Rational function systems
with applications in signal processing

Theses of the PhD Dissertation

Levente Lócsi

Supervisor: Ferenc SCHIPP, professor emeritus, DSc

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Introduction

The topic of our dissertation is the study of complex rational function systems and their application in the field of digital signal processing, with emphasis on the analysis of ECG signals. I have carried out the research work forming the basis of this PhD dissertation as a doctoral student of the ELTE PhD School of Informatics starting from 2008, and as an assistant lecturer at the Department of Numerical Analysis of ELTE Faculty of Informatics with my supervisor being Professor Ferenc Schipp.

By complex rational functions we mean complex valued functions of a complex variable expressible in the form of a simple fraction. For signal processing purposes we will consider their values on the complex unit circle, and compare them with the signal to process. From a mathematical point of view, it is interesting to study and develop orthogonal systems and FFT-like algorithms related to these rational systems.

The ECG signals or ElectroCardioGrams register the electric changes arising from the functioning of the human heart. Their medical analysis is a widespread tool for doctors to diagnose many types of illnesses and conditions. Nowadays besides or even instead of the regular paper-based solution the ECG recordings are frequently stored and transmitted in digital form. The efficient and reliable processing of these signals is a common and very complex task involving compression, noise filtering, segmentation and analysis.

The study of the application potential of complex rational functions and function systems in the case of ECG curves is motivated by the mathematical results related to rational approximation in the last decades, their more and more intense application in signal processing, and also the fact that the plot of simple rational functions is already quite similar to the typical parts of the ECG. With the proper choice of a few parameters we may obtain a function approximating an ECG segment remarkably well.

Naturally we need some numerical optimization techniques to determine the best choice for the parameters at hand. Finding appropriate algorithms, the study and improvement of these is another direction of our research. The functions at hand are defined analytically, in the continuous case, but their application in digital signal processing requires discretization. The related special properties and potentials are also subject to our investigations.
Mathematical background

Let \( \mathbb{C} \) denote the set of complex numbers, \( \mathbb{D} := \{ z \in \mathbb{C} : |z| < 1 \} \) the disk, i.e. the interior of the complex unit circle, \( \mathbb{T} := \{ z \in \mathbb{C} : |z| = 1 \} \) the torus, i.e. the complex unit circle, and \( \mathbb{D}^* := \mathbb{C} \setminus (\mathbb{D} \cup \mathbb{T}) \). Natural numbers shall be considered as the set \( \mathbb{N} := \{ 1, 2, 3, \ldots \} \). Let us denote by \( \mathcal{A} \) the class of functions analytic on \( \mathbb{D} \) and continuous on \( \mathbb{D} \cup \mathbb{T} \), i.e. the disk algebra. Furthermore, let us introduce the scalar product

\[
\langle \cdot, \cdot \rangle : \mathcal{A} \times \mathcal{A} \to \mathbb{C}, \quad \langle f, g \rangle = \frac{1}{2\pi} \int_{0}^{2\pi} f(e^{it}) \cdot \overline{g}(e^{it}) \, dt = \int_{\mathbb{T}} f(z) \cdot \overline{g}(z) \, dz,
\]

such that the product integral on \( \mathbb{T} \) shall be calculated.

We shall start with the following elementary rational functions

\[
\varphi_n(z) := \frac{1}{(1 - \alpha_n z)^{m_n}} \quad (z \in \mathbb{C} \setminus \{1/\alpha_n\}; \ m \in \mathbb{N}; \ n = 1, \ldots, m),
\]

with parameters \( \alpha_n \in \mathbb{D} \) \((n = 1, \ldots, m)\) and \( m_n = \sum_{i\leq n} \chi(a_i = \alpha_n) \) the multiplicity of the parameter \( \alpha_n \). One may observe that \( \varphi_n \) has a pole of order \( m_n \) at \( 1/\alpha_n \in \mathbb{D}^* \), and furthermore \( \Phi := (\varphi_n : n = 1, \ldots, m) \subset \mathcal{A} \). E.g. with the parameters \( a, a, b, b \in \mathbb{D} \) we will gain the following functions:

\[
\varphi_1(z) = \frac{1}{1 - az}, \quad \varphi_2(z) = \frac{1}{(1 - az)^2}, \quad \varphi_3(z) = \frac{1}{1 - bz}, \quad \varphi_4(z) = \frac{1}{(1 - bz)^2}.
\]

For the purpose of signal processing these functions shall be examined on the unit circle, so in case of a function \( f \), the values \( f(e^{it}) \ (t \in [0, 2\pi]) \), their real and imaginary parts will be considered. It turns out that with the appropriate choice of 3–4 parameters and the joint coefficients we may arrive at a very good approximation of ECG segments. However, the question is, how to find the appropriate parameters, and how to determine the unknown coefficients?

It would be easy to calculate the coefficients, if we started with an orthogonal or orthonormal system. So let us apply the Gram–Schmidt orthogonalization procedure to the linearly independent system \( \Phi \). We shall arrive at an orthonormal system with respect to the above scalar product, a so-called Malmquist–Takenaka system denoted by \( \Psi := (\psi_n : n = 1, \ldots, m) \). The
elements of this system can be expressed using the Blaschke-functions

\[ B_a(z) := \frac{z - a}{1 - a z} \quad (a \in \mathbb{D}; \; z \in \mathbb{C} \setminus \{1/\overline{a}\}), \]

namely

\[ \psi_n(z) = \frac{\sqrt{1 - |a_n|^2}}{1 - \overline{a_n} z} \prod_{k=1}^{n-1} B_{a_k}(z) \quad (z \in \mathbb{C}; \; n = 1, \ldots, m). \]

This form suggests a more convenient approach for the calculation of the values of \( \psi_n \) instead of the actual realization of the orthogonalization procedure. We can assert that the \( m \)-dimensional subspaces spanned by \( \Phi \) and \( \Psi \) coincide, i.e. \( \text{span } \Phi = \text{span } \Psi \).

By the orthonormality of the Malmquist–Takenaka system, we can calculate the orthogonal projection \( P_\Psi f = P_{a_1, \ldots, a_m} f \) of some function \( f \in \mathcal{A} \) on the subspace \( \text{span } \Psi \) according to the following formula:

\[ P_\Psi f = \sum_{n=1}^{m} \langle f, \psi_n \rangle \psi_n, \]

so the unknown coefficients are given by the scalar products \( \langle f, \psi_n \rangle \).

It is another task to find the appropriate parameters. This problem can be formulated as follows. Denote by \( \mathcal{E}_\Psi f = \mathcal{E}_{a_1, \ldots, a_m} f \) the best approximation of function \( f \) in \( \| \cdot \|_2 \) on the subspace \( \text{span } \Psi \), i.e.

\[ \mathcal{E}_\Psi f := \| f - P_\Psi f \|_2 = \sqrt{\langle f - P_\Psi f, f - P_\Psi f \rangle}. \]

Our goal is to minimize the value \( \mathcal{E}_\Psi f \) in case of a function \( f \in \mathcal{A} \) (given by its values on \( \mathbb{T} \)) — we can consider an ECG segments as such — and fixed dimension \( m \in \mathbb{N} \) by choosing the parameters \( a_1, a_2, \ldots, a_m \) of the system well. The Nelder–Mead simplex algorithm can be applied to realize the optimization of the parameters.

Naturally in applied calculations the functions and the scalar product should be all handled as discretized along uniform sampling points of \( \mathbb{T} \). E.g. an ECG segment usually consists of 200-300 sampled data points.
Theses of the Dissertation

Below we will summarize the results of our research work in the form of theses. After the statement of our theses we will shortly summarize our relevant results, referring to the appropriate chapters of the dissertation with further details, and also listing our related publications.

**Thesis 1: Rational approximation of ECG curves**

_We have elaborated the method of processing ECG signals based on complex rational function systems. We have successfully applied the Nelder–Mead simplex algorithm for the approximation and identification of the parameters of the systems. This method enables the concise and elegant representation of ECG segments, furthermore with its application we also gain noise filtering and efficient, low-loss compression._

This thesis summarizes the central goal and result of our research work. The plots of the function systems at hand already seemed quite similar to the different parts of ECG curves, and it turned out that by choosing their parameters well we can obtain great approximation, furthermore the parameters may even carry direct diagnostic information. During our research we have encountered the Nelder–Mead method, which was the first one to give suitable approximation results for the optimal values of the parameters at hand without using any a priori information on their location.

We have discussed the applied rational function system, among them the orthogonal Malmquist–Takenaka systems and the Blaschke functions in detail in Chapter 2. (Blaschke functions also play an important role in several further constructions.) The Nelder–Mead method is introduced in Chapter 5. And finally in Chapter 6 we have presented how to process actual ECG signals using these methods.

In our first related work we have examined the application of our approach in the case of artificially generated signals [1]. We have presented the proposed method on several international conferences and journal papers [4, 8, 11]. The
related MATLAB tools have been also published [9, 10]. On Figure 1 we show an example of one segment of a real ECG recording and its approximation acquired using rational functions.

**Thesis 2: Hyperbolic Nelder–Mead-algorithm**

*We have developed the variants of the Nelder–Mead simplex method adapted to the hyperbolic plane and space, using the Poincaré disk model and its three-dimensional analogue. We have translated several known properties of the original method to the new hyperbolic versions. Also the implementation of the algorithm is in our hands.*

While the original algorithm works in the $n$-dimensional Euclidean space, our problem is bound to the complex unit disk. Apart from the bijective mapping of these sets on one another, also the adaptation of the Nelder–Mead method to the Poincaré disk model of the Bolyai–Lobachevsky hyperbolic geometry seemed to be a promising path. After realizing the constructions of hyperbolic geometry, the algorithm itself has also been implemented, its hyperbolic version in 2 and 3 dimensions has been developed. The original algorithm is very widespread applied in practice, but there are very few properties known about its convergence. We have proven the validity of several published statements also in the hyperbolic case.
We have devoted the entire Chapter 5 to the presentation of the hyperbolic Nelder–Mead algorithm. We review the original algorithm, give a summary about the realization of the hyperbolic constructions, we introduce the new hyperbolic variant of the algorithm itself, and prove some related propositions.

This work has appeared independently in [13], and it is also mentioned in works [4, 8]. Figure 2 illustrates some basic elements in the Poincaré disk model of hyperbolic geometry, and also the progress of the simplex method on the unit disk. In this case the simplex is a hyperbolic triangle.

**Thesis 3: Discretizational properties**

We have examined several properties related to the discretization of the rational systems at hand. We have given an algorithm for the efficient sequential calculation of the non-equidistant discretizations related to these systems. We have implemented FFT-like constructions. We have introduced novel orthogonal systems as slices of the Dirichlet-kernel.

The discussed systems are defined in the continuous case, but their discretization is important both from the theoretical and practical points of view. It turns out, that while uniform sampling points should be used in the case of...
the traditional trigonometric system, in the case of these more general systems one has to consider non-equidistant discretizations.

We examine the notions and results related to the discretization properties in Chapter 3. In this topic rational product systems and the argument functions (of Blaschke functions) also play an important role.

In paper [2] we have given an introduction to this topic, with an example application in the case of ECG signals. Our method for the efficient sequential numerical calculation of the points of the non-equidistant discretization has been published in [5]. The mentioned novel orthogonal systems are described in [6]. The implementation of the FFT-like constructions is documented in [7].

**Thesis 4: Solution of an inverse problem**

We have specified and proved, in which cases can a prescribed set of four points arise as the zeros of the composition of two two-factor Blaschke products. In an admissible setting we have proved the existence of infinitely many solutions, and given their construction. We have introduced reciprocal Blaschke functions, and proved their existence and uniqueness.

Compositions of Blaschke products enable the construction of product systems and FFT algorithms. If we know the parameters of some Blaschke products, then the zeros of their composition — another Blaschke product — can be calculated by solving polynomial equations. The question arises in the inverse setting: when we are given zeros of a composition, can we find the parameters for the original functions?

We have solved this inverse problem in the most simple non-trivial case, namely considering the composition of two two-factor Blaschke products. The details of the solution are discussed in Chapter 4 based on the paper [12]. A Blaschke function, and an example for the composition in question can be observed on Figure 3. The lines of different shades mark the absolute values and angles of the complex function values inside the unit circle.
Figure 3: Left: A Blaschke function. Right: The composition of two two-factor Blaschke products. The zeros are also marked on both images.

**Thesis 5: Toolboxes for the MATLAB system**

_During our research work we have implemented many programs using the Matlab numerical mathematical software environment. We have published the essence of the developed programs in the form of toolboxes, which are free to download. Our tools enable the more thorough understanding, teaching, investigation and high quality visualization of the named topics._

Generally we have used the MATLAB system during our research work both to aid the treatment of theoretical problems and to tackle practical signal processing tasks. We have collected the essential parts of our programs for each topic, organized them into so-called toolboxes and made them available for open access. All along we have taken special care of the beautiful visualization of the constructions investigated, and to make even difficult notions easy to handle.

Such toolboxes include the following.

- **RAIT: Rational Approximation and Interpolation Toolbox.** This package contains the implementation of the applied rational systems and the algorithms for optimization among others. Our relevant publications are [9, 10]. The toolbox can be downloaded from the following page.
• **FFTRatSys.** The programs related to rational product systems and FFT-like algorithms are collected here, these are documented in [7]. We emphasize that this toolbox highly utilizes the object-oriented approaches now available at language level in the newer MATLAB releases. It is available at

http://numanal.inf.elte.hu/~locsi/fftratsys/

• **HypNM.** Puts the programming and visualization of different constructions on the Poincaré disk model of hyperbolic geometry and its three-dimensional analogue at your fingertips, and also provides the hyperbolic variant of the Nelder–Mead algorithm. It is published in [13].

http://numanal.inf.elte.hu/~locsi/hypnm/

We have given examples for the use of many of our MATLAB programs through the solution of some simple tasks in Appendix B of the dissertation. Appendix A contains some calculations and proofs in order to support the mathematical statements mentioned in the main text.

**References**


(In Hungarian)


