need to describe this G-Brownian motion by its martingale properties. That is the martingale characterization theorem for G-Brownian motion.

In [29], the authors give the martingale characterization theorem for G-Brownian motion under Markovian condition, but this condition is not so convenient in applications.

In this paper, we define capacity via G-expectation, and state that G-expectation is not filtration consistent. Then we investigate the relationship between symmetric G-martingales and martingales under linear expectation, when the corresponding G-heat equation is uniformly parabolic. Based on these results, we give the martingale characterization theorem for G-Brownian motion without Markovian condition which improves the related result in [29]. The current result extends the classical Lèvy's theorem. Additionally, we give a different method to prove Lèvy's theorem.

The main contribution of this work is to investigate the properties for symmetric G-martingales, and give the martingale characterization of G-Brownian motion when the corresponding G-heat equation is uniformly parabolic, see in section 5. G-expectation theory has received more attention since Peng's basic paper appeared, see [23], [24], [25]. Soner et al [21] study the G-martingale problem under some condition, and investigate the representation theorem for all G-martingales including the non-symmetric martingale based on a class of backward stochastic differential equations. In the current work, we are focused on symmetric G-martingale, our method is totally different from the method in [21]. As for the martingale characterization for G-Brownian motion, our method is different from [29], which is based on viscosity solution theory for nonlinear parabolic equation. But in this paper, the path wise analysis is important to underly our result, which is a different approach from [29]. Meanwhile, the result in this paper is very useful for further application in finance, especially for option pricing problems with volatility uncertainty. By using the result in this paper, we can prove the Girsanov type theorem under G-framework and the pricing formula.

The rest of the paper is organized as follows. In section 2, we review the G-framework established in [23] and adapt it according to our objective. In section 3, we investigate some properties of G-expectation by using stochastic control, define the capacity via G-expectation, and show that it is not filtration consistent. In section 4, we investigate the relation between the symmetric G-martingale and the martingale under each probability measure P_v, when the corresponding G-heat equation is uniformly parabolic. In section 5, based on path wise analysis, we give the martingale characterization for G-Brownian motion. The last section is conclusion and discussion about this work and future work.

2 G-Framework

In this section, we recall the G-framework established in [23]. Let $\Omega = C_0(R^+)$ be the space of all R-valued continuous paths functions $(\omega_t)_{t\in R^+}$, with $\omega_0 = 0$. For any $\omega^1, \omega^2 \in \Omega$, we define

$$\rho(\omega^1, \omega^2) := \sum_{i=1}^{\infty} 2^{-i}[(\max_{t\in[0,i]} |\omega_t^1 - \omega_t^2|) \wedge 1].$$

We set, for each $t \in [0, \infty)$,

$$\begin{aligned} W_t &:= \{\eta_{\cdot\wedge t} : \eta \in \Omega\}, \\ \mathscr{F}_t &:= \mathscr{B}_t(W) = \mathscr{B}(W_t), \\ \mathscr{F}_{t+} &:= \mathscr{B}_{t+}(W) = \bigcap_{s>t} \mathscr{B}_s(W), \\ \mathscr{F} &:= \bigvee_{s>0} \mathscr{F}_s. \end{aligned}$$

Then $(\Omega, \mathscr{F})$ is the canonical space with the natural filtration.

This space is used throughout the rest of this paper. For each $T > 0$, consider the following spaces of ąřrandom variablesąś

$$L_{ip}^0(\mathscr{F}_T) := \left\{X = \varphi(\omega_{t_1}, \omega_{t_2} - \omega_{t_1}, \cdots, \omega_{t_m} - \omega_{t_{m-1}}), \forall m \geq 1, t_1, \cdots, t_m \in [0, T], \forall \varphi \in lip(R^m)\right\},$$

where $lip(R^m)$ is the collection of bounded Lipschitz continuous functions on R^m.

Obviously, it holds that $L^0_{ip}(\mathscr{F}_t) \subseteq L^0_{ip}(\mathscr{F}_T)$, for any $t \leq T < \infty$. We notice that $X, Y \in L^0_{ip}(\mathscr{F}_t)$ means $XY \in L^0_{ip}(\mathscr{F}_t)$ and $|X| \in L^0_{ip}(\mathscr{F}_t)$. We further denote

$$L^0_{ip}(\mathscr{F}) := \bigcup_{n=1}^{\infty} L^0_{ip}(\mathscr{F}_n).$$

We set $B_t(\omega) = \omega_t$, and define

$$E_G[\varphi(B_t + x)] = u(t, x),$$

where $u(t, x)$ is the viscosity solution to the following G-heat equation

$$\begin{cases} \dfrac{\partial u}{\partial t} - G(\dfrac{\partial^2 u}{\partial x^2}) = 0, \\ \\ u(0, x) = \varphi(x). \end{cases}$$

$$\varphi(\cdot) \in lip(R), G(a) = \frac{1}{2} \sup_{\sigma_0^2 \leq \sigma^2 \leq 1} a\sigma^2, \sigma_0 \in [0, 1].$$

For any $X(\omega) = \varphi(B_{t_1}, B_{t_2} - B_{t_1}, \cdots, B_{t_m} - B_{t_{m-1}}) \in L^0_{ip}(\mathscr{F})$, $0 < t_1 < \cdots < t_m < \infty$,

$$E_G[\varphi(B_{t_1}, B_{t_2} - B_{t_1}, \cdots, B_{t_m} - B_{t_{m-1}})] = \varphi_m,$$

where φ_m is obtained via the backward deduction:

$$\begin{aligned}
&\varphi_1(x_1, \cdots, x_{m-1}) = E_G[\varphi(x_1, \cdots, x_{m-1}, B_{t_m} - B_{t_{m-1}})], \\
&\varphi_2(x_1, \cdots, x_{m-2}) = E_G[\varphi_1(x_1, \cdots, x_{m-2}, B_{t_{m-1}} - B_{t_{m-2}})], \\
&\qquad \vdots \\
&\varphi_{m-1}(x_1) = E_G[\varphi_{m-2}(x_1, B_{t_2} - B_{t_1})], \\
&\varphi_m = E_G[\varphi_{m-1}(B_{t_1})].
\end{aligned}$$

And

$$E_G[X|\mathscr{F}_{t_j}] = \varphi_{m-j}(B_{t_1}, B_{t_2} - B_{t_1}, \cdots B_{t_j} - B_{t_{j-1}}).$$

Definition 2.1. *The expectation $E_G[\cdot]$, introduced through the above procedure is called G-expectation. The corresponding canonical process B is called a G-Brownian motion under $E_G[\cdot]$.*

Remark 2.2. *As in [23], the G-expectation satisfies*

(a) Monotonicity: $E_G[Y] \geq E_G[Y']$, if $Y \geq Y'$, $Y, Y' \in L^0_{ip}(\mathscr{F})$.

(b) Self dominated property: $E_G[X] - E_G[Y] \leq E_G[X - Y]$, $X, Y \in L^0_{ip}(\mathscr{F})$.

(c) Positive homogeneity $E_G[\lambda Y] = \lambda E_G[Y]$, $\lambda \geq 0$, $Y \in L^0_{ip}(\mathscr{F})$.

(d) Constant translatability $E_G[X + c] = E_G[X] + c$, $X \in L^0_{ip}(\mathscr{F})$, c is a constant.

Besides $(a) \sim (d)$, the conditional G-expectation still satisfies the properties:

(e) *Time consistency* $E_G[E_G[X|\mathscr{F}_t]|\mathscr{F}_s] = E_G[X|\mathscr{F}_{t\wedge s}]$, $X \in L_{ip}^0(\mathscr{F})$.

(f) $E_G[XY|\mathscr{F}_s] = X^+E_G[Y|\mathscr{F}_s] + X^-E_G[-Y|\mathscr{F}_s]$, $X \in L_{ip}^0(\mathscr{F}_s)$, $Y \in L_{ip}^0(\mathscr{F})$.

(g) $E_G[X+Y|\mathscr{F}_s] = X + E_G[Y|\mathscr{F}_s]$, $X \in L_{ip}^0(\mathscr{F}_s)$, $Y \in L_{ip}^0(\mathscr{F})$.

Remark 2.3. *By properties* (c) *and* (d), $E_G[\cdot]$ *satisfies* $E_G[c] = c$ *where* c *is a constant.*

Remark 2.4. *By theory of stochastic control as in [30], we know that, for any fixed* $T > 0$,

$$E_G[X] = \sup_{v.\in\Lambda'} E\left[\varphi\left(\int_{t_1}^{t_2} v_s dB_s, \cdots, \int_{t_{m-1}}^{t_m} v_s dB_s\right)\right] = \sup_{P_v\in\Lambda} E_{P_v}[X], \tag{2.1}$$

where $X = \varphi(\omega_{t_1}, \cdots, \omega_{t_m} - \omega_{t_{m-1}}) \in L_{ip}^0(\mathscr{F}_T)$, E *is the linear expectation under weiner measure, and G-expectation becomes linear expectation when* $\sigma_0 = 1$, $\{B_t\}_{t\geq 0}$ *is Brownian motion under Weiner measure.*

$$\int_0^\cdot v dB_s(\cdot) : C[0,T] \longrightarrow C[0,T].$$

Here

$$\begin{aligned}
\Lambda' &= \{v, v \text{ is progressively measurable and quadratic integrable } s.t., \\
&\quad \sigma_0^2 \leq v^2(t) \leq 1, a.s. \text{ with respect to Weiner measure}, 0 \leq t \leq T\}, \\
\Lambda &= \{P_v : P_v \text{ is the distribution of} \int_0^\cdot v_s dB_s, v. \in \Lambda'\}.
\end{aligned}$$

Lemma 2.5. *For any* $X \in L_{ip}^0(\mathscr{F})$, *if* $E_G[|X|] = 0$, *then for any* $\omega \in \Omega$, *we have* $X(\omega) = 0$.

Proof: $X = \varphi(\omega_t)$, $E_G[X] = \sup_{v\in\Lambda'} E[|\varphi(\int_0^t v_s dB_s)|] = 0$. Let $v \equiv 1$, then $E[|\varphi(B_t)|] = 0$. Since $\varphi(\cdot)$ is Lipschitz continuous, then $\varphi(\cdot) \equiv 0$. Similarly, we can prove that for any $X \in L_{ip}^0(\mathscr{F})$ satisfying $E_G[|X|] = 0$, we have $X(\omega) = 0, \forall \omega \in \Omega$. ∎

Remark 2.6. *Denote* $\|X\| = E_G[|X|], \forall X \in L_{ip}^0(\mathscr{F})$. *By Lemma 2.5, we can prove that* $(L_{ip}^0(\mathscr{F}), \|\cdot\|)$ *is a normed space. Let* $(L_G^1(\mathscr{F}), \|\cdot\|)$ *be the completion of* $(L_{ip}^0(\mathscr{F}), \|\cdot\|)$, *then G-expectation and related conditional expectation can be continuously extended to the Banach space-*$(L_G^1(\mathscr{F}), \|\cdot\|)$. *G-expectation satisfies properties* $(a) \sim (d)$, *and conditional G-expectation satisfies properties* $(a) \sim (g)$. *In the completion space, property* (f) *holds for any bounded random variable* X, *but we have* $E_G[XY|\mathscr{F}_s] = XE_G[Y]$, $\forall Y \in L_G^1(\mathscr{F}_s^T)$, $\forall X \in L_G^1(\mathscr{F}_s)$, *and* $X \geq 0$. *Similarly, we can define* $\|X\|_p = E^{1/p}[|X|^p], p \geq 1$. *Let* $L_G^p(\mathscr{F})$ *be the completion space of* $L_{ip}^0(\mathscr{F})$ *under norm* $\|\cdot\|_p$. *Obviously* $L_G^{p'}(\mathscr{F}) \subset L_G^p(\mathscr{F})$ *for any* $1 \leq p \leq p'$, *and it holds for any* $L_G^p(\mathscr{F}_t)$ *(the proof can be found in [23]).*

Definition 2.7. $X, Y \in L_G^p(\mathscr{F}_T)$, $X \leq Y$ *in* L_G^p, *if* $E_G[((X-Y)^+)^p] = 0$, $p \geq 1$.

Definition 2.8. $X_n, X \in L_G^p$, $p \geq 1$, $X_n \longrightarrow X$ *in* L_G^p, *if* $E_G[|X_n - X|^p] \longrightarrow 0$, *as* $n \to \infty$.

For $p \geq 1$ and $0 < T < \infty$ (fixed T). Consider the following type of simple processes

$$M_G^{p,0}(0,T) = \{\eta : \eta_t(\omega) = \sum_{j=0}^{N-1} \xi_{t_j}(\omega) I_{[t_j, t_{j+1})}(t),$$
$$\forall N \geq 1,\ 0 = t_0 < \cdots < t_N = T,\ \xi_{t_j}(\omega) \in L_G^p(\mathscr{F}_{t_j}),\ j = 0.\cdots, N-1\}.$$

For each $\eta_t(\omega) = \sum_{j=0}^{N-1} \xi_{t_j}(\omega) I_{[t_j, t_{j+1})}(t) \in M_G^{p,0}(0,T)$, the related Bochner's integral is defined as

$$\int_0^T \eta_t(\omega) dt = \sum_{j=0}^{N-1} \xi_{t_j}(\omega)(t_{j+1} - t_j),$$

and

$$\widetilde{E}_G[\eta] = \frac{1}{T} \int_0^T E_G[\eta_t] dt = \frac{1}{T} \sum_{j=0}^{N-1} E_G[\xi_{t_j}(\omega)](t_{j+1} - t_j).$$

We can easily check that $\widetilde{E}_G[\cdot] : M_G^{1,0}(0,T) \to R$ satisfies $(a) \sim (d)$ in Section 2.

$$\|\eta\|_p = (\frac{1}{T} \int_0^T \|\eta_t^p\| dt)^{1/p} = \left(\frac{1}{T} \sum_{j=0}^{N-1} E_G[|\xi_{t_j}(\omega)|^p](t_{j+1} - t_j) \right)^{1/p}.$$

As discussed in Remark 2.6, $\|\cdot\|_p$ forms a norm in $M_G^{p,0}(0,T)$. Let $M_G^p(0,T)$ be the completion of $M_G^{p,0}(0,T)$ under this norm.

3 Capacity Under G-Framework

3.1 G-Expectation and Related Properties

G-expectation is a kind of time consistent nonlinear expectations, which has the properties of linear expectation in Weiner space except linearity. In this section, we prove some fundamental properties of G-expectation.

We set

$$\mathscr{S}^p = \{M | M : R^+ \times \Omega \to R, M(t,\omega) \in L_G^p(\mathscr{F}_t), \forall T > 0,\ \{M_t\}_{t \in [0,T]} \in M_G^p(0,T)\}.$$

First we will prove that an important class of random variable-bounded continuous functions belongs to the completion space- $L_G^1(\mathscr{F})$

Remark 3.1. *If $X \in L_G^1(\mathscr{F}_T)$, there is a sequence $f_n \in L_{ip}^0(\mathscr{F}_T)$, such that f_n converges to f in $L_G^1(\mathscr{F}_T)$, then for each $P_v \in \Lambda$, f_n converges to f in $L^1(\Omega, P_v)$, and this convergence is uniform with respect to P_v. We have $E_G[f] = \sup_{P_v \in \Lambda} E_{P_v}[f]$ (see Proposition 2.2 in [9]).*

Next we will prove the tightness of Λ.

Lemma 3.2. *Λ is tight, that is, for any $\varepsilon > 0$, there exists a compact set $K \subset C([0, T]) \subset \Omega$, such that for any $P_v \in \Lambda$, $P_v(K^c) < \varepsilon$, where K^c is the complement of K.*

Proof: For any continuous function $x(t),\ t \in [0, T]$, define

$$\omega_x(\delta) = \sup_{|s-t|\le\delta, s,t\in[0,T]} |x_t - x_s|.$$

By Arzela-Ascoli theorem and Prokhrov theorem (see Theorem 4.4.11 in [4]), to prove the tightness of Λ, we only need to prove that for any $\alpha > 0$, for any $P_v \in \Lambda$, we have

$$\lim_{\delta\to 0} P_v\left(\{x : \omega_x(\delta) \ge \alpha\}\right) = 0.$$

For $\alpha > 0$, by Proposition 7.2 in the Appendix we know that

$$\lim_{\delta\to 0} P_v\left(\{x : \omega_x(\delta) \ge \alpha\}\right) \le \lim_{\delta\to 0} \frac{E_{P_v}[\omega_x^2(\delta)]}{\alpha^2} = 0,$$

then Λ is tight. ∎

Remark 3.3. *For any fixed T, $C[0, T]$ is a polish space, by Prokhrov theorem, we know that Λ is weakly compact. For $X_n \in L^0_{ip}(\mathscr{F}_T)$, $X_n \downarrow 0$ pointwise. As in the Appendix of [9], by Dinni lemma, $E_G[X_n] \downarrow 0$.*

Lemma 3.4. *For any fixed $T > 0$, $0 < t < T$, $f \in C_b(W_t)$, we have $f \in L^1_G(\mathscr{F}_t)$.*

Proof: For any bounded continuous $f \in C_b(W_t)$, $|f| \le M,\ M > 0$, there exists a sequence of random variables $f_n \in L^0_{ip}(\mathscr{F}_t)$, such that f_n monotonically converges to f.
As Λ is tight, for any $\varepsilon > 0$, there exists a compact set $K \in \mathscr{F}_T$, such that

$$\sup_{P_v\in\Lambda} E_{P_v}[I_{K^c}] < \varepsilon,$$

where K^c is the complement of K. Since a compact set is closed in any metric space, we know that K^c is an open set, hence

$$E_G[I_{K^c}] = \sup_{P_v\in\Lambda} E_{P_v}[I_{K^c}] < \varepsilon.$$

And by Dini's theorem on any compact set K, f_n converges to f uniformly,

$$\lim_{n\to\infty} E_G[|f_n - f|] \le \lim_{n\to\infty} [E_G[|f_n - f|I_K] + E_G[|f_n - f|I_{K^c}]] \le M\varepsilon,$$

then $\lim_{n\to\infty} E_G[|f_n - f|] = 0$, note that this convergence is uniform with respect to t, so $f \in L^1_G(\mathscr{F}_T)$.
∎

3.2 Capacity under G-Framework

Since the publication of Kolomogrov's famous book on probability, the study of "the nonlinear probability" theory named "capacity" has been studied intensively in the past decades, see [5], [8], [12], [16], [18], [22], [26]. Hence it is meaningful to extend such a theory to G-framework, and this section contributes to such an extension. We shall now define a nonlinear measure through G-expectation and investigate its properties.

Definition 3.5. *$P_G(A) = \sup_{P_v \in \Lambda} P_v(A)$, for any Borel set A, where P_v is the distribution of $\int_0^t v dB_s$ and v is a bounded adapted process, and Λ is the collection of all such P_v.*

By Remark 3.1 and Remark 3.3, P_G is a regular Choquet capacity (we call it G-capacity), that is, it has the following properties:

(1) For any Borel set A, $0 \le P_G(A) \le 1$;

(2) If $A \subset B$, then $P_G(A) \le P_G(B)$;

(3) If A_n is a sequence of Borel sets, then $P_G(\bigcup_n A_n) \le \sum_n P_G(A_n)$;

(4) If A_n is an increasing sequence of Borel sets, then

$$P_G(\bigcup_n A_n) = \lim_n P_G(A_n).$$

Remark 3.6. *Here property (4) dose not hold for the intersection of decreasing sets. There are two ways to define capacity under G framework, see [10].*
Let $\widetilde{\Lambda}$ be the closure of Λ under weak topology.

(1) $P_G(A) = \sup_{P_v \in \Lambda} P_v(A)$,

(2) $\overline{P}_G(A) = \sup_{P \in \widetilde{\Lambda}} P_v(A)$.

Then P_G and $\overline{P}_G$ all satisfy the properties in Remark 3.2.
We use the standard capacity related vocabulary: A set A is polar if $P_G(A) = 0$, a property holds ą̌rquasi-surelyąś (q.s.), if it holds outside a polar set. Here P_G quasi-surely is equivalent to $\overline{P}_G$ quasi-surely. Even in general, $P_G(A) \le \overline{P}_G(A)$, but if $P_G(A) = 0$, $I_A \in L_G^1(\mathscr{F}_T)$. Then by Theorem 59 in [10](page 24), $\overline{P}_G(A) = P_G(A) = 0$. Thus, a property holds P_G-quasi-surely if and only if it holds $\overline{P}_G$-quasi-surely.

Remark 3.7. *As in the Appendix of [9], we consider the Lebesgue extension of G-expectation, we can define G-expectation for a large class of measurable functions, such as all the functions with $\sup_{P_v \in \Lambda} E_{P_v}[|X|] < \infty$, but we can not define G conditional expectation. So far we can only define G conditional expectation for the random variables in $L_G^1(\mathscr{F})$, in the rest of the paper, we denote $\sup_{P_v \in \Lambda} E_{P_v}[X] = E_G[X]$, but that dose not mean $X \in L_G^1(\mathscr{F})$.*

First we give the property of this Choquet capacity-P_G:

Proposition 3.8. *Let $p \ge 1$.*

(1) If A is a polar set, then for any $\xi \in L_G^p(\mathscr{F}_T)$, $E_G[I_A \xi] = 0$.

(2) $P_G\{|\xi| > a\} \le \frac{E_G[|\xi|^p]}{a^p}$, $\xi \in L_G^p(\mathscr{F}_T)$, $a > 0$.

(3) If X_n, $X \in L_G^p(\mathscr{F})$, $E_G[|X_n - X_p|^p] \to 0$, then there exists a sub-sequence X_{n_k} of X_n, such that $X_{n_k} \to X, q.s.$

Proof: (1)Without loss of generality, let $p=1$. If $\xi \in L_{ip}^0(\mathscr{F}_T)$, then $E_G[I_A\xi]=0$. If $\xi \in L_G^1(\mathscr{F}_T)$, then there exists a sequence of random variables ξ_n, which satisfies $E_G[|\xi_n-\xi|] \longrightarrow 0$, and $E_G[|\xi_n I_A - \xi I_A|] \longrightarrow 0$, hence we obtain

$$E_G[\xi I_A] = \lim_{n\to\infty} E_G[\xi_n I_A] = 0.$$

(2) From

$$E_G[|\xi|^p] = E_G[|\xi|^p I_{\{|\xi|>a\}} + |\xi|^p I_{\{|\xi|\le a\}}] \ge a^p P_G\{|\xi|>a\},$$

we get the result.

(3) By (2), we know that for any $\varepsilon>0$,

$$\lim_{n\to\infty} P_G\{|X_n - X| > \varepsilon\} = 0.$$

Then for every positive integer k, there exists $n_k>0$, such that

$$P_G\{|X_n - X| \ge \frac{1}{2^k}\} < \frac{1}{2^k}, \quad \forall n \ge n_k.$$

Suppose $n_1 < n_2 < \ldots < n_k < \ldots$, let $X_k' = X_{n_k}$ be a sub-sequence of X_n. Then

$$\begin{aligned} P_G\{X_k' \nrightarrow X\} &= P_G\left\{\bigcup_m \bigcap_k \bigcup_v |X_{k+v}' - X| \ge \varepsilon_m\right\} \\ &\le \sum_m P_G\left\{\bigcap_k \bigcup_v |X_{k+v}' - X| \ge \varepsilon_m\right\} \\ &\le \sum_m P_G\left\{\bigcup_v |X_{k_0+m+v}' - X| \ge \varepsilon_m\right\} \\ &\le \sum_m \sum_v P_G\left\{|X_{k_0+m+v}' - X| \ge \frac{1}{2^{k_0+m+v}}\right\} \\ &\le \sum_m \sum_v \frac{1}{2^{k_0+m+v}} = \frac{1}{2^{k_0}} \to 0. \end{aligned}$$

Therefore, $P_G\{X_k' \nrightarrow X\} = 0$. ∎

Next we investigate the relation between $X \le Y$ in L_G^p, and $X \le Y$, q.s.

Lemma 3.9. $E_G[(X-Y)^+]=0$ *if and only if* $X \le Y$*, q.s.*

Proof: Without loss of generality, suppose that $p=1$, if $X \le Y$, q.s., $\{X-Y\ge 0\}$ is a polar set. Then by Proposition 3.8, we know that

$$E_G[(X-Y)^+] = E_G[(X-Y)I_{\{X-Y\ge 0\}}] = 0.$$

If $X \le Y$ in L_G^1, which means $E_G[(X-Y)^+]=0$, then by Proposition 3.8, for any $\varepsilon>0$, we have

$$P_G[(X-Y)^+ > \varepsilon] \le \frac{E_G[(X-Y)^+]}{\varepsilon} = 0.$$

Let $\varepsilon \downarrow 0$, for P_G is a Choquet capacity, we know

$$P_G[(X-Y)^+ > 0] = \lim_{\varepsilon \to 0} P_G[(X-Y)^+ > \varepsilon] = 0,$$

so we have $P_G[X \geq Y] = P_G[(X-Y)^+ > 0] = 0$, that is $X \leq Y$ q.s. ∎

Remark 3.10. *After defining the capacity, one important issue is whether $I_A \in L_G^1(\mathscr{F})$, for any Borel set $A \in \mathscr{F}$. Here we give a counter example that there exists a sequence of Borel sets which do not belong to $L_G^1(\mathscr{F})$.*

Example 3.11. *Let*

$$A_n = \left\{ \omega : \overline{\lim}_{t\to 0} \frac{B_{t+s} - B_s}{\sqrt{2t \log\log 1/t}} \in (1 - 1/2n, 1) \right\}, \ n = 1, 2, \cdots$$

Then $A_n \downarrow \phi$, for any fixed n, let $v \equiv 1 - \dfrac{1}{3n}$, Ôò

$$P_v(A_n) = P\left\{ \omega : (1 - \frac{1}{3n}) \overline{\lim}_{t\to 0} \frac{B_{t+s} - B_s}{\sqrt{2t \log\log 1/t}} \in (1 - 1/2n, 1) \right\} = 1,$$

which means $\lim_{n\to\infty} P_G(A_n) = 1$.

For any sequences of random variables $\{X_n\} \subset L_G^1(\mathscr{F})$, satisfying $X_n \downarrow 0$ q.s., we have $E_G[X_n] \downarrow 0$, see Theorem 26 in [10]. So the sets A_n do not belong to $L_G^1(\mathscr{F})$.

Actually, the next lemma tells us even not all the open Borel sets belong to $L_G^1(\mathscr{F})$.

Lemma 3.12. *There exists an open set $A \in \mathscr{F}_T$, such that I_A dose not belongs to $L_G^1(\mathscr{F}_T)$.*

Proof: We prove this result by contradiction. If for any open set $A \in \mathscr{F}_T$, $I_A \in L_G^1(\mathscr{F}_T)$ holds. For all the open sets satisfying $A_n \downarrow \phi$, we have $P_G(A_n) \downarrow 0$. Because for any Borel set B, there exists compact sets $\{F_n\} \subset B$ satisfying $P_G(B \setminus F_n) \downarrow 0$, see [16]. Then we have for any Borel set $B \in \mathscr{F}_T$, $I_B \in L_G^1(\mathscr{F}_T)$. But Example 3.11 shows that not all the Borel sets belong to $L_G^1(\mathscr{F}_T)$, which is a contradiction. ∎

Then we can not define conditional G-expectation for I_A, where A is any Borel set, and even for any open set, that means we can not define conditional G-capacity, and that is why we claim that G-expectation is not filtration consistent.

4 Symmetric Martingale in G-Framework

We begin with the definition of martingale in G-framework.

Definition 4.1. *$M \in \mathscr{S}^2$ is called a martingale, if for any $0 \leq s \leq t < \infty$, it satisfies $E[M_t|\mathscr{F}_s] = M_s$; if furthermore M is symmetric, that is $E[-M_t|\mathscr{F}_s] = -E[M_t|\mathscr{F}_s]$, then it is called a symmetric martingale.*

In this section, when the corresponding G-heat equation is uniformly parabolic, which means $\sigma_0 > 0$, we prove that the symmetric martingale is a martingale under each probability measure P_v, and give the Doob's martingale inequality for symmetric martingales.

4.1 Path analysis

In this section, we give some path properties of quadratic variation process $\langle B\rangle_t$ and the related stochastic integral $\int_0^t \eta(s)dB_s$, $\eta \in M_G^2(0,T)$ and $\int_0^t \eta_1(s)d\langle B\rangle_s$, $\eta_1(s) \in M_G^1(0,T)$.

From the definition of $E_G[\cdot]$, we know that the canonical process B_t is a quadratic integrable martingale under each P_v. So they have a universal version of "quadratic variation process of B_t", and by the definition of stochastic integral with respect to G-Brownian motion, for any $\eta \in M_G^2(0,T)$, $\int_0^T \eta_s dB_s$ is well defined, which means $\int_0^T \eta_s dB_s$ is a P_v local martingale. Similar arguments can be found in Lemma 2.10 in [10]. Then, due to the Doob's martingale inequality for each P_v, the following result holds.

Lemma 4.2. *For any $\eta \in M_G^2(0,T)$, $\int_0^t \eta(s)dB_s$ is quasi-sure continuous.*

Proof: If $\eta \in M_G^{2,0}(0,T)$, then the result is true. If $\eta \in M_G^2(0,T)$, then there exist $\{\eta_n\} \subset M_G^{2,0}(0,T)$ such that $\int_0^T E_G[|\eta(s)-\eta_n(s)|^2]ds \longrightarrow 0$.

For we have

$$\begin{aligned} &\sup_{P_v\in\Lambda} E_{P_v}[\sup_{0\le t\le T}|\int_0^t(\eta-\eta_n)dB_s|^2] \le K\sup_{P_v\in\Lambda}E_{P_v}[\int_0^T(\eta-\eta_n)^2d\langle B\rangle_s] \\ \le\ & K\sup_{P_v\in\Lambda}E_{P_v}[\int_0^T(\eta-\eta_n)^2ds] \le K\int_0^T\sup_{P_v\in\Lambda}E_{P_v}[(\eta-\eta_n)^2]ds \longrightarrow 0. \end{aligned}$$

Hence, $\int_0^t \eta_n(s)dB_s$ uniformly converges to $\int_0^t \eta(s)dB_s$ q.s. Therefore, $\int_0^t \eta(s)dB_s$ is continuous q.s. ∎

By similar argument we can get the following lemma.

Lemma 4.3. *For any $\eta \in M_G^1(0,T)$, $\int_0^t \eta(s)d\langle B\rangle_s$ and $\int_0^t \eta(s)ds$ are quasi-surely continuous.*

4.2 Representation theorem

In this section, we are concerned with the G-martingale when the corresponding G-heat equation is uniformly parabolic ($\sigma_0 > 0$). In the following, for any $X \in L_{ip}^0(\mathscr{F}_T)$, we will give X a representation in terms of stochastic integral. For this part, Peng gives the conjecture for representation theorem of G-martingales. Soner et al [21] prove this theorem for a large class of G-martingales by BSDE method. Here for the ease of exposition, we prove the theorem separately for some special martingales.

First we prove a lemma.

Lemma 4.4. *Suppose $\sigma_0 > 0$, if u is the solution of G-heat equation, then we have*

$$u(r, B_{t-r}) = u(t,0) + \int_0^{t-r} u_x(B_v)dB_v + \int_0^{t-r} u_{xx}(B_v)d\langle B\rangle_v - \int_0^{t-r}(u_{xx}^+ - \sigma_0^2 u_{xx}^-)dv.$$

Proof: The proof follows that of Itô's formula. Due to the regularity of parabolic equation, see [18], we know u, u_x, u_{xx} are all uniformly continuous. Since Lipschitz continuous functions are dense in uniform continuous functions, we assume that u, u_x, u_{xx} are Lipschitz continuous.

Let $\delta_n = \dfrac{t-r}{n}$, we have

$$\begin{aligned} u(r,B_{t-r}) - u(t,0) &= \sum_{k=0}^{n-1}\left[u(t-(k+1)\delta_n, B_{(k+1)\delta_n}) - u(t-k\delta_n, B_{k\delta_n})\right] \\ &= \sum_{k=0}^{n-1}\left[u(t-(k+1)\delta_n, B_{(k+1)\delta_n}) - u(t-k\delta_n, B_{(k+1)\delta_n})\right] \\ &+ \sum_{k=0}^{n-1}\left[u(t-k\delta_n, B_{(k+1)\delta_n}) - u(t-k\delta_n, B_{k\delta_n})\right] \\ &= \sum_{k=0}^{n-1}\Big[-u_t(t-k\delta_n, B_{k\delta_n})\delta_n + u_x(t-k\delta_n, B_{k\delta_n})(B_{(k+1)\delta_n} - B_{k\delta_n}) \\ &\quad + \frac{1}{2}u_{xx}(t-k\delta_n, B_{k\delta_n})(B_{(k+1)\delta_n} - B_{k\delta_n})^2\Big] - \xi_n + \eta_n, \end{aligned}$$

where

$$\begin{aligned} \eta_n &= \frac{1}{2}\sum_{k=0}^{n-1}[u_{xx}(t-k\delta_n, B_{k\delta_n} + \theta_1(B_{(k+1)\delta_n} - B_{k\delta_n})) - u_{xx}(t-k\delta_n, B_{k\delta_n})](B_{(k+1)\delta_n} - B_{k\delta_n})^2, \\ \xi_n &= \sum_{k=0}^{n-1}\left[u_t(t-k\delta_n + \theta_2\delta_n, B_{(k+1)\delta_n}) - u_t(t-k\delta_n, B_{(k+1)\delta_n})\right]\delta_n, \\ &\quad + \sum_{k=0}^{n-1}\left[u_t(t-k\delta_n, B_{(k+1)\delta_n}) - u_t(t-k\delta_n, B_{k\delta_n})\right]\delta_n, \end{aligned}$$

and θ_1, θ_2 are constants in $[0,1]$, which depend on ω, t and n. Hence,

$$\begin{aligned} E_G[|\eta_n|] &\leq \sum_{k=0}^{n-1} E_G\Big[|u_{xx}(t-k\delta_n, B_{k\delta_n} + \theta(B_{(k+1)\delta_n} - B_{k\delta_n})) \\ &\quad - u_{xx}(t-k\delta_n, B_{k\delta_n})|(B_{(k+1)\delta_n} - B_{k\delta_n})^2\Big] \\ &\leq K\sum_{k=0}^{n-1} E_G[|B_{(k+1)\delta_n} - B_{k\delta_n}|^3] \leq K\sum_{k=0}^{n-1}\delta_n^{3/2} \longrightarrow 0. \end{aligned}$$

Here, K is the Lipschitz constant of u_{xx}, and by similar argument we get $E_G[|\xi_n|] \longrightarrow 0$ as $n \to \infty$.

Then, we have

$$E_G[|\sum_{k=1}^{n} -u_t(t-k\delta_n, B_{k\delta_n})I_{[k\delta_n,(k+1)\delta_n)}(v) - u_t(t-v, B_v)|] \le C(\delta_n + \delta_n^{1/2}) \longrightarrow 0.$$

Therefore,

$$\sum_{k=0}^{n-1} -u_t(t-k\delta_n, B_{k\delta_n})\delta_n \longrightarrow \int_0^{t-r} -u_t(t-v, B_v)dv.$$

Similarly we get

$$\sum_{k=0}^{n-1} u_x(t-k\delta_n, B_{k\delta_n})(B_{(k+1)\delta_n} - B_{k\delta_n}) \longrightarrow \int_0^{t-r} u_x(t-v, B_v)dB_v.$$

Note

$$\begin{aligned}
& E_G[|\sum_{k=0}^{n-1} u_{xx}(t-k\delta_n, B_{k\delta_n})(B_{(k+1)\delta_n} - B_{k\delta_n})^2 - \sum_{k=0}^{n-1} u_{xx}(t-k\delta_n, B_{k\delta_n})(\langle B\rangle_{(k+1)\delta_n} - \langle B\rangle_{k\delta_n})|^2] \\
\le\ & \sum_{k=0}^{n-1} E_G[|u_{xx}(t-k\delta_n, B_{k\delta_n})|^2|(B_{(k+1)\delta_n} - B_{k\delta_n})^2 - (\langle B\rangle_{(k+1)\delta_n} - \langle B\rangle_{k\delta_n})|^2] \\
+\ & 2\sum_{j\neq i} E_G[u_{xx}(t-j\delta_n, B_{j\delta_n})u_{xx}(t-i\delta_n, B_{i\delta_n})((B_{(j+1)\delta_n} - B_{j\delta_n})^2 - (\langle B\rangle_{(j+1)\delta_n} - \langle B\rangle_{j\delta_n})) \\
& ((B_{(i+1)\delta_n} - B_{i\delta_n})^2 - (\langle B\rangle_{(i+1)\delta_n} - \langle B\rangle_{i\delta_n}))] \\
\le\ & \sum_{k=0}^{n-1} E_G[(c + c|B_{k\delta_n}|^2)|\int_{k\delta_n}^{(k+1)\delta_n} (B_v - B_{k\delta_n})dB_v|^2] \\
\le\ & \sum_{k=0}^{n-1} CE_G[|\int_{k\delta_n}^{(k+1)\delta_n} (B_v - B_{k\delta_n})dB_v|^2] \le C\sum_{k=0}^{n-1} \delta_n^2 \longrightarrow 0,
\end{aligned}$$

and

$$\sum_{k=0}^{n-1} u_{xx}(t-k\delta_n, B_{k\delta_n})(\langle B\rangle_{(k+1)\delta_n} - \langle B\rangle_{k\delta_n}) \longrightarrow \int_0^{t-r} u_{xx}(t-v, B_v)d\langle B\rangle_v.$$

Since u solves G-heat equation, we get

$$\begin{aligned}
u(r, B_{t-r}) - u(t,0) &= \int_0^{t-r} -u_t(t-v, B_v)dv + \int_0^{t-r} u_x(t-v, B_v)dB_v \\
&\quad + \int_0^{t-r} u_{xx}(t-v, B_v)d\langle B\rangle_v \\
&= \int_0^{t-r} u_x(t-v, B_v)dB_v + \int_0^{t-r} u_{xx}(t-v, B_v)d\langle B\rangle_v \\
&\quad -\frac{1}{2}\int_0^{t-r} (u_{xx}^+ - \sigma_0^2 u_{xx}^-)dv.
\end{aligned}$$

∎

Theorem 4.5. *When $\sigma_0 > 0$, then for any $X \in L^0_{ip}(\mathscr{F}_T)$, we have*

$$X = E_G[|\varphi(\cdot)|] + \int_0^{t_m} Z_s dB_s + \int_0^{t_m} \eta(s) d\langle B\rangle_s - \int_0^{t_m} (\eta^+ - \sigma_0^2 \eta^-) ds.$$

Proof: When $m = 1$, for the regularity of the u, $\lim_{r\to 0} u(r, B_{t-r}(\omega)) = \varphi(B_t(\omega))$.
By Lemma 4.4, and path analysis in Section 3, we know

$$\begin{aligned}\varphi(B_t) &= u(t,0) + \int_0^t u_x(t-v, B_v) dB_v \\ &+ \int_0^t u_{xx}(t-v, B_v) d\langle B\rangle_v - \int_0^t (u^+_{xx} - \sigma_0^2 u^-_{xx}) dv.\end{aligned}$$

Due to the definition of G-expectation, $E_G[\varphi(B_t)] = u(t,0)$, so the result holds when $m = 1$.
By similar argument, we get

$$\begin{aligned}\varphi(B_T - B_t) &= u(T-t,0) + \int_t^T u_x(T-v, B_v - B_t) dB_v \\ &+ \int_t^T u_{xx}(T-v, B_v - B_t) d\langle B\rangle_v - \int_t^T (u^+_{xx} - \sigma_0^2 u^-_{xx}) dv.\end{aligned}$$

When $m = 2$, for each x

$$\begin{aligned}\varphi(x, B_T - B_t) &= u(T-t, x, 0) + \int_t^T u_y(T-v, x, B_v - B_t) dB_v \\ &+ \int_t^T u_{yy}(T-v, x, B_v - B_t) d\langle B\rangle_v \\ &- \int_t^T (u^+_{yy}(T-v, x, B_v - B_t) - \sigma_0^2 u^-_{yy}(T-v, x, B_v - B_t)) dv.\end{aligned}$$

By continuous dependence estimate theorem in [13], we know for each fixed t, $u(T-t, x, 0)$ is lipschitz continuous and bounded with respect to x, then there exist $\eta \in M^1_G(0,T)$ and $z \in M^2_G(0,T)$, such that

$$u(T-t, B_t, 0) = E_G[|u(T-t, B_t, 0)|] + \int_0^t z_s dB_s + \int_0^t \eta_s d\langle B\rangle_s - \int_0^t (\eta^+_s - \sigma_0^2 \eta^-_s) ds. \tag{4.2}$$

That is

$$\varphi(B_t, B_T - B_t) = E_G[\varphi(B_t, B_T - B_t)] + \int_0^T z_s dB_s + \int_0^T \eta_s d\langle B\rangle_s - \int_0^T (\eta^+_s - \sigma_0^2 \eta^-_s) ds.$$

Here η and z are different from (4.2).
Then by induction, we know the result is true for any $X \in L^0_{ip}(\mathscr{F}_T)$. ∎

4.3 Properties for the Symmetric Martingale

From [23], we know $E_G[(\langle B\rangle_t - t)^+] = E_G[(\sigma_0^2 t - \langle B\rangle_t)^+] = 0$. Then $\sigma_0^2 t \le \langle B\rangle_t \le t$, and $\sigma_0^2(T - t) \le \langle B\rangle_T - \langle B\rangle_t \le T - t$, for any $t \le T$.

Lemma 4.6. *For any $\eta \in M_G^1(0,T)$, we have*

$$\int_0^T \eta^+(s)d\langle B\rangle_s - \int_0^T \eta^+(s)ds \le 0 q.s., \text{and} \quad \int_0^T \eta^-(s)d\langle B\rangle_s - \sigma_0^2 \int_0^T \eta^-(s)ds \ge 0.$$

Proof: If $\eta \in M_G^{1,0}(0,T)$, then

$$\int_0^T \eta^+(s)d\langle B\rangle_s = \sum_{j=0}^{n-1} \xi_j^+(\langle B\rangle_{t_{j+1}} - \langle B\rangle_{t_j}) \le \sum_{j=0}^{n-1} \xi_j^+(t_{j+1} - t_j) = \int_0^T \eta^+(s)ds,$$

and

$$\int_0^T \eta^-(s)d\langle B\rangle_s = \sum_{j=0}^{n-1} \xi_j^-(\langle B\rangle_{t_{j+1}} - \langle B\rangle_{t_j}) \ge \sigma_0^2 \sum_{j=0}^{n-1} \xi_j^+(t_{j+1} - t_j) = \sigma_0^2 \int_0^T \eta^-(s)ds.$$

If $\eta \in M_G^1(0,T)$, then there exists $\eta_n \in M_G^{1,0}(0,T)$, such that

$$\begin{aligned} E_G\left[\left|\int_0^T (\eta_n - \eta)d\langle B\rangle_s\right|\right] &\longrightarrow 0, \\ E_G\left[\left|\int_0^T (\eta_n - \eta)ds\right|\right] &\longrightarrow 0. \end{aligned}$$

Then there exists a subsequence $\eta_k \subset \eta_n$ such that

$$\int_0^T (\eta_k - \eta)d\langle B\rangle_s \longrightarrow 0,\ q.s.,$$

and

$$\int_0^T (\eta_k - \eta)ds \longrightarrow 0,\ q.s..$$

Therefore

$$\begin{aligned} \int_0^T \eta^+(s)d\langle B\rangle_s - \int_0^T \eta^+(s)ds &\le 0,\ q.s. \\ \int_0^T \eta^-(s)d\langle B\rangle_s - \sigma_0^2 \int_0^T \eta^-(s)ds &\ge 0,\ q.s. \end{aligned}$$

∎

Theorem 4.7. *Suppose $\sigma_0 > 0$, then for any $X \in L_G^1(\mathscr{F}_T)$, $\eta = E_G[X \mid \mathscr{F}_t]$ satisfies*

$$\sup_{P_v \in \Lambda} E_{P_v}[(X - \eta)I_A] = 0, \quad \forall A \in \mathscr{F}_t.$$

Proof: If $X \in L^0_{ip}(\mathscr{F}_T)$, then

$$X_T = E_G[X_T] + \int_0^T z_s dB_s + \int_0^T \eta_s d\langle B\rangle_s - \int_0^T (\eta^+ - \sigma_0^2\eta^-)ds.$$

By Theorem 4.1.42 in [24], we have

$$\eta = E_G[X_T \mid \mathscr{F}_t] = E_G[X_T] + \int_0^t z_s dB_s + \int_0^t \eta_s d\langle B\rangle_s - \int_0^t (\eta^+ - \sigma_0^2\eta^-)ds.$$

Since $\int_0^t z_s dB_s$ is a quadratic integrable martingale for each P_v, then

$$\begin{aligned}
\sup_{P_v\in\Lambda} E_{P_v}[(X-\eta)I_A] &= \sup_{P_v\in\Lambda} E_{P_v}\left[(\int_t^T \eta_s d\langle B\rangle_s - \int_t^T (\eta^+ - \sigma_0^2\eta^-)ds + \int_t^T z_s dB_s)I_A \right] \\
&= \sup_{P_v\in\Lambda} E_{P_v}\left[(\int_t^T \eta_s d\langle B\rangle_s - \int_t^T (\eta^+ - \sigma_0^2\eta^-)ds)I_A \right] \le 0.
\end{aligned}$$

On the other hand

$$\sup_{P_v\in\Lambda} E_{P_v}\left[(\int_t^T \eta_s d\langle B\rangle_s - \int_t^T (\eta^+ - \sigma_0^2\eta^-)ds)I_A \right] \ge \sup_{P_v\in\Lambda} E_{P_v}\left[\int_t^T \eta_s d\langle B\rangle_s - \int_t^T (\eta^+ - \sigma_0^2\eta^-)ds \right] = 0.$$

Therefore $\sup_{P_v\in\Lambda} E_{P_v}[(X-\eta)I_A] = 0$.

If $X \in L^1_G(\mathscr{F}_T)$, then there exist $X_n \in L^0_{ip}(\mathscr{F}_T)$, such that $E_G[|X - X_n|] \to 0$ and

$$E_G[|E_G[X \mid \mathscr{F}_t] - E_G[X_n \mid \mathscr{F}_t]|] \to 0.$$

Hence,

$$\begin{aligned}
\sup_{P_v\in\Lambda} E_{P_v}[(X - E_G[X \mid \mathscr{F}_t])I_A] &= \sup_{P_v\in\Lambda} E_{P_v}[(X - X_n + E_G[X_n \mid \mathscr{F}_t] - E_G[X \mid \mathscr{F}_t])I_A \\
&+ (X_n - E_G[X_n \mid \mathscr{F}_t])I_A] \\
&\le E_G[|X - X_n|] + E_G[|E_G[X \mid \mathscr{F}_t] - E_G[X_n \mid \mathscr{F}_t]|] \\
&+ \sup_{P_v\in\Lambda} E_{P_v}[(X_n - E_G[X_n \mid \mathscr{F}_t])I_A] \longrightarrow 0.
\end{aligned}$$

On the other hand

$$\begin{aligned}
\sup_{P_v\in\Lambda} E_{P_v}[(X - E_G[X \mid \mathscr{F}_t])I_A] &\ge \sup_{P_v\in\Lambda} E_{P_v}[(X_n - E_G[X_n \mid \mathscr{F}_t])I_A] \\
&- E_G[|X - X_n|] - E_G[|E_G[X \mid \mathscr{F}_t] - E_G[X_n \mid \mathscr{F}_t]|] \longrightarrow 0.
\end{aligned}$$

So $\sup_{P_v\in\Lambda} E_{P_v}[(X - E_G[X \mid \mathscr{F}_t])I_A] = 0$. ∎

The next corollary states that there is a universal version of the conditional expectation for the symmetric random variables under each P_v.

Corollary 4.8. *If $\sigma_0 > 0$, $M_T \in L_G^1(\mathscr{F}_T)$, and $-E_G[M_T \mid \mathscr{F}_t] = E_G[-M_T \mid \mathscr{F}_t]$, then*

$$E_G[M_T \mid \mathscr{F}_t] = E_{P_v}[M_T \mid \mathscr{F}_t], \quad P_v.a.s.$$

Then consequently we have the main result of this section.

Theorem 4.9. *If M is a symmetric martingale under G expectation with $\sigma_0 > 0$, then M is a martingale under each P_v.*

Remark 4.10. *We can not hope that*

$$E_{P_v}[X|\mathscr{F}_t] \leq E_G[X|\mathscr{F}_t] = -E_G[-X|\mathscr{F}_t] \leq -E_{P_v}[-X|\mathscr{F}_t] = E_{P_v}[X|\mathscr{F}_t]$$

Obviously for any $X \in L_G^1(\mathscr{F})$, $E_G[X] \geq E_{P_v}[X]$. But this is not trivial for conditional G expectation, since it is defined as the "Markovian" version, so $E_{P_v}[X|\mathscr{F}_t] \leq E_G[X|\mathscr{F}_t]$ does not hold for each P_v.

As a corollary, we can get the Doob's Martingale inequality for "symmetric G-martingale".

Corollary 4.11. *When $\sigma_0 > 0$, ff $M_T \in L_G^2(\mathscr{F}_T)$, and $-E_G[M_T \mid \mathscr{F}_t] = E_G[-M_T \mid \mathscr{F}_t]$ and M_t is quasi-surely continuous then we have $P_G[\sup_{0\leq t\leq T}|M_t| \geq \lambda] \leq \frac{1}{\lambda^p}E_G[|M_T|^p]$, for any $p \geq 1$, $T \geq 0$, as a special case $E_G[\sup_{0\leq t\leq T}|M_t|^2] \leq 2E_G[|M_T|^2]$.*

Proof: By the Doob's martingale inequality in probability space, we have

$$\begin{aligned} & P_G[\sup_{0\leq t\leq T}|M_t| \geq \lambda] \\ = & \sup_{P_v\in\Lambda} P_v[\sup_{0\leq t\leq T}|M_t| \geq \lambda] \\ \leq & \sup_{P_v\in\Lambda} \frac{1}{\lambda^p}E_{P_v}[|M_T|^p] \\ = & \frac{1}{\lambda^p}E_G[|M_T|^p]. \end{aligned}$$

∎

5 Martingale Characterization of G-Brownian Motion

We are ready to prove the Martingale Characterization theorem for G-Brownian Motion.

Theorem 5.1. *Let $M \in \mathscr{S}^2$. Suppose that*

(I) M is a symmetric martingale;

(II) $M_t^2 - t$ is a martingale;

(III) for $\sigma_0 > 0$, $\sigma_0^2 t - M_t^2$ is a martingale; and

(IV) M is continuous, which means for every $\omega \in \Omega$, $M(t,\omega)$ is a continuous function.

Then M is a G-Brownian motion in the sense that M has the same finite distribution as the G-Brownian motion B.

Remark 5.2. *The Lèvy's martingale characterization of Brownian motion in a probability space states that, B_t is a continuous martingale with respect to $\mathscr{F}_t$, and $B_t^2 - t$ is a $\mathscr{F}_t$ martingale, iff B_t is a Brownian motion. Our martingale characterization of G-Brownian motion covers Lèvy's martingale characterization of Browninan motion, when $\sigma_0 = 1$. In a probability space, the quadratic process $\langle B\rangle_t$ of Brownian motion B_t almost sure equals to t with respect to the probability measure P. But in the G-framework, the quadratic process $\langle B\rangle_t$ of G -Brownian motion B_t is not a fixed function any more. Instead, it is a stochastic process in G-framework. The criteria (III) is the description of the nonsymmetric property for $\langle B\rangle_t$. Condition (IV) is reasonable, thanks to [10], for any G-Brownian motion, it has continuous path.*

In a probability space, thanks to the characteristic function of normal distribution, Lèvy's martingale characteristic theorem of Brownian motion holds. But in G-framework, there is no characteristic function of "G-normal distribution"(see the definition in [23]), we have to find a different method to solve this problem.

Next we shall prove Theorem 5.1 in 4 steps.

Proof of Theorem 5.1:

Step 1. $\forall \eta \in M_G^2(0,T)$, we define the stochastic integral of Itô's type $\int_0^T \eta dM_t$, and prove that $\int_0^t \eta dM_s$ is a symmetric martingale under $E_G[\cdot]$.

By conditions $(I) \sim (IV)$ in Theorem 5.1, we have the following proposition.

Proposition 5.3. *for any $t_j \le t_{j+1} < t_i \le t_{i+1}$,*

(1) $E_G[(M_{t_{j+1}} - M_{t_j})\xi_{t_j}(M_{t_{i+1}} - M_{t_i})\xi_{t_i}] = 0$, $\xi_{t_j} \in L_G^p(\mathscr{F}_{t_j})$, $\xi_{t_i} \in L_G^p(\mathscr{F}_{t_i})$,

(2) $E_G[(M_{t_{j+1}} - M_{t_j})^2|\mathscr{F}_{t_j}] = E_G[(M_{t_{j+1}} - M_{t_j})^2] = t_{j+1} - t_j$,

(3) $E_G[X + \xi_{t_j}(M_{t_{j+1}} - M_{t_j})^2] = E_G[X + \xi_{t_j}((M_{t_{j+1}})^2 - (M_{t_j})^2)]$, $X \in L_G^p(\mathscr{F}_T)$, $\xi_{t_j} \in L_G^p(\mathscr{F}_{t_j})$.

Let $\int_0^t \eta dM_s = \int_0^T \eta I_{[0,t]}(s) dM_s$. The next lemma shows that $\int_0^t \eta dM_s$ is a symmetric martingale.

Lemma 5.4. *For any $\eta \in M_G^2(0,T)$, $t \in [0,T]$, $\int_0^t \eta dM_s$ is a symmetric martingale.*

Step 2: The quadratic variation process $\langle M\rangle_t$ of M_t exists. Defining the stochastic integral with respect to $\langle M\rangle_t$ in $M_G^1(0,T)$, we can get the isometric formula in the G-framework.

Now we give the isometry formula in the G-framework.

First, we give a proposition.

Proposition 5.5. *For all $0 \le s \le t$, $\xi \in L_G^1(\mathscr{F}_s)$ and $X \in L_G^1(\mathscr{F})$ we have*

$$E_G[X+\xi(M_t^2-M_s^2)]=E_G[X+\xi(M_t-M_s)^2]=E_G[X+\xi(\langle M\rangle_t-\langle M\rangle_s)].$$

Lemma 5.6. *For $\eta \in M_G^2(0,T)$, it holds that*

$$E_G[(\int_0^T \eta(s)dM_s)^2]=E_G[\int_0^T \eta^2(s)d\langle M\rangle_s]\le \int_0^T E_G[\eta^2(s)]ds.$$

Remark 5.7. *We can prove Lemma 5.4 and 5.6 by the similar method to that in [23] and [29]. Thus we omit the proof.*

Next, we investigate the property of the quadratic process $\langle M\rangle_t$.

Lemma 5.8. *(1)*

$$E_G\left[\left(\int_s^t (M_v-M_s)dM_v\right)^2\middle|\mathscr{F}_s\right]=E_G\left[\int_s^t (M_v-M_s)^2 d\langle M\rangle_v\right]=\frac{1}{2}(t-s)^2.$$

(2) For any $\eta \in M_G^1(0,T)$, $\frac{1}{2}\int_0^t \eta_s d\langle M\rangle_s-\int_0^t G(\eta_s)ds$ is a martingale.

Proof: (1) Let $s=t_0<t_1<\ldots<t_N=t$, and $t_{j+1}-t_j=\frac{t-s}{N}$. Note that

$$\sum_{j=0}^{N-1}(M_{t_j}-M_s)(M_{t_{j+1}}-M_{t_j})\longrightarrow \int_s^t (M_v-M_s)dM_v \text{ in } L_G^2(\mathscr{F}_t).$$

Then

$$\left(\sum_{j=0}^{N-1}(M_{t_j}-M_s)(M_{t_{j+1}}-M_{t_j})\right)^2 \longrightarrow \left(\int_s^t (M_v-M_s)dM_v\right)^2 \text{ in } L_G^1.$$

And we have

$$\begin{aligned}
&E_G\left[\left|E_G\left[\left(\int_s^t (M_v-M_s)dM_v\right)^2\middle|\mathscr{F}_s\right]-E_G\left[\left(\sum_{j=0}^{N-1}(M_{t_j}-M_s)(M_{t_{j+1}}-M_{t_j})\right)^2\middle|\mathscr{F}_s\right]\right|\right]\\
\le\;& E_G\left[E_G\left[\left|\left(\int_s^t (M_v-M_s)dM_v\right)^2-\left(\sum_{j=0}^{N-1}(M_{t_j}-M_s)(M_{t_{j+1}}-M_{t_j})\right)^2\right|\middle|\mathscr{F}_s\right]\right]\\
=\;& E_G\left[\left|\left(\int_s^t (M_v-M_s)dM_v\right)^2-\left(\sum_{j=0}^{N-1}(M_{t_j}-M_s)(M_{t_{j+1}}-M_{t_j})\right)^2\right|\right]\longrightarrow 0.
\end{aligned}$$

So

$$E_G\left[\left(\sum_{j=0}^{N-1}(M_{t_j}-M_s)(M_{t_{j+1}}-M_{t_j})\right)^2\middle|\mathscr{F}_s\right]\longrightarrow E_G\left[\left(\int_s^t(M_v-M_s)dM_v\right)^2\middle|\mathscr{F}_s\right]\text{ in }L_G^1.$$

Now

$$\begin{aligned}&E_G\left[\left(\sum_{j=0}^{N-1}(M_{t_j}-M_s)(M_{t_{j+1}}-M_{t_j})\right)^2\middle|\mathscr{F}_s\right]\\ =\;& E_G\left[\left(\sum_{j=0}^{N-1}(M_{t_j}-M_s)^2(M_{t_{j+1}}-M_{t_j})\right)^2\middle|\mathscr{F}_s\right]\\ =\;&\sum_{j=0}^{N-1}(t_j-s)(t_{j+1}-t_j).\end{aligned}$$

We know

$$E_G\left[\left(\int_s^t(M_v-M_s)dM_v\right)^2\middle|\mathscr{F}_s\right]=\lim_{N\to\infty}\sum_{j=0}^{N-1}(t_j-s)(t_{j+1}-t_j)=\int_s^t vdv=\frac{1}{2}(t-s)^2.$$

(2) For any $\eta\in M_G^{1,0}(0,T)$, $\eta_t=\sum_{j=0}^{N-1}\xi_{t_j}(\omega)I_{[t_j,t_{j+1})}(t)$, for any $t\in[0,T]$, suppose $t=t_{N-1}$. Then,

$$\begin{aligned}&E_G\left[\int_0^T\eta_s d\langle M\rangle_s-2\int_0^T G(\eta_s)ds\middle|\mathscr{F}_t\right]\\ =\;&E_G\left[\sum_{j=0}^{N-1}\xi_j(\langle M\rangle_{t_{j+1}}-\langle M\rangle_{t_j})-2\sum_{j=0}^{N-1}G(\xi_j)(t_{j+1}-t_j)\middle|\mathscr{F}_{t_{N-1}}\right]\end{aligned}$$

$$\begin{aligned}=\;&\sum_{j=0}^{N-2}\xi_j(\langle M\rangle_{t_{j+1}}-\langle M\rangle_{t_j})-2\sum_{j=0}^{N-2}G(\xi_j)(t_{j+1}-t_j)+\xi_{t_{N-1}}^+E_G\left[\langle M\rangle_{t_N}-\langle M\rangle_{t_{N-1}}\middle|\mathscr{F}_{t_{N-1}}\right]\\ &+\xi_{t_{N-1}}^-E_G\left[-(\langle M\rangle_{t_N}-\langle M\rangle_{t_{N-1}})\middle|\mathscr{F}_{t_{N-1}}\right]-2G(\xi_{N-1})(t_N-t_{N-1})\\ =\;&\int_0^t\eta_s d\langle M\rangle_s-2\int_0^t G(\eta_s)ds+2G(\xi_{N-1})(t_N-t_{N-1})-2G(\xi_{N-1})(t_N-t_{N-1})\\ =\;&\int_0^t\eta_s d\langle M\rangle_s-2\int_0^t G(\eta_s)ds.\end{aligned}$$

If $\eta\in M_G^1(0,T)$, then there exists a sequence $\eta^N\in M_G^{1,0}(0,T)$, such that $\eta^N\longrightarrow\eta$ in $M_G^1(0,T)$, and $(\eta^N)^+\longrightarrow(\eta)^+$, $(\eta^N)^-\longrightarrow(\eta)^-$, then $G(\eta^N)\longrightarrow G(\eta)$.

Now

$$
\begin{aligned}
& E_G\left[\int_0^T \eta_s d\langle M\rangle_s - 2\int_0^T G(\eta_s)ds\middle|\mathscr{F}_t\right] - \left(\int_0^t \eta_s d\langle M\rangle_s - 2\int_0^t G(\eta_s)ds\right) \\
= \;& E_G\left[\int_0^T (\eta_s-\eta_s^N)d\langle M\rangle_s - 2\int_0^T (G(\eta_s)-G(\eta_s^N))ds + \int_0^T \eta_s^N d\langle M\rangle_s - 2\int_0^T G(\eta_s^N)ds\middle|\mathscr{F}_t\right] \\
& -\left(\int_0^t \eta_s d\langle M\rangle_s - 2\int_0^t G(\eta_s)ds\right) \\
\leq \;& E_G\left[\int_0^T |\eta_s-\eta_s^N|d\langle M\rangle_s - 2\int_0^T |G(\eta_s)-G(\eta_s^N)|ds\middle|\mathscr{F}_t\right] \\
& +\int_0^t |\eta_s-\eta_s^N|d\langle M\rangle_s - 2\int_0^t |G(\eta_s)-G(\eta_s^N)|ds \longrightarrow 0 \text{ in } L_G^1.
\end{aligned}
$$

Similarly we have

$$
\begin{aligned}
& E_G\left[\int_0^T \eta_s d\langle M\rangle_s - 2\int_0^T G(\eta_s)ds\middle|\mathscr{F}_t\right] - \left(\int_0^t \eta_s d\langle M\rangle_s - 2\int_0^t G(\eta_s)ds\right) \\
\geq \;& E_G\left[-\left|\int_0^T (\eta_s-\eta_s^N)d\langle M\rangle_s - 2\int_0^T (G(\eta_s)-G(\eta_s^N))ds\right|\middle|\mathscr{F}_t\right] \\
& -\left|\int_0^t (\eta_s-\eta_s^N)d\langle M\rangle_s - 2\int_0^t (G(\eta_s)-G(\eta_s^N))ds\right| \longrightarrow 0 \text{ in } L_G^1.
\end{aligned}
$$

So

$$
E_G\left[\int_0^T \eta_s d\langle M\rangle_s - 2\int_0^T G(\eta_s)ds\middle|\mathscr{F}_t\right] = \int_0^t \eta_s d\langle M\rangle_s - 2\int_0^t G(\eta_s)ds.
$$

∎

Lemma 5.9. *For any $s<t\in[0,T]$, $E_G[(M_t-M_s)^4]<K(t-s)^2$, for some $K>0$.*

Proof: We first prove that $\langle M\rangle_t$ is continuous in $[0,T]$ quasi-surely, which means $\langle M\rangle_t$ has continuous path outside a polar set. To prove the continuity of $\langle M\rangle_t$, we only need to prove the continuity of $\int_0^t M_s dM_s$.

Note that $\int_0^t \eta dM_s$ is a symmetric martingale, see step 1. If $\eta\in M_G^{2,0}(0,T)$, then $\int_0^t \eta dM_s$ has continuous path. If $\eta\in M_G^2(0,T)$, then there exits $\eta_n\in M_G^{2,0}(0,T)$, such that

$$
E_G[|I_n(t)-I(t)|^2]\longrightarrow 0,
$$

where $I_n(t)=\int_0^t \eta_n dM_s$. By Lemma 4.11, we have

$$
E_G[\sup_{0\leq t\leq T}|I_n(t)-I_m(t)|^2]\leq E_G[\int_0^T |\eta_n-\eta_m|^2dt]\longrightarrow 0.
$$

Then by Proposition 3.8, we know that $\sup_{0\le t\le T}|I_n(t)-I_m(t)|^2$ is convergence q.s., so $I(t)$ is continuous quasi-surely.

Since M is continuous, $\langle M\rangle$ is continuous quasi-surely. And

$$\langle M\rangle_t^2=\sum_{j=0}^{N-1}(\langle M\rangle_{t_{j+1}^N\wedge t}-\langle M\rangle_{t_j^N\wedge t})^2+2\sum_{j=0}^{N-1}\langle M\rangle_{t_j^N\wedge t}(\langle M\rangle_{t_{j+1}^N\wedge t}-\langle M\rangle_{t_j^N\wedge t}).$$

We know

$$\sum_{j=0}^{N-1}\langle M\rangle_{t_j^N\wedge t}(\langle M\rangle_{t_{j+1}^N\wedge t}-\langle M\rangle_{t_j^N\wedge t})\longrightarrow\int_0^t\langle M\rangle_s d\langle M\rangle_s.\text{ in } L_G^1.$$

So $\sum_{j=0}^{N-1}(\langle M\rangle_{t_{j+1}^N\wedge t}-\langle M\rangle_{t_j^N\wedge t})$ is convergent in L_G^1. Since $\langle M\rangle$ is continuous in $[0,T]$ quasi-surely, $\sum_{j=0}^{N-1}(\langle M\rangle_{t_{j+1}^N\wedge t}-\langle M\rangle_{t_j^N\wedge t})^2$ converges to 0 quasi-surely. Then we get

$$E_G[(\langle M\rangle_t)^2]\le E_G[\lim_{N\to\infty}\sum_{j=0}^{N-1}(\langle M\rangle_{t_{j+1}^N\wedge t}-\langle M\rangle_{t_j^N\wedge t})]+\int_0^t E_G[\langle M\rangle_s]ds=\frac{1}{2}t^2.$$

Similarly we can get that $E_G[(\langle M\rangle_t-\langle M\rangle_s)^2]\le\frac{1}{2}(t-s)^2$.

Since $(M_t-M_s)^4=\left(\langle M\rangle_t-\langle M\rangle_s+2\int_s^t(M_v-M_s)dM_v\right)^2$, we get

$$E_G[(M_t-M_s)^4]\le E_G[(\langle M\rangle_t-\langle M\rangle_s)^2]+2\int_s^t E_G[(M_v-M_s)^2]dv\le K(t-s)^2,$$

where K is a constant. ∎

Step 3.

Lemma 5.10. *Let $u(t,x)\in C^{1,2}([0,T]\times R)$ be bounded. Suppose that $u(t,x)$ and the partial derivatives u_t and $u_{x,x}$ are all uniformly continuous. Then for any $0<s\le t<T, 0<s+\delta<t+\delta<T$, we have*

$$\begin{aligned}&u(t,M_{t+\delta}-M_\delta)-u(s,M_{s+\delta}-M_\delta)\\=&\int_{s+\delta}^{t+\delta}\frac{1}{2}u_{xx}(v-\delta,M_v-M_\delta)d\langle M\rangle_v+\int_{s+\delta}^{t+\delta}u_x(v-\delta,M_v-M_\delta)dM_v\\&+\int_{\delta+s}^{\delta+t}u_t(v-\delta,M_v-M_\delta)dv.\end{aligned}$$

Proof: We know $u(t,x)$, $u_t(t,x)$ and $u_{xx}(t,x)$ are all uniformly continuous. Without loss of generality, suppose $u(t,x)$, $u_t(t,x)$ and $u_{xx}(t,x)$ are all Lipschitz continuous, and suppose the Lipschitz constant is C. Since $u_x(t,x)$ is bounded and continuous and we only consider the problem in the finite time horizon, by the proof of Lemma 3.4, we can also suppose that $u_t(t,x)$ is uniformly continuous. For every $N>0$, we have

$$\pi_{s,t}^N=\{s,s+\delta_N,\cdots,s+N\delta_N=t\},\quad \delta_N=\frac{t-s}{N},$$

and

$$
\begin{aligned}
& u(t, M_{t+\delta}-M_\delta)-u(s, M_{s+\delta}-M_\delta) \\
= & \sum_{j=0}^{N-1}\Big[\Big(u(t_{j+1}, M_{\delta+t_{j+1}}-M_\delta)-u(t_j, M_{t_{j+1}+\delta}-M_\delta)\Big) \\
& +\Big(u(t_j, M_{\delta+t_{j+1}}-M_\delta)-u(t_j, M_{t_j+\delta}-M_\delta)\Big)\Big] \\
= & \sum_{j=0}^{N-1}\Big(u(v_{j+1}-\delta, M_v-M_\delta)-u(v_j-\delta, M_v-M_\delta)\Big)+\sum_{j=0}^{N-1} u_x(v_j-\delta, M_{v_j}-M_\delta)(M_{v_{j+1}}-M_{v_j}) \\
& \quad +\frac{1}{2}\sum_{j=0}^{N-1} u_{xx}(v_j-\delta, M_{v_j}-M_\delta)(M_{v_{j+1}}-M_{v_j})^2 \\
+ & \frac{1}{2}\sum_{j=0}^{N-1}\Big[u_{xx}(v_j-\delta, M_{v_j}-M_\delta+\theta_j(\omega)(M_{v_{j+1}}-M_{v_j}))-u_{xx}(v_j-\delta, M_{v_j}-M_\delta)\Big](M_{v_{j+1}}-M_{v_j})^2 \\
= & \sum_{j=0}^{N-1}\int_{v_j}^{v_{j+1}} u_t(v-\delta, M_{v_{j+1}}-M_\delta)dv+\sum_{j=0}^{N-1} u_x(v_j-\delta, M_{v_j}-M_\delta)(M_{v_{j+1}}-M_{v_j}) \\
& \quad +\frac{1}{2}\sum_{j=0}^{N-1} u_{xx}(v_j-\delta, M_{v_j}-M_\delta)(M_{v_{j+1}}-M_{v_j})^2+ \\
& \frac{1}{2}\sum_{j=0}^{N-1}\Big[u_{xx}(v_j-\delta, M_{v_j}-M_\delta+\theta_j(\omega)(M_{v_{j+1}}-M_{v_j}))-u_{xx}(v_j-\delta, M_{v_j}-M_\delta)\Big](M_{v_{j+1}}-M_{v_j})^2,
\end{aligned}
$$

where $\theta_j(\omega)\in(0,1)$, $v_j=\delta+t_j$.
Now that

$$
\begin{aligned}
& E_G\left[\left|\sum_{j=0}^{N-1}\int_{v_j}^{v_{j+1}} u_t(v-\delta, M_{v_j}-M_\delta)dv-\int_{s+\delta}^{t+\delta} u_t(v-\delta, M_v-M_\delta)dv\right|\right] \\
\leq\ & E_G\left[\left|\sum_{j=0}^{N-1}\int_{v_j}^{v_{j+1}} C|M_v-M_{v_j}|dv\right|\right] \\
\leq\ & \sum_{j=0}^{N-1}\frac{2}{3}|v_{j+1}-v_j|^{3/2}=\sum_{j=0}^{N-1}\frac{2}{3}\delta_N^{3/2}\longrightarrow 0,
\end{aligned}
$$

and

$$
\sum_{j=0}^{N-1} u_x(v_j-\delta, M_{v_j}-M_\delta)I_{[t_j,t_{j+1})}\longrightarrow u_x(v-\delta, M_v-M_\delta),
$$

we have

$$
\sum_{j=0}^{N-1} u_x(v_j-\delta, M_{v_j}-M_\delta)(M_{v_{j+1}}-M_{v_j})\longrightarrow\int_{s+\delta}^{t+\delta} u_x(v-\delta, M_v-M_\delta)dM_v \ \text{ in } \ L_G^2.
$$

Since

$$E_G\left[\left|\sum_{j=0}^{N-1}u_{xx}(v_j-\delta,M_{v_j}-M_\delta)(M_{v_{j+1}}-M_{v_j})^2-\sum_{j=0}^{N-1}u_{xx}(v_j-\delta,M_{v_j}-M_\delta)(\langle M\rangle_{v_{j+1}}-\langle M\rangle_{v_{j+1}})\right|^2\right]$$

$$\begin{aligned}
&= E_G\left[\left|\sum_{j=0}^{N-1}u_{xx}(v_j-\delta,M_{v_j}-M_\delta)\int_{v_j}^{v_{j+1}}(M_v-M_{v_j})dM_v\right|^2\right]\\
&\le E_G\left[\sum_{j=0}^{N-1}u_{xx}^2(v_j-\delta,M_{v_j-M_\delta})\left(\int_{v_j}^{v_{j+1}}(M_v-M_{v_j})dM_v\right)^2\right]+2E_G\left[\sum_{j\neq i}u_{xx}(v_j-\delta,M_{v_j}-M_\delta)\right.\\
&\quad\left. u_{xx}(v_i-\delta,M_{v_i}-M_\delta)\int_{v_j}^{v_{j+1}}(M_v-M_{v_j})dM_v\int_{v_i}^{v_{i+1}}(M_v-M_{v_i})dM_v\right]\\
&\le \sum_{j=0}^{N-1}E_G[u_{xx}^2(v_j-\delta,M_{v_j})-M_\delta]\delta_N^2\le C\sum_{j=0}^{N-1}E_G[1+\delta+|M_{v_j}-M_\delta|]\delta_N^2\longrightarrow 0,
\end{aligned}$$

we have

$$\sum_{j=0}^{N-1}u_{xx}(v_j-\delta,M_{v_j}-M_\delta)(\langle M\rangle_{v_{j+1}}-\langle M\rangle_{v_j})\longrightarrow\int_{s+\delta}^{t+\delta}u_{xx}(v-\delta,M_v-M_\delta)d\langle M\rangle_v.$$

So the result holds. ∎

Step 4.

Lemma 5.11. *If $u(t,x)$ is the viscosity solution to the G-heat equation, and $\sigma_0>0$, then*

$$E_G[\varphi(M_t+x)]=u(t,x),\quad \varphi\in lip(R).$$

Proof: For $0<\sigma_0\le 1$, then by Theorem 4.13 in [17], $u(t,x)\in C^{1,2}((0,T]\times R)$, and $u(t,x)$, $u_t(t,x)$ and $u_{xx}(t,x)$ are all uniformly continuous, $u_x(t,x)$ is bounded and continuous, see in [28]. Then by Lemma 5.10, for any $0<\varepsilon<v\le T$, we have

$$\begin{aligned}
&u(\varepsilon,M_{\delta+v-\varepsilon}-M_\delta+x)-u(v,x)=\int_\delta^{v-\varepsilon+\delta}-u_t(v+\delta-s,M_s-M_\delta+x)ds\\
&+\int_\delta^{v-\varepsilon+\delta}u_x(v+\delta-s,M_s-M_\delta+x)dM_s+\frac{1}{2}\int_\delta^{v-\varepsilon+\delta}u_{xx}(v+\delta-s,M_s-M_\delta+x)d\langle M\rangle_s.
\end{aligned}$$

By Lemma 5.8, it is easy to check that

$$
\begin{aligned}
& E_G[u(\varepsilon, M_{\delta+v-\varepsilon} - M_\delta + x)] - u(v,x) \\
= \ & E_G\left[\int_\delta^{v-\varepsilon+\delta} -u_t(v+\delta-s, M_s - M_\delta + x)ds \right. \\
& \left. + \frac{1}{2}\int_\delta^{v-\varepsilon+\delta} u_{xx}(v+\delta-s, M_s - M_\delta + x)d\langle M\rangle_s \right] \\
= \ & E_G\left[\frac{1}{2}\int_\delta^{v-\varepsilon+\delta} u_{xx}(v+\delta-s, M_s - M_\delta + x)d\langle M\rangle_s \right. \\
& \left. - \int_\delta^{v-\varepsilon+\delta} G(u_{xx}(v+\delta-s, M_s - M_\delta + x))ds \right] \\
= \ & 0.
\end{aligned}
$$

Then $\lim_{\varepsilon\to 0} E_G[u(\varepsilon, M_{\delta+v-\varepsilon} - M_\delta + x)] = u(v,x)$.

We also have

$$E_G[|u(\varepsilon, M_{\delta+v-\varepsilon} - M_\delta + x) - u(0, M_{\delta+v} - M_\delta + x)] \leq 2CE_G[\sqrt{\varepsilon} + |M_{\delta+v-\varepsilon} - M_{\delta+v}|] \leq 4C\sqrt{\varepsilon},$$

where C is the Lipschitz constant of φ.

Then $E_G[\varphi(M_{\delta+t} - M_\delta + x)] = u(t,x) = E_G[\varphi(M_t + x)]$.

Without loss of generality, by the definition of G-Brownian motion, see [23], we only need to prove the case for $m = 2$:

$$
\begin{aligned}
& E_G[\varphi(M_{t_1}, M_{t_2} - M_{t_1})] \\
= \ & E_G[E_G[\varphi(M_{t_1}, M_{t_2} - M_{t_1})|\mathscr{F}_{t_1}]] \\
= \ & E_G[E_G[\varphi(x, M_{t_2} - M_{t_1})|\mathscr{F}_{t_1}]_{x=M_{t_1}}] \\
= \ & E_G[E_G[\varphi(x, M_{t_2-t_1})]_{x=M_{t_1}}].
\end{aligned}
$$

Based on Step 1 $\sim$ Step 4, M has the same finite distribution with G-Brownian motion B, we complete the proof of the Theorem 5.1. ∎

6 Discussion

In this paper, we investigate the properties of capacity defined by G-expectation, and prove that G-expectation is not filtration consistent. Meanwhile by path-wise analysis, we prove that symmetric G-martingales are martingales under each P_v. Based on these arguments, we obtain the martingale characterization theorem for a G-Brownian motion. The application of this framework in finance can be found in [9]. Our martingale characterization theorem of G-Brownian motion includes Lèvy's martingale characterization theorem of Brownian motion. In the proof of classical Lèvy's martingale characterization theorem, the characterization function of normal distribution plays an important role. But in G-framework the characterization function of "G-normal" distribution dose not hold.

Our proof of martingale characterization of G-Brownian motion gives a totally different method to prove Lèvy's martingale characterization of Brownian motion. In our proof the compactness of Λ is essential. Based on our result one can investigate some elementary problems such as the martingale representation with respect to G-Brownian motion in G-framework.

7 APPENDIX

Lemma 7.1. *If $M \in L_G^1(\mathscr{F}_t)$, $\varphi(x)$ is Lipschitz continuous, then $\varphi(M(t)) \in L_G^1(\mathscr{F}_t)$.*

Proof: Without loss of generality, suppose $\varphi(x) \geq 0$. We know that there exists a sequence $\varphi_n(x) \in lip(R)$ such that $\varphi_n(x) \longrightarrow \varphi(x)$ monotonically. By the Lipschitz continuous of $\varphi(x)$, and the fact that $E_G[|\varphi(M_t)|^2] < \infty$, we have

$$E_G[|\varphi(M_t) - \varphi_n(M_t)|] \leq E_G[|\varphi(M_t)| I_{\{|M_t| \geq n\}}]$$
$$\leq \quad E_G^{1/2}[|\varphi(M_t)|^2] E_G^{1/2}[I_{\{|M_t| \geq n\}}] \leq E_G^{1/2}[|\varphi(M_t)|^2] \frac{E_G^{1/2}[M_t^2]}{n} \longrightarrow 0.$$

∎

Proposition 7.2. *For any $v \in \Lambda'$, set $I_\delta = E[\sup_{|s-t|\leq\delta} |\int_s^t v_\tau dB_\tau|]$, then $\lim_{\delta\to 0} I_\delta = 0$, where E refers to the expectation under Weiner measure, and B is the canonical process which is the Brownian motion under Weiner measure. (see the definition of Λ' in §3)*

Proof: The proof is trivial, and hence omitted. ∎

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