SOME OPERATIONS OVER ATANASSOV’S INTUITIONISTIC FUZZY SETS BASED ON EINSTEIN T-NORM AND T-CONORM

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We define some operations over Atanassov’s intuitionistic fuzzy sets (AIFSs), such as Einstein intersection, Einstein product, Einstein scalar multiplication and Einstein exponentiation, etc., and then define new concentration and dilation of AIFSs. These definitions will be useful while dealing with various linguistic hedges like “very”, “more or less”, “highly”, “very very” etc., involved in the problems under intuitionistic fuzzy environment. We also prove some propositions and present some examples in this context.

Keywords: Atanassov’s intuitionistic fuzzy sets (AIFSs); Einstein operation; CON\(\varepsilon\); DIL\(\varepsilon\); linguistic variables.

1. Introduction

Atanassov’s Intuitionistic fuzzy set (AIFS),\(^1\) a generalization of fuzzy set,\(^2\) has been found to be highly useful to deal with vagueness. Each element in an AIFS is expressed by an ordered pair, and each ordered pair is characterized by a membership degree and a non-membership degree, the sum of the membership degree and the non-membership degree of each ordered pair is less than or equal to one. Since it was first introduced by Atanassov in 1986,\(^1\) the AIFS theory has been widely investigated and applied to a variety.\(^3\)–\(^7\) The operation over AIFSs is an interesting and important research topic in AIFS theory that has been receiving more and more attention in recent years.\(^1,8\)–\(^17\) Atanassov\(^1,8,9\) defined some basic operations and relations of AIFSs, including intersection, union, complement, algebraic sum and algebraic product, etc., and proved an equality between AIFSs. Atanassov\(^10\) determined the pseudo-fixed points of all operators defined over AIFSs. Atanassov\(^11\)

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formulated and proved two theorems related to the relations between some of the operators over AIFSs. De et al. further defined the concentration, dilation and normalization of AIFSs, and proved some propositions. Deschrijver and Kerre introduced some aggregation operators on the lattice L, and defined the special classes of binary aggregation operators, based on triangular norms (t-norms) on the unit interval (Deschrijver and Kerre showed that AIFSs could be seen as L-fuzzy sets). However, there is only one type of the operational laws of AIFS, that is to say, these operations are based on the product t-norm and its dual t-conorm, which are widely used in decision making. Although Deschrijver and Kerre have introduced a generalised union and a generalised intersection of AIFSs using a general t-norm and t-conorm, the extensions of the scalar multiplication and exponentiation operations on AIFS are limited for the structures of various t-norms are different. The Einstein t-norm is the same strict t-norm as the product t-norm, and the Einstein t-norm and its dual t-conorm have unique structures. Therefore, this paper focuses mainly on another type of operations on AIFS, which are based on the Einstein t-norm and its dual t-conorm.

In the present paper, motivated by the generalised union and intersection over AIFSs, we introduce some operations over AIFSs, such as Einstein intersection, Einstein product. Then we deduce Einstein scalar multiplication and Einstein exponentiation over AIFSs. Finally, we define new concentrated AIFS, dilated AIFS and make some characterizations.

2. Preliminaries

In this section, the basic concepts of AIFSs and some existing operations over AIFSs will be introduced below to facilitate future discussions.

**Definition 1.** Let a set $E$ be fixed, an AIFS $A$ of $E$ is defined as:
$$A = \{(x, \mu_A(x), \nu_A(x))| x \in E\}$$
where the mappings $\mu_A : E \to [0,1]$ and $\nu_A : E \to [0,1]$ satisfy the condition $0 \leq \mu_A(x) + \nu_A(x) \leq 1, \forall x \in E$, and they denote the degrees of membership and nonmembership of element $x \in E$ to set $A$, respectively.

Let $\pi_A(x) = 1 - \mu_A(x) - \nu_A(x)$, then it is usually called the Atanassov’s intuitionistic fuzzy index of $x \in A$, representing the degree of indeterminacy or hesitation of $x$ to $A$. It is obvious that $0 \leq \pi_A(x) \leq 1$ for every $x \in E$.

Atanassov described some operations over AIFSs as follows.

**Definition 2.** If $A$ and $B$ are two AIFSs of the set $E$, then
- $A \subset B$ iff $\forall x \in E, \mu_A(x) \leq \mu_B(x)$ & $\nu_A(x) \geq \nu_B(x)$,
- $A = B$ iff $A \subset B$ & $B \subset A$,
- $\bar{A} = \{(x, \mu_A(x), \nu_A(x))| x \in E\}$,
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...further defined the following operations over AIFSs.

Definition 3. The scalar multiplication operation over an AIFS $A$ of the universe $E$ is denoted by $nA$ and is defined by

$$nA = \{(x, 1 - [1 - \mu_A(x)]^n, [\nu_A(x)]^n)|x \in E\}$$

where $n$ is any positive real number.

Definition 4. The exponentiation operation over an AIFS $A$ of the universe $E$ is denoted by $A^n$ and is defined by

$$A^n = \{(x,[\mu_A(x)]^n, 1 - [1 - \nu_A(x)]^n)|x \in E\}$$

where $n$ is any positive real number.

Definition 5. The concentration of an AIFS $A$ of the universe $E$, is denoted by $\text{CON}(A)$ and is defined by

$$\text{CON}(A) = \{(x, \mu_{\text{CON}(A)}(x), \nu_{\text{CON}(A)}(x))|x \in E\}$$

where $\mu_{\text{CON}(A)}(x) = [\mu_A(x)]^2$, $\nu_{\text{CON}(A)}(x) = 1 - [1 - \nu_A(x)]^2$.

In other words, concentration of an AIFS is defined by $\text{CON}(A) = A^2$. The linguistic variable “very” is viewed as labels of the concentration on AIFSs of the universe of discourse.$^{12}$

Definition 6. The dilation of an AIFS $A$ of the universe $E$, is denoted by $\text{DIL}(A)$ and is defined by

$$\text{DIL}(A) = \{(x, \mu_{\text{DIL}(A)}(x), \nu_{\text{DIL}(A)}(x))|x \in E\}$$

where $\mu_{\text{DIL}(A)}(x) = [\mu_A(x)]^{1/2}$, $\nu_{\text{DIL}(A)}(x) = 1 - [1 - \nu_A(x)]^{1/2}$.

In other words, dilation of an AIFS is defined by $\text{CON}(A) = A^{1/2}$. The linguistic variable “more or less” is viewed as labels of the dilation on AIFSs of the universe of discourse.$^{12}$

The set theoretical operations have had an important role since the beginning of fuzzy set (FS) theory.$^2$ Starting from Zadeh’s operations min and max, many other operations were introduced in the FS literature. All types of the particular operations were included in the general concepts of t-norms and t-conorms.$^{17,19,20}$
which satisfy the requirements of the conjunction and disjunction operators, respectively. Thus, the t-norm $T$ and t-conorm $S$ are the most general families of binary functions that map the unit square into the unit interval.

Next, we introduce some examples of t-norms and t-conorms,\cite{17,19,20} which satisfy the requirements of the conjunction and disjunction operators, respectively.

- Zadeh-intersection $T_M$ is a t-norm, Zadeh-union $S_M$ is a t-conorm, where
  \begin{align}
  T_M(a, b) &= \min\{a, b\}, \quad S_M(a, b) = \max\{a, b\}. 
  \end{align}

- Algebraic product $T_P$ is a t-norm and Algebraic sum $S_P$ is a t-conorm, where
  \begin{align}
  T_P(a, b) &= a \cdot b, \quad S_P(a, b) = a + b - a \cdot b. 
  \end{align}

- Einstein product $T_\varepsilon$ is a t-norm and Einstein sum $S_\varepsilon$ is a t-conorm, where
  \begin{align}
  T_\varepsilon(a, b) &= \frac{a \cdot b}{1 + (1 - a) \cdot (1 - b)}, \quad S_\varepsilon(a, b) = \frac{a + b}{1 + a \cdot b} 
  \end{align}

where each pair $(T, S)$ is related by the De Morgan duality: $S(a, b) = 1 - T(1 - a, 1 - b)$, $\forall (a, b) \in [0, 1]^2$.

Based on the general t-norm and t-conorm, Deschrijver and Kerre\cite{17} proposed the generalised intersection and union of AIFSs $A$ and $B$, respectively, as follows:

\begin{align}
A \cap_{T,S} B &= \{ (x, T(\mu_A(x), \mu_B(x)), S(\nu_A(x), \nu_B(x))) | x \in E \} \quad (4) \\
A \cup_{S,T} B &= \{ (x, S(\mu_A(x), \mu_B(x)), T(\nu_A(x), \nu_B(x))) | x \in E \} \quad (5)
\end{align}

where any pair $(T, S)$ can be used, $T$ denotes a t-norm and $S$ a so-called t-conorm dual to the t-norm $T$, defined by $S(a, b) = 1 - T(1 - a, 1 - b)$. Furthermore, the generalised intersection $(A \cap_{T,S} B)$ and union $(A \cup_{S,T} B)$ are AIFSs of $E$.

For instance, the Atanassov-intersection $A \cap B$ and the Atanassov-union $A \cup B$ of two AIFSs $A$ and $B$ can be obtained by the t-norm $T_M$ and the t-conorm $S_M$; the algebraic product $A \odot B$ and the algebraic sum $A \oplus B$ by the algebraic product $T_P$ and the algebraic sum $S_P$, i.e.,

- $A \cap B = \{ (x, T_M(\mu_A(x), \mu_B(x)), S_M(\nu_A(x), \nu_B(x))) | x \in E \}$,
- $A \cup B = \{ (x, S_M(\mu_A(x), \mu_B(x)), T_M(\nu_A(x), \nu_B(x))) | x \in E \}$,
- $A \odot B = \{ (x, T_P(\mu_A(x), \mu_B(x)), S_P(\nu_A(x), \nu_B(x))) | x \in E \}$,
- $A \oplus B = \{ (x, S_P(\mu_A(x), \mu_B(x)), T_P(\nu_A(x), \nu_B(x))) | x \in E \}$.

### 3. Some Operations over AIFS Based on Einstein t-norm and t-conorm

The set theoretical operators have had an important role since in the beginning of AIFS theory. In this section, let the t-norm $T$ and its dual t-conorm $S$ be Einstein product $T_\varepsilon$ and Einstein sum $S_\varepsilon$, respectively, then the generalised intersection (4) and union (5) over two AIFSs $A$ and $B$ become Einstein product (denoted by $A \odot_\varepsilon B$)
and Einstein sum (denoted by $A \otimes \varepsilon B$) over two AIFSs $A$ and $B$, respectively, as follows:

\[ A \otimes \varepsilon B = \{(x, T_{\varepsilon}(\mu_A(x), \mu_B(x)), S_{\varepsilon}(\nu_A(x), \nu_B(x))) | x \in E\} \tag{6} \]

\[ A \otimes \varepsilon B = \{(x, S_{\varepsilon}(\mu_A(x), \mu_B(x)), T_{\varepsilon}(\nu_A(x), \nu_B(x))) | x \in E\}. \tag{7} \]

Based on Einstein t-norm and its dual t-conorm (3), Einstein product (6) and Einstein sum (7) over two AIFSs $A$ and $B$ are further indicated as the following operations:

\[ A \otimes \varepsilon B = \left\{ \left( x, \frac{\mu_A(x) \cdot \mu_B(x)}{1 + (1 - \mu_A(x)) \cdot (1 - \mu_B(x))}, \frac{\nu_A(x) + \nu_B(x)}{1 + \nu_A(x) \cdot \nu_B(x)} \right) | x \in E \right\} \tag{8} \]

\[ A \otimes \varepsilon B = \left\{ \left( x, \frac{\mu_A(x) + \mu_B(x)}{1 + \mu_A(x) \cdot \mu_B(x)}, \frac{\nu_A(x) \cdot \nu_B(x)}{1 + (1 - \nu_A(x)) \cdot (1 - \nu_B(x))} \right) | x \in E \right\} \tag{9} \]

**Theorem 1.** If $n$ is any positive integer and $A$ is an AIFS of $E$, then the exponentiation operation $\wedge_n$ is a mapping from $Z^+ \times E$ to $E$:

\[ A \wedge_n = \left\{ \left( x, \frac{2[\mu_A(x)]^n}{[2 - \mu_A(x)]^n + [\mu_A(x)]^n}, \frac{[1 + \nu_A(x)]^{n+1} - [1 - \nu_A(x)]^{n+1}}{[1 + \nu_A(x)]^{n+1} + [1 - \nu_A(x)]^{n+1}} \right) | x \in E \right\} \tag{10} \]

where $A \wedge_n = A \otimes \varepsilon A \otimes \varepsilon \ldots \otimes \varepsilon A$.

**Proof.** Mathematical induction can be used to prove that the above Eq. (10) holds for all positive integers $n$. The Eq. (10) is called $P(n)$.

Basis: Show that the statement $P(n)$ holds for $n = 1$, and the statement $P(n)$ amounts to the statement (1), i.e.,

\[ A \wedge_1 = \left\{ \left( x, \frac{2[\mu_A(x)]}{[2 - \mu_A(x)] + [\mu_A(x)]}, \frac{[1 + \nu_A(x)] - [1 - \nu_A(x)]}{[1 + \nu_A(x)] + [1 - \nu_A(x)]} \right) | x \in E \right\} \]

In the left-hand side of the equation, $A \wedge_1 = A = \{(x, \mu_A(x), \nu_A(x)) | x \in E\}$.

In the right-hand side of the equation,

\[ A \wedge_1 = \left\{ \left( x, \frac{2[\mu_A(x)]}{[2 - \mu_A(x)] + [\mu_A(x)]}, \frac{[1 + \nu_A(x)] - [1 - \nu_A(x)]}{[1 + \nu_A(x)] + [1 - \nu_A(x)]} \right) | x \in E \right\} \]

The two sides are equal, so the statement $P(n)$ is true for $n = 1$. Thus it has been shown the statement $P(1)$ holds.

Inductive step: Show that if $P(n)$ holds, then also $P(n + 1)$ holds.

Assume $P(n)$ holds (for some unspecified value of $n$). It must then be shown that $P(n + 1)$ holds, that is:

\[ A \wedge_{n+1} = \left\{ \left( x, \frac{2[\mu_A(x)]^{n+1}}{[2 - \mu_A(x)]^{n+1} + [\mu_A(x)]^{n+1}}, \frac{[1 + \nu_A(x)]^{n+1} - [1 - \nu_A(x)]^{n+1}}{[1 + \nu_A(x)]^{n+1} + [1 - \nu_A(x)]^{n+1}} \right) | x \in E \right\} \]
Using the induction hypothesis that $P(n)$ holds, the left-hand side can be rewritten to:

$$A^{\wedge,n}(n+1) = A^{\wedge,n} \otimes \varepsilon A,$$

and based on the Einstein sum operation over two AIFSs, we have

$$A^{\wedge,n} \otimes \varepsilon A = \left\{ \left( x, \frac{2[\mu_A(x)]^{n+1} + [1 + \nu_A(x)]^{n+1} - [1 - \nu_A(x)]^{n+1}}{[2 - \mu_A(x)]^{n+1} + [\mu_A(x)]^{n+1}} \right) | x \in E \right\},$$

thereby showing that indeed $P(n+1)$ holds.

Since both the basis and the inductive step have been proved, it has now been proved by mathematical induction that $P(n)$ holds for any positive integer $n$.

Next, we prove the result of $A^{\wedge,n}$ is also an AIFS even if $n$ is any positive real number, i.e.,

$$0 \leq \frac{2[\mu_A(x)]^{n}}{[2 - \mu_A(x)]^{n} + [\mu_A(x)]^{n}} + \frac{[1 + \nu_A(x)]^{n} - [1 - \nu_A(x)]^{n}}{[1 + \nu_A(x)]^{n} + [1 - \nu_A(x)]^{n}} \leq 1.$$

The left-hand side of the inequality holds obviously, we prove the right-hand side of the inequality.

Since $0 \leq \mu_A(x) \leq 1$, $0 \leq \nu_A(x) \leq 1$, $0 \leq \mu_A(x) + \nu_A(x) \leq 1$, then $1 - \nu_A(x) \geq \nu_A(x) \geq 0$, thus

$$0 \leq \frac{2[\mu_A(x)]^{n}}{[2 - \mu_A(x)]^{n} + [\mu_A(x)]^{n}} \leq \frac{2[\mu_A(x)]^{n}}{[1 + (1 - \mu_A(x))]^{n} + [\mu_A(x)]^{n}} \leq \frac{2[\mu_A(x)]^{n}}{[1 + \nu_A(x)]^{n} + [\mu_A(x)]^{n}}.$$

Also since $1 - \nu_A(x) \geq \mu_A(x) \geq 0$, then

$$\frac{[1 + \nu_A(x)]^{n} - [1 - \nu_A(x)]^{n}}{[1 + \nu_A(x)]^{n} + [1 - \nu_A(x)]^{n}} \leq \frac{1 + \nu_A(x)\nu_A(x)]^{n} - [1 - \nu_A(x)]^{n}}{[1 + \nu_A(x)]^{n} + [\mu_A(x)]^{n}} \leq \frac{1 + \nu_A(x)]^{n} - [\mu_A(x)]^{n}}{[1 + \nu_A(x)]^{n} + [\mu_A(x)]^{n}}.$$

Thus

$$\frac{2[\mu_A(x)]^{n}}{[2 - \mu_A(x)]^{n} + [\mu_A(x)]^{n}} + \frac{[1 + \nu_A(x)]^{n} - [1 - \nu_A(x)]^{n}}{[1 + \mu_A(x)]^{n} + [1 - \nu_A(x)]^{n}} \leq \frac{2[\mu_A(x)]^{n}}{[1 + \nu_A(x)]^{n} + [\mu_A(x)]^{n}} + \frac{[1 + \nu_A(x)]^{n} - [\mu_A(x)]^{n}}{[1 + \nu_A(x)]^{n} + [\mu_A(x)]^{n}} = 1.$$

That is to say, the AIFS $A^{\wedge,n}$ defined above is an AIFS.

Similarly, we can get new scalar multiplication operation over AIFSs as follows:

**Theorem 2.** If $n$ is any positive number and $A$ is an AIFS of E, then the scalar multiplication operation $\cdot \varepsilon$ is a mapping from $R^+ \times E$ to $E$

$$n \cdot \varepsilon A = \left\{ \left( x, \frac{1 + \nu_A(x)]^{n} - [1 - \nu_A(x)]^{n}}{[1 + \mu_A(x)]^{n} + [1 - \mu_A(x)]^{n}}, \frac{2[\mu_A(x)]^{n}}{[2 - \nu_A(x)]^{n} + [\mu_A(x)]^{n}} \right) | x \in E \right\}$$
where \( n_xA = A \oplus A \oplus \cdots \oplus A \), and the result of \( n_xA \) is also an AIFS even if \( n \) is any positive real number.

**Example 1.** Consider the universe \( E = \{1, 2, 3, 4, 5\} \), and let \( A \) be an AIFS of \( E \) given by \( A = \{(0.2, 0.6), (0.3, 0.6), (0.5, 0.5), (0.6, 0.3), (0.9, 0)\} \).

Then we see that

\[
A^{\land 2} = \{(0.0244, 0.8824), (0.0604, 0.8824), (0.2, 0.8), (0.3103, 0.5505), (0.802, 0)\},
\]

\[
A^{\lor 0.5} = \{(0.5, 0.3333), (0.5916, 0.3333), (0.7321, 0.2679), (0.7913, 0.1535), (0.9499, 0)\},
\]

\[
2_{x}A = \{(0.3846, 0.3103), (0.5505, 0.3103), (0.8, 0.2), (0.8824, 0.6004), (0.9945, 0)\},
\]

\[
0.5_{x}A = \{(0.101, 0.7913), (0.1535, 0.7913), (0.2679, 0.7321), (0.3333, 0.5916), (0.6268, 0)\}.
\]

**Proposition 1.** For an AIFS \( A \) of the set \( E \) and for any positive number \( n \).

(i) \( \Box A^{\land n} = (\Box A)^{\land n} \),

(ii) \( \Diamond A^{\land n} = (\Diamond A)^{\land n} \),

(iii) If \( \pi_A(x) = 0 \), then \( \pi_{A^{\lor n}}(x) = 0 \)

(iv) \( A^{\lor m} \subset A^{\land n} \), where \( m \) and \( n \) are both positive numbers and \( m \geq n \).

(v) If \( A \) is totally intuitionistic, then \( A^{\land n} \) is also so.

**Proof.**

(i) \( \Box A^{\land n} = \left\{ \left( x, \frac{2[\mu_A(x)]^n}{[2 - \mu_A(x)]^n + [\mu_A(x)]^n} \right) \mid x \in E \right\} = (\Box A)^{\land n} \)

(ii) \( \Diamond A^{\land n} = \left\{ \left( x, \frac{2[1 - \nu_A(x)]^n}{[1 + \nu_A(x)]^n + [1 - \nu_A(x)]^n} \right) \mid x \in E \right\} = (\Diamond A)^{\land n} \)

(iii) If \( \pi_A(x) = 0 \Rightarrow \mu_A(x) + \nu_A(x) = 1 \), then

\[
A^{\land n} = \left\{ \left( x, \frac{2[\mu_A(x)]^n}{[2 - \mu_A(x)]^n + [\mu_A(x)]^n}, \frac{[1 + \nu_A(x)]^n - [1 - \nu_A(x)]^n}{[1 + \nu_A(x)]^n + [1 - \nu_A(x)]^n} \right) \mid x \in E \right\}
\]

which gives \( \pi_{A^{\lor n}}(x) = 0 \).

(iv) Since

\[
A^{\land n} = \left\{ \left( x, \frac{2[\mu_A(x)]^n}{[2 - \mu_A(x)]^n + [\mu_A(x)]^n}, \frac{[1 + \nu_A(x)]^n - [1 - \nu_A(x)]^n}{[1 + \nu_A(x)]^n + [1 - \nu_A(x)]^n} \right) \mid x \in E \right\}
\]

and

\[
A^{\land m} = \left\{ \left( x, \frac{2[\mu_A(x)]^m}{[2 - \mu_A(x)]^m + [\mu_A(x)]^m}, \frac{[1 + \nu_A(x)]^m - [1 - \nu_A(x)]^m}{[1 + \nu_A(x)]^m + [1 - \nu_A(x)]^m} \right) \mid x \in E \right\}.
\]
Proof.

Similar to Proposition 2.

Proposition 3. Consider two AIFSs $A$ and $B$ of the set $E$, for any positive number $n$,

(i) If $A \subseteq B$, then $A^\wedge n \subseteq B^\wedge n$,

(ii) If $A \subseteq B$, then $\nu_\varepsilon A \subseteq \nu_\varepsilon B$,

(iii) $(A \cap B)^\wedge n = A^\wedge n \cap B^\wedge n$,

(iv) $(A \cup B)^\wedge n = A^\wedge n \cup B^\wedge n$,

(v) $\nu_\varepsilon (A \cap B) = \nu_\varepsilon A \cap \nu_\varepsilon B$,

(vi) $\nu_\varepsilon (A \cup B) = \nu_\varepsilon A \cup \nu_\varepsilon B$.

Proof.

(i) Since

$$A^\wedge n = \left\{ x, \frac{2[\mu_A(x)]^n}{[2 - \mu_A(x)]^n + [\mu_A(x)]^n}, \frac{[1 + \nu_A(x)]^n - [1 - \nu_A(x)]^n}{[1 + \nu_A(x)]^n + [1 - \nu_A(x)]^n} \right\} | x \in E \right\}.$$
Let

\[ f_2(x) = \frac{2x^n}{(2-x)^n+x^n}, \quad g_2(y) = \frac{(1+y)^n-(1-y)^n}{(1+y)^n+(1-y)^n} \]  

then we have

\[ f_2'(x) = \frac{4n x^{n-1}(2-x)^{n-1}}{([2-x]^n + x^n]^2} > 0, \quad x \in (0,1) \]  

and

\[ g_2'(y) = \frac{4n(1+y)^{n-1}(1-y)^{n-1}}{([1+y]^n + (1-y)^n]^2} > 0, \quad y \in (0,1). \]

It show that both \( f_2(x) \) and \( g_2(y) \) are increasing functions. So if \( A \subseteq B \), i.e., \( \forall x \in E, \mu_A(x) \leq \mu_B(x) \) \& \( \nu_A(x) \geq \nu_B(x) \), then \( f_2(\mu_A(x)) \leq f_2(\mu_B(x)) \) and \( g_2(\nu_A(x)) \geq g_2(\nu_B(x)) \), i.e., \( \mu_{A\wedge_n B}(x) \leq \mu_{B\wedge_n B}(x) \) and \( \nu_{A\wedge_n B}(x) \geq \nu_{B\wedge_n B}(x) \) thus \( A\wedge_n B \subseteq B\wedge_n B \),

(ii) Similar to (i),

(iii) Since \( A \cap B = \{x, \min\{\mu_A(x), \mu_B(x)\}, \max\{\nu_A(x), \nu_B(x)\}\}|x \in E\} \), then

\[ (A \cap B)^{\wedge_n} = \{(x, \mu_{(A \cap B)^{\wedge_n}}(x), \nu_{(A \cap B)^{\wedge_n}}(x))|x \in E\} \]

where

\[ \mu_{(A \cap B)^{\wedge_n}}(x) = \frac{2[\min\{\mu_A(x), \mu_B(x)\}]^n}{[2 - \min\{\mu_A(x), \mu_B(x)\}]^n + [\min\{\mu_A(x), \mu_B(x)\}]^n} \]

and

\[ \nu_{(A \cap B)^{\wedge_n}}(x) = \frac{[1 + \max\{\nu_A(x), \nu_B(x)\}]^n - [1 - \max\{\nu_A(x), \nu_B(x)\}]^n}{[1 + \max\{\nu_A(x), \nu_B(x)\}]^n + [1 - \max\{\nu_A(x), \nu_B(x)\}]^n}. \]

Also since the (11)–(13) hold, i.e., \( f_2(x) \) and \( g_2(y) \) are increasing functions, we have

\[ \mu_{(A \cap B)^{\wedge_n}}(x) = \frac{2[\min\{\mu_A(x), \mu_B(x)\}]^n}{[2 - \min\{\mu_A(x), \mu_B(x)\}]^n + [\min\{\mu_A(x), \mu_B(x)\}]^n} \]

\[ = \min\left\{ \frac{2[\mu_A(x)]^n}{[2 - \mu_A(x)]^n + [\mu_A(x)]^n}, \frac{2[\mu_B(x)]^n}{[2 - \mu_B(x)]^n + [\mu_B(x)]^n} \right\} \]

and

\[ \nu_{(A \cap B)^{\wedge_n}}(x) = \frac{[1 + \max\{\nu_A(x), \nu_B(x)\}]^n - [1 - \max\{\nu_A(x), \nu_B(x)\}]^n}{[1 + \max\{\nu_A(x), \nu_B(x)\}]^n + [1 - \max\{\nu_A(x), \nu_B(x)\}]^n} \]

\[ = \max\left\{ \frac{[1 + \nu_A(x)]^n - [1 - \nu_A(x)]^n}{[1 + \nu_A(x)]^n + [1 - \nu_A(x)]^n}, \frac{[1 + \nu_B(x)]^n - [1 - \nu_B(x)]^n}{[1 + \nu_B(x)]^n + [1 - \nu_B(x)]^n} \right\}. \]

That is to say \( (A \cap B)^{\wedge_n} = A^{\wedge_n} \cap B^{\wedge_n} \).
(iv) Since \( A \cup B = \{(x, \max\{\mu_A(x), \mu_B(x)\}, \min\{\nu_A(x), \nu_B(x)\})|x \in E\} \) then
\[
(A \cup B)^{\land n} = \{(x, \mu_{(A \cup B)^{\land n}}(x), \nu_{(A \cup B)^{\land n}}(x))|x \in E\}
\]
where
\[
\mu_{(A \cup B)^{\land n}}(x) = \frac{2[\max\{\mu_A(x), \mu_B(x)\}]^n}{[2 - \max\{\mu_A(x), \mu_B(x)\}]^n + [\max\{\mu_A(x), \mu_B(x)\}]^n}
\]
and
\[
\nu_{(A \cup B)^{\land n}}(x) = \frac{[1 + \min\{\nu_A(x), \nu_B(x)\}]^n - [1 - \min\{\nu_A(x), \nu_B(x)\}]^n}{[1 + \min\{\nu_A(x), \nu_B(x)\}]^n + [1 - \min\{\nu_A(x), \nu_B(x)\}]^n}
\]
Also since the (14)–(16) hold, i.e., \( f_2(x) \) and \( g_2(y) \) are increasing functions, we have
\[
\mu_{(A \cup B)^{\land n}}(x) = \max \left\{ \frac{2[\mu_A(x)]^n}{[2 - \mu_A(x)]^n + [\mu_A(x)]^n}, \frac{2[\mu_B(x)]^n}{[2 - \mu_B(x)]^n + [\mu_B(x)]^n} \right\}
\]
and
\[
\nu_{(A \cup B)^{\land n}}(x) = \min \left\{ \frac{[1 + \nu_A(x)]^n - [1 - \nu_A(x)]^n}{[1 + \nu_A(x), \nu_B(x)]^n + [1 - \nu_B(x)]^n}, \frac{[1 + \nu_B(x)]^n - [1 - \nu_B(x)]^n}{[1 + \nu_A(x), \nu_B(x)]^n + [1 - \nu_B(x)]^n} \right\}
\]
That is to say \( (A \cup B)^{\land n} = A^{\land n} \cup B^{\land n} \),

Proofs of (v) and (vi) are similar to those of (iii) and (iv).

Based on the above operations, we can define new concentration and dilation of an AIFS as follows.

**Definition 7.** The concentration of an AIFS \( A \) in the universe \( E \) is denoted by \( \text{CON}^c(A) \) and is defined by
\[
\text{CON}^c(A) = \{(x, \mu_{\text{CON}^c(A)}(x), \nu_{\text{CON}^c(A)}(x))|x \in E\}
\]
where
\[
\mu_{\text{CON}^c(A)}(x) = \frac{2[\mu_A(x)]^2}{[2 - \mu_A(x)]^2 + [\mu_A(x)]^2}, \nu_{\text{CON}^c(A)}(x) = \frac{[1 + \nu_A(x)]^2 - [1 - \nu_A(x)]^2}{[1 + \nu_A(x)]^2 + [1 - \nu_A(x)]^2}
\]
In other words, concentration of an AIFS is defined by \( \text{CON}^c(A) = A^{\land \frac{1}{2}} \).
Definition 8. The dilation of an AIFS $A$ in the universe $E$ is denoted by $\text{DIL}^\varepsilon(A)$ and is defined by

$$\text{DIL}^\varepsilon(A) = \{(x, \mu_{\text{DIL}^\varepsilon(A)}(x), \nu_{\text{DIL}^\varepsilon(A)}(x)) | x \in E\} \quad (18)$$

where

$$\mu_{\text{DIL}^\varepsilon(A)}(x) = \frac{2[\mu_A(x)]^{1/2}}{[2 - \mu_A(x)]^{1/2} + [\mu_A(x)]^{1/2}}, \quad \nu_{\text{DIL}^\varepsilon(A)}(x) = \frac{[1 + \nu_A(x)]^{1/2} - [1 - \nu_A(x)]^{1/2}}{[1 + \nu_A(x)]^{1/2} + [1 - \nu_A(x)]^{1/2}}$$

In other words, dilation of an AIFS is defined by $\text{DIL}^\varepsilon(A) = A \land \varepsilon^{1/2}$.

The following propositions are now straightforward.

Proposition 4. For an AIFS $A$ of the universe $E$.

(i) $\text{CON}^\varepsilon(A) \subseteq A \subseteq \text{DIL}^\varepsilon(A)$,

(ii) If $\pi_A(x) = 0$, then $\pi_{\text{CON}^\varepsilon(A)}(x) = 0$,

(iii) If $\pi_A(x) = 0$, then $\pi_{\text{DIL}^\varepsilon(A)}(x) = 0$,

(iv) $\Box \text{CON}^\varepsilon(A) = \text{CON}^\varepsilon(\Box A)$,

(v) $\Diamond \text{CON}^\varepsilon(A) = \text{CON}^\varepsilon(\Diamond A)$,

(vi) $\Box \text{DIL}^\varepsilon(A) = \text{DIL}^\varepsilon(\Box A)$,

(vii) $\Diamond \text{DIL}^\varepsilon(A) = \text{DIL}^\varepsilon(\Diamond A)$.

To represent the values of linguistic variables, some terms, such as the negation “not”, the connectives “and” and “or”, the hedges “very”, “highly” and “more or less”, etc., are viewed as labels of various operations over AIFSs in the universe of discourse as follows.\textsuperscript{12}

not($A$) = $\bar{A}$,

A and $B = A \cap B = \{(x, \min\{\mu_A(x), \mu_B(x)\}, \max\{\nu_A(x), \nu_B(x)\}) | x \in E\}$,

A or $B = A \cup B = \{(x, \max\{\mu_A(x), \mu_B(x)\}, \min\{\nu_A(x), \nu_B(x)\}) | x \in E\}$,

very ($A$) = $\text{CON}(A)$,

more or less ($A$) = $\text{DIL}(A)$.

Based on the Einstein operations, we define here “very”, “more or less”, “highly”, “plus”, “minus”, etc. If $A$ be an AIFS, then

$\text{CON}^\varepsilon(A)$ may be treated as very($A$),

$\text{DIL}^\varepsilon(A)$ may be treated as more or less($A$),

and furthermore we have

$$\text{plus}(A) = A^{\land 1.25},$$

$$\text{minus}(A) = A^{\land 0.75},$$

$$\text{highly}(A) = \text{minus very very}(A),$$

$$\text{plus plus}(A) = \text{minus very}(A),$$

$$\text{highly}(A) = \text{plus plus very}(A).$$
Example 2. Let the universe be given by $E = \{6, 7, 8, 9, 10\}$, an AIFS “LARGE” of $A$ may be defined by $\text{LARGE} = \{ (0.1, 0.8), (0.3, 0.5), (0.6, 0.2), (0.9, 0), (1, 0) \}$, then

Very LARGE $= \{(0.0055, 0.9756), (0.0604, 0.8), (0.3103, 0.3846), (0.802, 0), (1, 0) \}$,

Not very LARGE $= \{(0.9756, 0.0055), (0.8, 0.0604), (0.3846, 0.3103), (0, 0.802), (0, 1) \}$,

Very very LARGE $= \{(0, 0.9997), (0.0019, 0.9756), (0.0653, 0.6701), (0.6189, 0), (1, 0) \}$.

More or less LARGE $= \{(0.3732, 0.5), (0.5916, 0.2679), (0.7913, 0.101), (0.9499, 0), (1, 0) \}$.

Example 3. Let $A$ be the AIFS denoting the linguistic variable “YOUNG” with the universe $E = [0, 100]$, and furthermore

$$A(\text{YOUNG}) = \{(x, \mu_{\text{YOUNG}}(x), \nu_{\text{YOUNG}}(x)) | x \in E \}$$

where

$$\mu_{\text{YOUNG}}(x) = \begin{cases} 1, & x \in [0, 25], \\ \left[ 1 + \left( \frac{x - 25}{5} \right)^2 \right]^{-1}, & x \in [25, 100], \\ 0, & x \in [0, 28], \\ \end{cases}$$

$$\nu_{\text{YOUNG}}(x) = \begin{cases} 0, & x \in [0, 28], \\ 1 - \left[ 1 + \left( \frac{x - 28}{5} \right)^2 \right]^{-1}, & x \in [28, 100]. \\ \end{cases}$$

then

very YOUNG $= \{(x, \mu_{\text{very YOUNG}}(x), \nu_{\text{very YOUNG}}(x)) | x \in E \}$

where

$$\mu_{\text{very YOUNG}}(x) = \begin{cases} 1, & x \in [0, 25], \\ \frac{2 \cdot [(1 + (\frac{x - 28}{5})^2)^{-1}]^2 + \left[ \left( 1 + \left( \frac{x - 25}{5} \right)^2 \right)^{-1} \right]^2}{2 - (1 + (\frac{x - 28}{5})^2)^{-1} \cdot [(1 + (\frac{x - 25}{5})^2)^{-1}]^2}, & x \in [25, 100] \\ \end{cases}$$

and

$$\nu_{\text{very YOUNG}}(x) = \begin{cases} 0, & x \in [0, 28], \\ \frac{[2 - (1 + (\frac{x - 28}{5})^2)^{-1}]^2 - [(1 + (\frac{x - 25}{5})^2)^{-1}]^2}{2 - (1 + (\frac{x - 28}{5})^2)^{-1} \cdot [(1 + (\frac{x - 25}{5})^2)^{-1}]^2}, & x \in [28, 100]. \\ \end{cases}$$

and more or less YOUNG is defined by

more or less YOUNG $= \{(x, \mu_{\text{more or less YOUNG}}(x), \nu_{\text{more or less YOUNG}}(x)) | x \in E \}$.
where

\[ \mu_{\text{more or less YOUNG}}(x) = \begin{cases} 
1, & x \in [0, 25], \\
\frac{2 \cdot [(1 + (\frac{x-25}{5})^2)^{-1}]^{1/2}}{2 - (1 + (\frac{x-25}{5})^2)^{-1/2} + [(1 + (\frac{x-25}{5})^2)^{-1}]^{1/2}}, & x \in [25, 100] 
\end{cases} \]

\[ \nu_{\text{more or less YOUNG}}(x) = \begin{cases} 
0, & x \in [0, 28], \\
\frac{2 - (1 + (\frac{x-28}{5})^2)^{-1/2} - [(1 + (\frac{x-28}{5})^2)^{-1}]^{1/2}}{2 - (1 + (\frac{x-28}{5})^2)^{-1/2} + [(1 + (\frac{x-28}{5})^2)^{-1}]^{1/2}}, & x \in [28, 100]. 
\end{cases} \]

4. Conclusion

In this paper we have defined new concentration (CON$^\varepsilon(A)$) and dilation (DIL$^\varepsilon(A)$), which will be useful in the calculus of linguistic variables under intuitionistic fuzzy environment. We see that CON$^\varepsilon(A)$ ⊆ A ⊆ DIL$^\varepsilon(A)$, CON$^\varepsilon(A)$ could be viewed as “very(A)” while DIL$^\varepsilon(A)$ as “more or less (A)”. We have made the study with suitable examples. It is worth to point out that the proposed Einstein operations over AIFSs will be applied to aggregating intuitionistic fuzzy information in the future.

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