Note

On monochromatic paths and monochromatic 4-cycles
in edge coloured bipartite tournaments

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Abstract

We call the digraph $D$ an $m$-coloured digraph if the arcs of $D$ are coloured with $m$ colours. A directed path (or a directed cycle) is called monochromatic if all of its arcs are coloured alike.

A set $N \subseteq V(D)$ is said to be a kernel by monochromatic paths if it satisfies the following two conditions:

(i) For every pair of different vertices $u, v \in N$, there is no monochromatic directed path between them.

(ii) For every vertex $x \in (V(D) - N)$, there is a vertex $y \in N$ such that there is an $xy$-monochromatic directed path.

In this paper it is proved that if $D$ is an $m$-coloured bipartite tournament such that every directed cycle of length 4 is monochromatic, then $D$ has a kernel by monochromatic paths.

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1. Introduction

For general concepts we refer the reader to [1]. Let $D$ be a digraph $V(D)$ and $A(D)$ will denote the sets of vertices and arcs of $D$, respectively. An arc $(u_1,u_2) \in A(D)$ is called asymmetrical (resp. symmetrical) if $(u_2,u_1) \notin A(D)$ (resp. $(u_2,u_1) \in A(D)$). The asymmetrical part of $D$ (resp. symmetrical part of $D$) which is denoted Asym($D$) (resp. Sym($D$)) is the spanning subdigraph of $D$ whose arcs are the asymmetrical (resp. symmetrical) arcs of $D$; $D$ is called an asymmetrical digraph if Asym($D$) = $D$. We recall that a subdigraph $D_1$ of $D$ is a spanning subdigraph if $V(D_1) = V(D)$. If $S$ is a nonempty set of $V(D)$ then the subdigraph $D[S]$ induced by $S$ is the digraph having vertex set $S$, and whose arcs are all those arcs of $D$ joining vertices of $S$. An arc $(u_1,u_2)$ of $D$ will be called an $S_1S_2$-arc whenever $u_1 \in S_1$ and $u_2 \in S_2$.

A set $I \subseteq V(D)$ is independent if $A(D[I]) = \emptyset$. A kernel $N$ of $D$ is an independent set of vertices such that for each $z \in V(D) - N$ there exists a $zN$-arc in $D$. A digraph $D$ is called a kernel-perfect digraph or KP-digraph when every induced subdigraph of $D$ has a kernel. A digraph $D$ is called a bipartite tournament if its vertices can be partitioned into two sets $V_1$ and $V_2$ such that:

(i) Every arc of $D$ has an endpoint in $V_1$ and the other endpoint in $V_2$.

(ii) For all $x_1 \in V_1$ and for all $x_2 \in V_2$, we have $|\{(x_1,x_2),(x_2,x_1)\} \cap A(D)| = 1$. We will write $D = (V_1,V_2)$ to indicate the partition.

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If \( T = (z_0, z_1, \ldots, z_n) \) is a directed path, we denote by \( \ell (T) = n \) its length and if \( z_i, z_j \in V (T) \) with \( i \leq j \), we denote \((z_i, T, z_j)\) the \( z_i, z_j \)-directed path contained in \( T \). For a directed cycle \( \gamma \), \( \ell (\gamma) \) will denote its length; a directed cycle is quasi-monochromatic if with at most one exception, all of its arcs are coloured alike.

If \( D \) is an \( m \)-coloured digraph then the closure of \( D \), denoted \( \mathcal{C}(D) \) is the \( m \)-coloured multidigraph defined as follows:

\[
V(\mathcal{C}(D)) = V(D),
\]

\[
A(\mathcal{C}(D)) = A(D) \cup \{(u,v) \text{ with colour } i \mid \text{there exists a } uv \text{-monochromatic directed path coloured } i \text{ contained in } D\}.
\]

Notice that for any digraph \( D \), \( C(\mathcal{C}(D)) \cong \mathcal{C}(D) \) and \( D \) has a kernel by monochromatic paths if and only if \( \mathcal{C}(D) \) has a kernel.

In [7] Sands et al. have proved that any 2-coloured digraph has a kernel by monochromatic paths. In particular they proved that any 2-coloured tournament has a kernel by monochromatic paths. They also raised the following problem: Let \( T \) be a 3-coloured tournament such that every directed cycle of length 3 is quasi-monochromatic; must \( \mathcal{C}(T) \) have a kernel? In [6] Shen Minggang proved that if in the problem we ask that every transitive tournament of order 3 be quasi-monochromatic, the answer will be yes. In [4] it was proved that if \( T \) is an \( m \)-coloured tournament such that every directed cycle of length at most 4 is quasi-monochromatic then \( \mathcal{C}(T) \) is kernel-perfect and hence \( T \) has a kernel by monochromatic paths. Results similar to those in [6] and [4] were proved for the digraph obtained from a tournament by the deletion of a single arc, in [5] and [3], respectively. The known sufficient conditions for the existence of a kernel by monochromatic paths in \( m \)-coloured \((m \geq 3)\) tournaments (or nearly tournaments), ask for the monochromaticity or quasi-monochromaticity of small subdigraphs as directed cycles of length at most 4 or transitive tournaments of order 3.

In this paper it is proved that if \( D \) is an \( m \)-coloured bipartite tournament such that every directed cycle of length 4 is monochromatic then \( D \) has a kernel by monochromatic paths and the result is best possible.

We will need the following result.

**Theorem 1.1** (Duchet [2]). If \( D \) is a digraph such that every directed cycle has at least one symmetrical arc, then \( D \) is a kernel-perfect digraph.

2. The main result

First we prove the following lemmas which will be useful in the proof of the main result:

**Lemma 2.1.** Let \( D = (V_1, V_2) \) be a bipartite tournament and \( C = (u_0, u_1, \ldots, u_n) \) a directed walk in \( D \). For \( \{i, j\} \subseteq \{0, 1, \ldots, n\} \), \( (u_i, u_j) \in A(D) \) or \( (u_j, u_i) \in A(D) \) if and only if \( f - i \equiv 1 \text{ (mod 2)} \).

**Proof.** Without loss of generality we may assume \( u_0 \in V_1 \), then we clearly have \( u_i \in V_1 \) iff \( i \equiv 0 \text{ (mod 2)} \) and \( u_i \in V_2 \) iff \( i \equiv 1 \text{ (mod 2)} \).

**Lemma 2.2.** For a bipartite tournament \( D = (V_1, V_2) \), every closed directed walk of length at most 6 in \( D \) is a directed cycle of \( D \).

**Proof.** Let \( C \) be a closed directed walk with \( \ell (C) \leq 6 \). We will prove that \( C \) is a directed cycle. Since \( D \) is bipartite \( \ell (C) \) is even (as every closed odd directed walk contains an odd directed cycle); \( \ell (C) = 2 \) is impossible as a bipartite tournament is an asymmetrical digraph. Suppose \( \ell (C) = 4 \), and let \( C = (u_0, u_1, u_2, u_3, u_0) \) we may assume w.l.o.g. \( u_i \in V_1 \) for \( i \in \{0, 2\} \)

and \( u_i \in V_2 \) for \( j \in \{1, 3\} \) which implies \( u_i \neq u_j \) for \( i \in \{0, 2\}, j \in \{1, 3\} \). Since \( (u_1, u_2) \in A(D) \) and \( (u_2, u_3) \in A(D) \) we have \( u_1 \neq u_3 \) (as \( D \) is an asymmetrical digraph) and analogously \( u_0 \neq u_2 \); so \( C \) is a directed cycle. Finally suppose \( \ell (C) = 6 \) and let \( C = (u_0, u_1, u_2, u_3, u_4, u_5, u_0) \), clearly we may assume w.l.o.g. that \( u_i \in V_1 \) for \( i \in \{0, 2, 4\} \) and \( u_i \in V_2 \) for \( j \in \{1, 3, 5\} \) which implies \( u_i \neq u_j \) for \( i \in \{0, 2, 4\} \) and \( j \in \{1, 3, 5\} \).

Moreover, since \( \{(u_1, u_3), (u_3, u_5, u_2)\} \subseteq A(D) \) for \( i \in \{0, 1, \ldots, 5\} \) (notation (mod 6)) and \( D \) is asymmetrical, we have \( u_i \neq u_{i+2} \) for \( i \in \{0, 1, \ldots, 5\} \).

**Lemma 2.3.** Let \( D \) be an \( m \)-coloured bipartite tournament such that every directed cycle of length 4 is monochromatic and \( u, v \in V(D) \). If there exists a \( uv \)-monochromatic directed path and there is no \( vu \)-monochromatic directed path
(in $D$), then at least one of the two following conditions holds:

(i) $(u, v) \in A(D);
(ii) there exists (in $D$) a $w$-directed path of length 2.

**Proof.** Let $D$, $u, v \in V(D)$ be as in the hypothesis. We proceed by induction on the length of a $uv$-monochromatic directed path. Clearly Lemma 2.3 holds when there exists a $uv$-monochromatic directed path of length at most 2. Suppose that Lemma 2.3 holds when there exists a $uv$-monochromatic directed path of length $\ell$ with $2 \leq \ell \leq n$. Now assume that there exists a $uv$-monochromatic directed path $T = (u = u_0, u_1, \ldots, u_{n+1} = v)$ with $\ell(T) = n + 1$; we may assume w.l.o.g. $T$ is coloured 1.

**Claim 1.** If $(u_i, v) \in A(D)$ for some $i \in \{0, 1, \ldots, n-2\}$ then $(u, v) \in A(D)$ or there exists a $w$-directed path of length 2.

Assume $(u_i, v) \in A(D)$ for some $i \in \{0, 1, \ldots, n-2\}$ and let $i_0 = \min\{i \in \{0, 1, \ldots, n-2\} \mid (u_i, v) \in A(D)\}$. If $i_0 = 0$ then $(u, v) \in A(D)$ and if $i_0 = 1$ then $(u, u_1, v)$ is a $w$-directed path of length 2, so we can assume $i_0 \in \{2, \ldots, n-2\}$.

Since $i_0 \equiv i_0 - 2 \pmod{2}$ and $i_0 \not\equiv n + 1 \pmod{2}$ (as $(u_{i_0}, v) \in A(D)$) we have $i_0 - 2 \not\equiv n + 1 \pmod{2}$ and it follows from Lemma 2.1 that $(u_{i_0 - 2}, v) \in A(D)$ or $(v, u_{i_0 - 2}) \in A(D)$. Now the choice of $i_0$ implies $(v, u_{i_0 - 2}) \in A(D)$ and hence $C_4 = (u_{i_0 - 2}, u_{i_0 - 1}, u_{i_0}, u_{i_0 - 2})$ is a directed cycle of length 4 which by hypothesis is monochromatic, moreover, since $(u_{i_0 - 1}, u_{i_0})$ is coloured 1 (as it is an arc of $T$), it follows that $C_4$ is coloured 1. Then we obtain that $T' = (u, T, u_{i_0}) \cup (u_{i_0}, v)$ is a $w$-monochromatic directed path with $\ell(T') < n + 1$; and the inductive hypothesis implies that $(u, v) \in A(D)$ or there exists a $u$-directed path of length 2.

Now, it follows from Lemma 2.1 that for each $i \in \{0, 1, \ldots, n-2\}$ $(u_i, u_{i+3}) \in A(D)$ or $(u_{i+3}, u_i) \in A(D)$ (as $i \not\equiv i + 3 \pmod{2}$).

We will analyze two possible cases:

Case a: There exists $i \in \{0, 1, \ldots, n-2\}$ such that $(u_i, u_{i+3}) \in A(D)$. Let $j_0 = \max\{j \in \{i + 3, \ldots, n + 1\} \mid (u_j, u_i) \in A(D)\}$ (notice that Lemma 2.1 implies $i \not\equiv j_0 \pmod{2}$).

Case a.1: $j_0 = n + 1$.

Is this case the result follows from Claim 1.

Case a.2: $j_0 = n$ and $i = 0$.

We have $(u_0 = u_i, u_{j_0} = u_{i+3}, u_{i+1})$ is a $w$-directed path of length 2.

Case a.3: $j_0 = n$ and $i \geq 1$.

Since $i \not\equiv j_0 \pmod{2}$, we have $i - 1 \not\equiv j_0 + 1 = n + 1 \pmod{2}$ and it follows from Lemma 2.1 that $(u_{i-1}, u_{i+1}) \in A(D)$ or $(v, u_{i-1}) \in A(D)$, the affirmation of Lemma 2.3 follows from Claim 1. When $(v, u_{i-1}) \in A(D)$ we obtain $C_4 = (u_{i-1}, u_i, u_{j_0} = u_{i+3}, u_{i-1})$ a directed cycle of length 4 which by hypothesis is monochromatic; in fact $C_4$ is coloured 1 (as $(u_{i-1}, u_i) \in A(T) \cap A(C_4)$); and then $T' = (u, T, u_i) \cup (u_i, u_{j_0}) \cup (u_{j_0}, T, v)$ is a $w$-monochromatic directed path with $\ell(T') \leq n$. Now it follows from the inductive hypothesis that $(u, v) \in A(D)$ or there exists a $w$-directed path of length 2.

Case a.4: $j_0 \leq n - 1$.

If $i \not\equiv j_0 + 2 \pmod{2}$ (as $i \not\equiv j_0 \pmod{2}$), so it follows from Lemma 2.1 that $(u_i, u_{i+2}) \in A(D)$ or $(u_{i+2}, u_i) \in A(D)$; now the choice of $j_0$ implies $(u_{i+2}, u_i) \in A(D)$. Thus $C_4 = (u_i, u_{i+2}, u_{i+1}, u_{i+2}, u_i)$ is a directed cycle of length 4 which by hypothesis is monochromatic and coloured 1 (as $(u_{i+2}, u_{i+1}) \in A(T) \cap A(C_4)$); in particular $(u_i, u_{i+2})$ is coloured 1 and then $T' = (u, T, u_i) \cup (u_i, u_{i+2}) \cup (u_{i+2}, T, v)$ is a $w$-monochromatic directed path with $\ell(T') \leq n$ and the inductive hypothesis implies $(u, v) \in A(D)$ or there exists a $w$-directed path of length 2.

Case b: For each $i \in \{0, 1, \ldots, n-2\}$, $(u_{i+3}, u_i) \in A(D)$.

$C_4 = (u_i, u_{i+1}, u_{i+2}, u_{i+3}, u_i)$ is a directed cycle of length 4 and by a directed path it is monochromatic, moreover $C_4$ is coloured 1 because $(u_i, u_{i+1}) \in A(T) \cap A(C_4)$, hence for each $i \in \{0, 1, \ldots, n-2\}$, $(u_{i+3}, u_i)$ is coloured 1. Let $k \in \{1, 2, 3\}$ such that $k \equiv n + 1 \pmod{3}$, then $(v = u_{k+1}, u_{k+2}, u_{k+3}, \ldots, u_i) \cup (u_i, T, u_k) \cup (u_k, u_0)$ is a $w$-monochromatic directed path, contradicting the hypothesis, thus this case is impossible. □

**Theorem 2.1.** Let $D$ be an $m$-coloured bipartite tournament. If every directed cycle of length 4 in $D$ is monochromatic, then $\mathcal{C}(D)$ is kernel-perfect.

**Proof.** During the proof we will use the fact that each closed directed walk of length at most 6 is a directed cycle (Lemma 2.2) without any more explanation.

In view of Theorem 1.1 it suffices to prove (and we will prove) that each directed cycle of $\mathcal{C}(D)$ has a symmetrical arc.
We proceed by contradiction; suppose that there exists a directed cycle of \( C \in \mathcal{G}(D) \), \( C = (u_0, u_1, \ldots, u_n, u_0) \) with \( C \subseteq \text{Asym}(\mathcal{G}(D)) \).

**Claim 2.** For each \( i \in \{0, 1, \ldots, n\} \), \((u_i, u_{i+1}) \in A(D)\) or there exists a \( u_i u_{i+1} \)-directed path of length 2 (notation mod \( n+1 \)).

Let \( i \in \{0, 1, \ldots, n\} \). Since \((u_i, u_{i+1}) \in A(\mathcal{G}(D))\) we have that there exists a \( u_i u_{i+1} \)-monochromatic directed path in \( D \), and the fact that \( C \) has no symmetrical arcs implies there is no \( u_{i+1} u_i \)-monochromatic directed path in \( D \), so Claim 2 follows from Lemma 2.3.

Now we consider two possible cases:

**Case a:** \( n = 2 \).

Since \( D \) has no odd directed cycles, we have that for some \( i \in \{0, 1, 2\} \), \((u_0, u_1) \notin A(D)\) (notation \( (\text{mod} \ 3) \)). W.l.o.g we may assume \((u_0, u_1) \notin A(D)\), then it follows from Claim 2 that there exists a \( u_0 u_1 \)-directed path of length 2 in \( D \), say \((u_0, v_0, u_1)\).

**Case a.1:** \( \{(u_0, u_2), (u_2, u_0)\} \subseteq A(D) \).

Since \((u_0, u_1) \notin A(D)\), \((u_0, v_0, u_1, u_2, v_2, u_0)\) is a directed cycle of length 4 in \( D \), which implies \((u_2, u_0)\) is a \( u_0 u_2 \)-monochromatic directed path in \( D \); thus \((u_0, u_1)\) is a symmetrical arc of \( C \) in \( \mathcal{G}(D) \), contradicting our assumption.

**Case a.2:** \( \{(u_1, u_2), (u_2, u_0)\} \notin A(D) \).

Now it follows from Claim 2 that there exists a \( u_1 u_2 \)-directed path of length 2 in \( D \), say \((u_1, v_1, u_2)\), and a \( u_2 u_0 \)-directed path of length 2 in \( D \), say \((u_2, v_2, u_0)\). Thus \((u_0, v_0, u_1, v_1, u_2, v_2, u_0)\) is a directed cycle of length 6 in \( D \), and it follows from Lemma 2.1 that \((u_0, v_1) \in A(D)\) or \((v_1, u_0) \in A(D)\). When \((u_0, v_1) \in A(D)\) we obtain \((u_0, v_1, u_2, v_2, u_0)\) is a directed cycle of length 4 in \( D \) and by hypothesis it is monochromatic, in particular \((u_0, v_1, u_2)\) is a \( u_0 u_2 \)-monochromatic directed path in \( D \) which implies \((u_2, u_0)\) is a symmetrical arc of \( C \) in \( \mathcal{G}(D) \), contradicting our assumption. When \((v_1, u_0) \in A(D)\) we have \((u_0, v_0, u_1, v_1, u_0)\) is a directed cycle of length 4 in \( D \) and by hypothesis is monochromatic, thus \((u_1, v_1, u_0)\) is a \( u_1 u_0 \)-monochromatic directed path in \( D \) and then \((u_0, u_1)\) is a symmetrical arc of \( C \) in \( \mathcal{G}(D) \), a contradiction.

**Case b:** \( n \geq 3 \).

In what follows the notation is taken modulo \( n+1 \).

In view of Claim 2, for each \( i \in \{0, 1, \ldots, n\} \) we can take a \( u_i u_{i+1} \)-directed path as follows:

\[
T_i = \{(u_i, u_{i+1}) \text{ when } (u_i, u_{i+1}) \in A(D)\} \\
\cup \{(u_i, u_{i+1}) \text{ when } (u_i, u_{i+1}) \notin A(D)\}
\]

Let \( C' = \bigcup_{i=1}^{n} T_i \). Then \( C' \) is a closed directed walk in \( D \), so we may let \( C' = (z_0, z_1, \ldots, z_{k}, z_0) \) and define the function \( \phi : \{0, 1, \ldots, k\} \to V(C) \) as follows: For each \( i \in \{0, 1, \ldots, n\} \) if \( T_i = (u_i = z_{i+1} = u_{i+1}) \) then \( \phi(i) = z_i = u_i \); and if \( T_i = (u_i = z_{i+1}, z_{i+2} = u_{i+2}) \) then \( \phi(i) = \phi(i+1) = z_{i+1} \).

We will say that an index \( j \in \{0, 1, \ldots, k\} \) is a principal index when \( \phi(j) = z_j \); and we will denote by \( I_p \) the set of principal indices. Notice that in \( C' \) the indexes are all different and also notice that a vertex \( u_j \) may correspond to a principal index \( \ell \) and also to a non principal index \( p \).

Suppose w.l.o.g. that \( u_0 = z_0 \). Since \( D \) is a bipartite tournament, we have \( k \equiv 1 \pmod{2} \) and by Lemma 2.1, for each \( i \in \{1, \ldots, \frac{k-1}{2}\} \), \((z_0, z_{2i+1}) \in A(D)\) or \((z_{2i+1}, z_0) \in A(D)\). We consider the following cases:

**Case b.1:** \( (z_0, z_0) \in A(D) \).

In this case we have \((z_0, z_1, z_2, z_3, z_0)\) is a directed cycle of length 4 and by hypothesis is monochromatic. The definition of \( C' \) implies \( z_1 = u_1 \) or \( z_2 = u_1 \). If \( z_1 = u_1 \) then \((u_1 = z_1, z_2, z_3, z_0 = u_0)\) is a \( u_0 u_1 \)-monochromatic directed path in \( D \) which implies \((u_0, u_1)\) is a symmetrical arc of \( C \) in \( \mathcal{G}(D) \), contradicting our assumption on \( C \). So \( z_1 \neq u_1 \), consequently \( z_2 = u_1 \) and then \((u_1 = z_2, z_3, z_0 = u_0)\) is a \( u_1 u_0 \)-monochromatic directed path in \( D \), thus \((u_0, u_1)\) is a symmetrical arc of \( C \) in \( \mathcal{G}(D) \), a contradiction.

**Case b.2:** \( (z_0, z_{k-2}) \in A(D) \).

The assumption in subcase b.2 implies \((z_0, z_{k-2}, z_{k-1}, z_k, z_0)\) is a directed cycle of length 4 which by hypothesis is monochromatic. The construction of \( C' \) implies \( z_k = u_0 \) or \( z_{k-1} = u_0 \). When \( z_k = u_0 \) we have that \((u_0, z_0, z_{k-2}, z_{k-1}, z_k, u_0)\)
is a \(u_0\text{to}u_0\)-monochromatic directed path in \(D\) which implies that \((u_0, u_0)\) is a symmetrical arc of \(C\) in \(\mathcal{E}(D)\), contradicting our assumption. Hence \(z_2 \neq u_0\) and then \(z_{k-2} = u_0\); now \((u_0 = z_0, z_{k-2}, z_{k-1} = u_0)\) is a \(u_0\text{to}u_0\)-monochromatic directed path in \(D\) which implies that \((u_0, u_0)\) is a symmetrical arc of \(C\) in \(\mathcal{E}(D)\), a contradiction.

**Case b.3:** \((z_0, z_1) \in \mathcal{A}(D)\) and \((z_{k-2}, z_{k-1}) \in \mathcal{A}(D)\).

Since \(\{(z_0, z_1), (z_0, z_3), (z_{k-2}, z_{k-1})\} \subseteq \mathcal{A}(D)\) we have \(k-2 \geq 5\) and there exists \(j \in \{1, \ldots, \frac{k-5}{2}\}\) such that \((z_0, z_{j+1}) \in \mathcal{A}(D)\) and \((z_{j+3}, z_0) \in \mathcal{A}(D)\). Let \(i_0 = \min\{j \in \{1, \ldots, \frac{k-5}{2}\} \mid ((z_0, z_{j+1}), (z_{j+3}, z_0)) \subseteq \mathcal{A}(D)\}\). Hence \(\tilde{C} = (z_0, z_{j+1}, z_{j+3}, z_0)\) is a directed cycle of length 4 in \(D\) which by hypothesis is monochromatic. Now we consider two possible cases.

**Case b.3.1:** \(2i_0 + 1 \in I_p\).

In this case \(z_{2i_0+1} = u_j\) for some \(j \in \{2, \ldots, n - 2\}\) (as \(3 \leq 2i_0 + 1 \leq k - 4\)). By the construction of \(C'\) we have \(z_{2i_0+2} = u_{j+1}\) or \(z_{2i_0+3} = u_{j+1}\). If \(z_{2i_0+2} = u_{j+1}\) then \((u_{j+1}, z_{2i_0+2}, z_{2i_0+3}, z_0, z_{2i_0+1} = u_j)\) is a \(u_{j+1}\text{to}u_{j+1}\)-monochromatic directed path in \(D\) which implies that \((u_{j+1}, u_{j+1})\) is a symmetrical arc of \(C\) in \(\mathcal{E}(D)\) contradicting our assumption. Hence \(z_{2i_0+2} \neq u_{j+1}\) and consequently \(z_{2i_0+3} = u_{j+1}\) thus \((u_{j+1}, z_{2i_0+3}, z_0, z_{2i_0+1} = u_j)\) is a \(u_{j+1}\text{to}u_{j+1}\)-monochromatic directed path in \(D\) and then \((u_j, u_{j+1})\) is a symmetrical arc of \(C\) in \(\mathcal{E}(D)\), a contradiction.

**Case b.3.2:** \(2i_0 + 1 \notin I_p\).

Now, by construction of \(C'\) we have that \(\{2i_0, 2i_0 + 2\} \subseteq I_p\), i.e. \(z_{2i_0} = u_{j-1}\) and \(z_{2i_0+2} = u_j\) for some \(j \in \{2, \ldots, n-1\}\). Lemma 2.1 implies \((z_{2i_0}, z_{2i_0+1}) \in \mathcal{A}(D)\) or \((z_{2i_0+2}, z_{2i_0+3}) \in \mathcal{A}(D)\). When \((z_{2i_0+2}, z_{2i_0+3}) \in \mathcal{A}(D)\) we obtain that \((z_{2i_0}, z_{2i_0+1}, z_{2i_0+2}, z_{2i_0+3}, z_{2i_0})\) is a directed cycle of length 4 and by hypothesis is monochromatic; thus \((u_j = z_{2i_0+2}, z_{2i_0+3}, z_{2i_0} = u_{j-1})\) is a \(u_{j-1}\text{to}u_{j-1}\)-monochromatic directed path and \((u_{j-1}, u_j)\) is a symmetrical arc of \(C\) in \(\mathcal{E}(D)\), a contradiction. So we have \((z_{2i_0}, z_{2i_0+1}) \in \mathcal{A}(D)\); observe that the choice of \(i_0\) implies \((z_0, z_{2i_0-1}) \in \mathcal{A}(D)\) (when \((z_{2i_0-1}, z_0) \in \mathcal{A}(D)\), the fact \((z_0, z_1) \in \mathcal{A}(D)\) implies that there exists \(j \leq i_0 - 2\) such that \((z_0, z_{j+1}) \in \mathcal{A}(D)\) and \((z_{j+3}, z_0) \in \mathcal{A}(D)\) contradicting the choice of \(i_0\)), thus \(C' = (z_0, z_{2i_0-1}, z_0, z_{2i_0+1}, z_0)\) is a directed cycle of length 4 which by hypothesis must be monochromatic; since \((z_{2i_0+3}, z_0) \in \mathcal{A}(\tilde{C} \cap C')\) we have that \(\tilde{C}\) and \(C'\) are of the same colour; so \((u_j = z_{2i_0+2}, z_{2i_0+3}, z_0, z_{2i_0-1}, z_0 = u_{j-1})\) is a monochromatic directed path in \(D\) and \((u_{j-1}, u_j)\) is a symmetrical arc of \(C\) in \(\mathcal{E}(D)\), a contradiction.

The following result is a direct consequence of Theorem 2.1:

**Theorem 2.2.** Let \(D\) be an \(m\)-coloured bipartite tournament. If every directed cycle of length 4 in \(D\) is monochromatic, then \(D\) has a kernel by monochromatic paths.

**Remark 2.1.** The hypothesis that every directed cycle of length 4 is monochromatic in Theorem 2.2 is tight.

Let \(D\) be the 3-coloured bipartite tournament defined as follows:

\[V(D) = \{u, v, w, x, y, z\}\] and \(A(D) = \{(u, x), (x, v), (t, y), (y, w), (w, z), (z, u), (x, w), (y, u), (z, v)\}\); the arcs \((x, w), (w, z)\) and \((z, u)\) are coloured 1; the arcs \((y, u), (u, x)\) and \((x, v)\) are coloured 2; and the arcs \((z, v), (v, y)\) and \((y, w)\) are coloured 3. The only directed cycles of length 4 in \(D\) are \((u, x, w, z, u), (v, y, u, x, v)\) and \((w, z, v, y, w)\) which are quasi-monochromatic and the digraph \(\mathcal{E}(D)\) is a complete digraph which has no kernel; hence \(D\) has no kernel by monochromatic paths. Moreover, we can construct an infinite family of digraphs all of whose directed cycles of length 4 are quasi-monochromatic and which have no kernel by monochromatic paths as follows: Let \(D_n\) be the digraph obtained from \(D\) by adding vertices \(x_1, x_2, \ldots, x_n\) and arcs coloured from 3 each one of these vertices to \(u, v\) and \(w\), respectively.

**Remark 2.2.** The assumption that every directed cycle of length 4 in a bipartite tournament \(D\) is monochromatic, does not imply that every directed cycle of length 6 in \(D\) is monochromatic.

**Remark 2.3.** For each \(m\) there exists an \(m\)-coloured Hamiltonian bipartite tournament such that every directed cycle of length 4 is monochromatic.

**Proof.** Let \(D\) be the \(m\)-coloured digraph defined as follows:

\[V(D) = X \cup Y \cup Z \cup W\] where: \(X = \{x_1, x_2, \ldots, x_m\}\), \(Y = \{y_1, y_2, \ldots, y_m\}\)

\[Z = \{z_1, z_2, \ldots, z_m\},\] \(W = \{w_1, w_2, \ldots, w_m\}\).

\[A(D) = X_Y \cup Y_Z \cup Z_W \cup W_X \cup X_W \cup W_Y\] where:

\[X_Y = \{(x_i, y_i) \mid i \in \{1, 2, \ldots, m\}\},\] \(Y_Z = \{(y_i, z_i) \mid i \in \{1, 2, \ldots, m\}\},\) \(Z_W = \{(z_i, w_i) \mid i \in \{1, 2, \ldots, m\}\},\)

\[W_X = \{(w_i, x_{i+1}) \mid i \in \{1, 2, \ldots, m - 1\}\} \cup \{(w_m, x_1)\}\].
$Z = \{(z_i, y_j) | i \in \{1, 2, \ldots, m\}, j \in \{1, 2, \ldots, m\}, i \neq j\}$,

$W = \{(w_i, z_j) | i \in \{1, 2, \ldots, m\}, j \in \{1, 2, \ldots, m\}, i \neq j\}$,

$X = \{(x_i, w_j) | i \in \{1, 2, \ldots, m\}, j \in \{1, 2, \ldots, m\}, i \neq j + 1\}$,

(notation mod $m$).

For each $i \in \{1, 2, \ldots, m\}$ the arc $(x_i, y_i)$ is colored $i$ and any other arc is coloured 1.

Clearly $D$ is an $m$-coloured bipartite tournament.

Claim 3. $D$ is Hamiltonian. It follows from the definition of $D$ that for each $i \in \{1, 2, \ldots, m\}$ we have the directed path

$T_i = (x_i, y_i, z_i, w_i, x_{i+1})$ and clearly $V(T_i) \cap V(T_j) = \emptyset$ for $j \neq i + 1$, and $V(T_i) \cap V(T_{i+1}) = \{x_{i+1}\}$. So $C = \bigcup_{i=1}^{m} T_i$ is a Hamiltonian directed cycle of $D$.

Claim 4. Every directed cycle of length 4 of $D$ is monochromatic. Proceeding by contradiction, suppose that $C_4 = (u_1, u_2, u_3, u_4, u_1)$ is a non monochromatic directed cycle of $D$, so $C_4$ must contain at least one arc coloured $i$ for some $i \in \{2, \ldots, m\}$, so we may assume that $u_1 = x_2$ and $u_2 = y_2$; it follows from the definition of $D$ that $u_3 = z_2$ and $(u_4 = w_2$ or $u_4 = y_i$ for some $i \neq 2$). When $u_4 = w_2$, we obtain that $(x_2, w_2) \in A(D)$ and hence $(w_2, x_2) \notin A(D)$, a contradiction. When $u_4 = y_i$ for some $i \neq 2$ we obtain that $(x_2, y_i) \in A(D)$ contradicting that $(u_4 = y_i, u_1 = x_2) \in A(D)$.

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References