DUALITY AND NORMAL PARTS OF OPERATOR MODULES

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Abstract. For an operator bimodule X over von Neumann algebras $A \subseteq \mathrm{B}(\mathcal{H})$ and $B \subseteq \mathrm{B}(\mathcal{K})$, the space of all completely bounded A, B-bimodule maps from X into $\mathrm{B}(\mathcal{K}, \mathcal{H})$, is the bimodule dual of X. Basic duality is developed. To X a normal bimodule X_{n} is associated such that each completely bounded A, B-bimodule map from X into a normal bimodule Y factorizes through X_{n}. X_{n} is described in terms of universal representations. Three differend maximal operator bimodule norm structures on a representable bimodule are considered. (A Banach bimodule is representable if it can be represented isometrically in $\mathrm{B}(\mathcal{H})$.) The normal part of the projective tensor product of central such bimodules (shown to be representable) is studied. The injective tensor product of normal representable bimodules is shown to be normal. The duality for the Haagerup tensor product is extended to operator bimodules.

1. Introduction

General duals of operator bimodules were considered in [36], [2] and [40]. The present paper shows that the classical theory effectively extends to the situation where a Banach space is replaced by a normal operator bimodule X over von Neumann algebras A and B. The role of the dual is played by the A', B'-bimodule $X^{\natural}$ consisting of all completely bounded A, B-bimodule maps from X into $\mathrm{B}(\mathcal{K}, \mathcal{H})$, where $\mathcal{H}$ and $\mathcal{K}$ are proper Hilbert modules over A and B, respectively. We build on fundamentals of operator spaces [20], [37], [39]; in particular operator versions of the Hahn - Banach ([3], [45]) and the bipolar theorem [21] have been a strong motivation for this work. In order to get a 'genuine' duality, we consider mainly bimodules over von Neumann algebras. This can be used as a tool to study bimodules X over general C*-algebras A, B since the operator space bidual $X^{\sharp\sharp}$ of X [4] is a normal operator bimodule over the von Neumann envelopes of A and B (Section 4).

In Section 2 we collect definitions of various (known) classes of bimodules, introduce abbreviations for their names and summarize some preliminary results.

In Section 3 we develop basic duality for normal operator bimodules by first noting the following: (i) $X^{\natural} = (\mathcal{H}^* \overset{h}{\otimes}_A X \overset{h}{\otimes}_B \mathcal{K})^{\sharp}$. In a C*-context a similar result was observed by Na in [36], but we deduce it as a corollary of a more general duality for the Haagerup tensor product of bimodules, extending the duality by Blecher and Smith [9]. (ii) The space $X^{A\sharp B}$ of all functionals $\rho \in X^{\sharp}$ such that for each $x \in X$ the maps $A \ni a \mapsto \rho(ax)$ and $B \ni b \mapsto \rho(xb)$ are normal can be identified with $\mathcal{H}^* \overset{h}{\otimes}_{A'} X^{\natural} \overset{h}{\otimes}_{B'} \mathcal{K}$. Reflexive normal bimodules X are characterized similarly as reflexive Banach spaces, by compactness of the unit ball of X in the topology induced by $X^{A\sharp B}$.

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For a general operator A, B-bimodule X we show that the closure $X_{\rm n}$ of the image of X in $X^{\natural\natural}$ is a normal A, B-bimodule having the following property: for each completely bounded A, B-bimodule map ϕ from X to a normal operator A, B-bimodule Y there exists a unique A, B-bimodule map $\tilde{\phi}$ from $X_{\rm n}$ into Y such that $\phi = \tilde{\phi}\iota$, where ι is the canonical map from X into $X_{\rm n}$. The bimodule $X_{\rm n}$ is described in Section 4 in a way that is somehow analogous to the description of the normal part of a linear functional [26, Section 10.1]; so $X_{\rm n}$ is called the normal part of X, although in general it is not contained in X.

The basic proofs are usually completely different from the classical ones since the range $\mathrm{B}(\mathcal{K}, \mathcal{H})$ of the 'functionals' here is computationally very different from $\mathbb{C}$. The essential role is played by the extended Haagerup tensor product and technically appropriate are strong bimodules. We shall recall the definition in Section 2, here we just note that in the special case $A = B = \mathrm{B}(\mathcal{H}) = \mathrm{M}_{\mathbb{I}}(\mathbb{C})$ strong are just bimodules of the form $\mathrm{M}_{\mathbb{I}}(V)$, where V is an operator space, a category equivalent to operator spaces. A distinction between the two categories, is that the quotient of strong bimodules is (in general) not normal. However, in Section 5 we prove that for a strong central bimodule X over an Abelian von Neumann algebra C and a subbimodule Y in X the quotient X/Y is normal if and only if Y is strong. (X is central if $xc = cx$ for $c \in C$, $x \in X$.) We describe the normal part of a central representable bimodule X and show that X is normal precisely when for each $x \in X$ the function $\Delta \ni t \mapsto \|x(t)\|$ on the spectrum Δ of C is continuous, where $x(t)$ is the coset of x in $X/[(\ker t)X]$.

Section 6 shows that there are different kinds of maximal operator bimodule norm structures (analogous to Paulsen's maximal operator space [38]) on a given normal representable A, B-bimodule X: the maximal operator A, B-bimodule is in general different from the maximal *normal* operator A, B-bimodule, but the latter is the normal part of the former. If X is a dual bimodule $V^\sharp$, we have in addition the maximal *dual normal* operator bimodule. The fact that on $V^\sharp$ the maximal normal operator and the maximal normal dual operator bimodule norm structures are different provides new examples of operator spaces $V^\sharp$ for which there is no operator space predual structure on V. (The first example of such a space was found by Le Merdy [28].) The duality relations, analogous to the Blecher duality [4] between minimal and maximal operator spaces, are also examined in Section 6.

Another natural example of normal part is presented in Section 7, where we study the projective tensor product $X \overset{\gamma}{\otimes}_C Y$ of central representable bimodules over an Abelian C^*-algebra C. ($X \overset{\gamma}{\otimes}_C Y$ is the quotient of the maximal Banach space tensor product $X \overset{\gamma}{\otimes} Y$ by $\overline{\mathrm{span}}\{xc \otimes y - x \otimes cy : \ x \in X, \ y \in Y, \ c \in C\}$.) We show that $X \overset{\gamma}{\otimes}_C Y$ is representable. However, if C is a von Neumann algebra and X and Y are normal, then $X \overset{\gamma}{\otimes}_C Y$ is not necessarily normal, but the strong completion of its normal part is a projective tensor product in the class of strong central representable C-bimodules.

For normal representable bimodules X (over A, B) and Y (over B, C) the injective norm of an element $w = \sum_{j=1}^n x_j \otimes_B y_j \in X \otimes_B Y$, as defined by Anantharaman and Pop in [2], is the supremum of $\| \sum_{j=1}^n \phi(x_j)\psi(y_j)\|$ over all contractive bimodule homomorphisms $\phi : X \to \mathrm{B}(\mathcal{K}, \mathcal{H})$ and $\psi : Y \to \mathrm{B}(l, \mathcal{K})$, where $\mathcal{H}$, $\mathcal{K}$ and l are cyclic Hilbert modules over A, B and C (resp.). Section 8 proves that the same norm is obtained if we restrict in this definition $\mathcal{H}$, $\mathcal{K}$ and l to be normal. (This is in contrast with the projective tensor product in the previous paragraph.) We also observe that the norm is independent of A and C.

Finally, in Section 9 the bimodule version of the normal Haagerup tensor product is introduced, extending the operator space version of Effros and Ruan [19], and some natural canonical isomorphisms of bimodules (extending some results of [14], [9]) are studied.

2. Basic classes of bimodules, notation and other preliminaries

Throughout A, B and C will be C*-algebras with unit 1, in fact von Neumann algebras most of the time. By a Banach A, B-bimodule we mean a Banach space X which is an A, B-bimodule such that $1x = x = x1$ and $\|axb\| \leq \|a\|\|x\|\|b\|$ for all $a \in A$, $b \in B$ and $x \in X$. The class of all such bimodules is denoted by ${}_A\mathrm{BM}_B$ (with ${}_A\mathrm{BM}$ and BM_A for left and right modules, respectively) and the space of all bounded A, B-bimodule maps from X to Y by $\mathrm{B}_A(X, Y)_B$.

A Hilbert A-module is just a Hilbert space $\mathcal{H}$ together with a $*$-representation π of A on $\mathcal{H}$. If π is cyclic then $\mathcal{H}$ is called cyclic. If each finite subset of $\mathcal{H}$ is contained in a closed cyclic submodule $[A\xi]$ then $\mathcal{H}$ is locally cyclic.

If A is a von Neumann algebra and π is normal then $\mathcal{H}$ is normal. If in addition $\mathcal{H}$ is faithful and all normal states on A and A' are vector states, then $\mathcal{H}$ is *proper*.

A proper A-module contains (up to a unitary equivalence) all normal cyclic Hilbert A-modules. If $A \subseteq \mathrm{B}(\mathcal{H})$ is in the standard form, then $\mathcal{H}$ is proper, hence locally cyclic by [23, Lemma 2.10] and [43, Lemma 2.3]. Separable proper modules are unique up to a unitary equivalence by [24, Lemma 2.7] and [26, Theorem 7.2.9].

For operator spaces, OS, $\mathrm{CB}(X, Y)$ denotes the set of completly bounded maps from X to Y. If $X, Y \in \mathrm{OS} \cap {}_A\mathrm{BM}_B$, let $\mathrm{CB}_A(X, Y)_B = \mathrm{B}_A(X, Y)_B \cap \mathrm{CB}(X, Y)$.

Definition 2.1. (i) The class ${}_A\mathrm{OM}_B$ of *operator A, B-bimodules* consists of all $X \in {}_A\mathrm{BM}_B \cap \mathrm{OS}$ such that for some Hilbert module $\mathcal{H}$ over A and B the space $\mathrm{CB}_A(X, \mathrm{B}(\mathcal{H}))_B$ contains a complete isometry.

(ii) If in (i) A and B are von Neumann algebras and $\mathcal{H}$ can be chosen to be normal over A and B, then X is a *normal operator A, B-bimodule* ($X \in {}_A\mathrm{NOM}_B$).

Operator bimodules are characterized by the CES theorem (see also [37, p. 234]):

Theorem 2.2. [10] *$X \in \mathrm{OS} \cap {}_A\mathrm{BM}_B$ is in ${}_A\mathrm{OM}_B$ if and only if $\mathrm{M}_n(X)$ is a Banach $\mathrm{M}_n(A), \mathrm{M}_n(B)$-bimodule for each $n = 1, 2 \ldots$.*

Normal operator bimodules are characterized as follows:

Theorem 2.3. [32], [34] *A bimodule $X \in {}_A\mathrm{OM}_B$ is normal if and only if for each $n \in \mathbb{N}$ and $x \in \mathrm{M}_n(X)$ the mappings $\mathrm{M}_n(A) \ni a \mapsto \|ax\|$ and $\mathrm{M}_n(B) \ni b \mapsto \|xb\|$ are weak* lower semi continuous. If A and B are σ-finite, this is the case if and only if for all $x \in \mathrm{M}_n(X)$ and sequences of projections (e_j) and (f_j) increasing to 1 in $\mathrm{M}_n(A)$ and $\mathrm{M}_n(B)$ (resp.) the equalities $\lim_j \|e_j x\| = \|x\| = \lim_j \|x f_j\|$ hold.*

We recall that a von Neumann algebra A is σ-finite if each orthogonal family of nonzero projections in A is countable.

An argument of Smith shows the following:

Theorem 2.4. [43] *If $\mathcal{H}$ over A and $\mathcal{K}$ over B are locally cyclic then $\|\phi\|_{\mathrm{cb}} = \|\phi\|$ for each $\phi \in \mathrm{B}_A(X, \mathrm{B}(\mathcal{K}, \mathcal{H}))_B$, where $X \in {}_A\mathrm{OM}_B$.*

Definition 2.5. (i) A *dual Banach A, B-bimodule* is a dual Banach space $X = V^\sharp \in {}_A\mathrm{BM}_B$ such that the maps $X \ni x \mapsto ax$ and $X \ni x \mapsto xb$ are weak* continuous for all $a \in A$ and $b \in B$. Then the preadjoints of these maps define a B, A-bimodule structure on V. Conversely, for every $V \in {}_B\mathrm{BM}_A$, $X = V^\sharp$ is the *dual Banach A, B-bimodule of V* by

$$\langle axb, v\rangle = \langle x, bva\rangle \ (x \in X,\ v \in V).$$

The category of such bimodules is denoted by ${}_A\mathrm{DBM}_B$ and the space of all weak* continuous (hence bounded) A, B-bimodule maps from X to Y by $\mathrm{N}_A(X, Y)_B$.

(ii) For von Neumann algebras A and B, $X = V^\sharp \in {}_A\mathrm{DBM}_B$ is a *normal dual Banach bimodule* ($X \in {}_A\mathrm{NDBM}_B$) if the maps $A \ni a \mapsto \langle ax, v\rangle$ and $B \ni b \mapsto \langle xb, v\rangle$ are weak* continuous for all $x \in X$ and $v \in V$.

Definition 2.6. $X \in {}_A\mathrm{OM}_B$ is *a dual operator A, B-bimodule* ($X \in {}_A\mathrm{DOM}_B$) if X is the Banach bimodule dual and the operator space dual of some $V \in {}_B\mathrm{BM}_A \cap \mathrm{OS}$. For such bimodules X, Y let $\mathrm{NCB}_A(X, Y)_B = \mathrm{N}_A(X, Y)_B \cap \mathrm{CB}(X, Y)$.

Remark 2.7. A Hilbert A-module $\mathcal{H}$ is an operator A-module if $\mathcal{H}$ is the column operator space. $\mathcal{H}$ is dual to the conjugate row Hilbert space $\mathcal{H}^*$ with the right module action of A by

$$\xi^* a = (a^*\xi)^*, \quad (\xi \in \mathcal{H}), \tag{2.1}$$

where ξ^* denotes ξ regarded as an element of $\mathcal{H}^*$. Here $\mathcal{H}$ will always mean a column Hilbert space and $\mathcal{H}^*$ the corresponding operator space dual.

Definition 2.8. For von Neumann algebras A and B, $X \in {}_A\mathrm{DOM}_B$ is *a normal dual operator A, B-bimodule* (${}_A\mathrm{NDOM}_B$) if there exist a normal Hilbert module $\mathcal{H}$ over A, B and a complete isometry in $\mathrm{NCB}_A(X, \mathrm{B}(\mathcal{H}))_B$.

The BEZ theorem [7] improves the characterization of ${}_A\mathrm{NDOM}_B$ [17] as follows.

Theorem 2.9. [7] *If $X \in {}_A\mathrm{OM}_B$ is a dual operator space, then the maps $X \ni x \mapsto ax$ and $x \mapsto xb$ are weak* continuous for all $a \in A$ and $b \in B$. If (for von Neumann algebras A and B) the maps $A \ni a \mapsto ax$ and $B \ni b \mapsto xb$ are also weak* continuous for all $x \in X$ then $X \in {}_A\mathrm{NDOM}_B$.*

Some considerations below will be independent of a particular choice of norms on $\mathrm{M}_n(X)$ for $n \geq 2$.

Definition 2.10. (i) [2] $X \in {}_A\mathrm{BM}_B$ is *representable* ($X \in {}_A\mathrm{RM}_B$) if for some Hilbert module $\mathcal{H}$ over A, B the space $\mathrm{B}_A(X, \mathrm{B}(\mathcal{H}))_B$ contains an isometry.

(ii) If in (i) A and B are von Neumann algebras and $\mathcal{H}$ can be chosen to be normal over A and B, then X is called *normal representable* (${}_A\mathrm{NRM}_B$).

(iii) If $X \in {}_A\mathrm{DBM}_B$ and for some normal Hilbert module $\mathcal{H}$ over A and B there exists an isometry in $\mathrm{N}_A(X, \mathrm{B}(\mathcal{H}))_B$, X is a *normal dual representable A, B-bimodule* ($X \in {}_A\mathrm{NDRM}_B$).

Definition 2.11. A subset S of $X \in {}_A\mathrm{BM}_B$ is *A, B-absolutely convex* if

$$\sum_{j=1}^{n} a_j x_j b_j \in S$$

for all $x_j \in S$ and $a_j \in A$, $b_j \in B$ satisfying $\sum_{j=1}^{n} a_j a_j^* \leq 1$, $\sum_{j=1}^{n} b_j^* b_j \leq 1$.

For an index set $\mathbb{J}$ and an $X \in \mathrm{OS}$ let $\mathrm{R}_{\mathbb{J}}(X)$ be the set $\mathrm{M}_{1,\mathbb{J}}(X)$ of all $1 \times \mathbb{J}$ bounded matrices with the entries in X and similarly $\mathrm{C}_{\mathbb{J}}(X) = \mathrm{M}_{\mathbb{J},1}(X)$. (An $\mathbb{I} \times \mathbb{J}$ matrix is bounded if the supremum of the norms of its finite submatrices is finite.)

A part of the following result was obtained also by B. E. Johnson (unpublished).

Theorem 2.12. *([32], [33, Theorem 1.1] or [40]) $X \in {}_A\mathrm{BM}_B$ is representable if and only if the unit ball B_X is A,B-absolutely convex. In this case the minimal operator A,B-bimodule norms on $\mathrm{M}_n(X)$ $(n = 1, 2, \dots)$ are given by*

$$\|x\|_{{}_A\mathrm{m}_B} = \sup \|axb\|, \tag{2.2}$$

where the supremum is over all a and b in the unit balls of $\mathrm{R}_n(A)$ and $\mathrm{C}_n(B)$.

The same norms (2.2) are also obtained by

$$\|x\|_{{}_A\mathrm{m}_B} = \sup \|\phi(x)\|, \tag{2.3}$$

where the supremum is over all contractions $\phi \in \mathrm{B}_A(X, \mathrm{B}(\mathcal{K}, \mathcal{H}))_B$ with $\mathcal{H}$ and $\mathcal{K}$ cyclic (or locally cyclic) Hilbert modules over A and B.

Definition 2.13. A bimodule $X \in {}_A\mathrm{NOM}_B$ is called *strong* ($X \in {}_A\mathrm{SOM}_B$) if

$$[a_i][x_{ij}](b_j) = \sum_{i\in\mathbb{I}, j\in\mathbb{J}} a_i x_{ij} b_j \in X \tag{2.4}$$

for all $[a_i] \in \mathrm{R}_{\mathbb{I}}(A)$, $[x_{ij}] \in \mathrm{M}_{\mathbb{I},\mathbb{J}}(X)$, $(b_j) \in \mathrm{C}_{\mathbb{J}}(A)$ and index sets $\mathbb{I}$ and $\mathbb{J}$.

As shown in [31], it suffices to require the condition (2.4) for orthogonal families of projections $(a_i) \subseteq A$ and $(b_j) \subseteq B$. Strong bimodules in $\mathrm{B}(\mathcal{H})$ are characterized as closed in the A,B-topology [33], the definition of which we shall avoid. Concerned with convex sets only, it suffices to note that a functional ρ on $\mathrm{B}(\mathcal{H})$ is A,B-continuous if and only if $\rho \in \mathrm{B}(\mathcal{H})^{A\sharp B}$, where $\mathrm{B}(\mathcal{H})^{A\sharp B}$ is defined as follows.

Definition 2.14. If A and B are von Neumann algebras and $X \in {}_A\mathrm{BM}_B$, $X^{A\sharp B}$ consists of all $\rho \in X^\sharp$ such that for each $x \in X$ the maps $A \ni a \mapsto \rho(ax)$ and $B \ni b \mapsto \rho(xb)$ are normal.

The argument from [31, Proposition 4.6] shows that bounded bimodule homomorphisms are continuous in the A,B-topology. Since the condition of being closed in the A,B-topology does not depend on norms on $\mathrm{M}_n(X)$ $(n > 1)$, we can define the subclass ${}_A\mathrm{SRM}_B$ of strong bimodules in class ${}_A\mathrm{NRM}_B$.

Occasionally we shall need the following version of the bipolar theorem.

Theorem 2.15. [33] *Let K be an A,B-absolutely convex subset of a bimodule $X \in {}_A\mathrm{SOM}_B$ (or merely $X \in {}_A\mathrm{SRM}_B$). If K is closed in the A,B-topology, then for each $x \in X \setminus K$ there exist normal cyclic Hilbert modules $\mathcal{H}$ over A and $\mathcal{K}$ over B and $\phi \in \mathrm{B}_A(X, \mathrm{B}(\mathcal{K}, \mathcal{H}))_B$ such that $\|\phi(y)\| \leq 1$ for all $y \in K$ and $\|\phi(x)\| > 1$. If $X \in {}_A\mathrm{NDOM}_B$ and K is weak* closed then ϕ can be chosen weak* continuous.*

The last part of Theorem 2.15 is [33, Remark 3.9] and also in [40].

Definition 2.16. For operator algebras $A \subseteq \mathrm{B}(\mathcal{H})$, $B \subseteq \mathrm{B}(\mathcal{K})$ and for $X \in {}_A\mathrm{OM}_B$, the *bimodule dual* of X is the A', B'-bimodule $X^\natural = \mathrm{CB}_A(X, \mathrm{B}(\mathcal{K}, \mathcal{H}))_B$, where

$$(a'\phi b')(x) = a'\phi(x)b' \quad (\phi \in X^\natural).$$

If $\mathcal{H}$ and $\mathcal{K}$ are proper, we emphasize this by writting $X^{\natural_\mathrm{p}}$ instead of $X^\natural$ and call $X^{\natural_\mathrm{p}}$ a *proper bimodule dual* of X.

If $X \in {}_A\mathrm{RM}_B$, we only define $X^{\natural_\mathrm{p}} = \mathrm{B}_A(X, \mathrm{B}(\mathcal{K}, \mathcal{H}))_B$, where $\mathcal{H}$ and $\mathcal{K}$ are proper.

Note that for $X \in {}_A\mathrm{OM}_B$ the proper duals of X in ${}_A\mathrm{OM}_B$ and ${}_A\mathrm{RM}_B$ agree (as Banach bimodules) by Theorem 2.4.

Definition 2.17. For von Neumann algebras $A \subseteq \mathrm{B}(\mathcal{H})$ and $B \subseteq \mathrm{B}(\mathcal{K})$ and a bimodule $X \in {}_A\mathrm{DOM}_B$, the A', B'-bimodule $X_\natural = \mathrm{NCB}_A(X, \mathrm{B}(\mathcal{K},\mathcal{H}))_B$ is called the *bimodule predual* of X. If $\mathcal{H}$ and $\mathcal{K}$ are proper then $X_\natural$ is denoted also by $X_{\natural_\mathrm{p}}$.

Note that if $X \in {}_A\mathrm{NOM}_B$, $X^\natural$ is a dual operator A', B'-bimodule (see Corollary 3.5(i) below), hence $(X^\natural)_\natural$ is defined. The following theorem was proved in [35] in the case $B = A$, but the same proof works in general.

Theorem 2.18. *If $X \in {}_A\mathrm{NRM}_B$, then $(X^{\natural_\mathrm{p}})_{\natural_\mathrm{p}}$ is the smallest strong A, B-bimodule containing X. In particular, $(X^{\natural_\mathrm{p}})_{\natural_\mathrm{p}} = X$ if and only if X is strong.*

Now to the Haagerup tensor product. For $X \in \mathrm{NOM}_B$ and $Y \in {}_B\mathrm{NOM}$ the completion of the algebraic tensor product $X \otimes_B Y$ with the norm

$$h(w) = \inf\{\|\sum_{j=1}^n x_j x_j^*\|^{1/2} \|\sum_{j=1}^n y_j^* y_j\|^{1/2} : \ w = \sum_{j=1}^n x_j \otimes_B y_j\}$$

is the *Haagerup tensor product* $X \overset{h}{\otimes}_B Y$. A typical element $w \in X \overset{h}{\otimes}_B Y$ can be represented as $w = \sum_{j=1}^\infty x_i \otimes_B y_i$, where the two series $\sum_{j=1}^\infty x_j x_j^*$ and $\sum_{j=1}^\infty y_j^* y_j$ are norm convergent. We write this as

$$w = x \odot_B y, \tag{2.5}$$

where $x \in \mathrm{R}_\mathbb{J}(X)$, $y \in \mathrm{C}_\mathbb{J}(Y)$ and $\mathbb{J} = \{1, 2, \ldots\}$.

The *extended Haagerup tensor product* $X \overset{eh}{\otimes}_B Y$ consists of all 'formal expressions' (2.5), where $x \in \mathrm{R}_\mathbb{J}(X)$ and $y \in \mathrm{C}_\mathbb{J}(Y)$ for some (infinite) index set $\mathbb{J}$. To avoid the term 'formal expression', we may assume that $X, Y, B \subseteq \mathrm{B}(\mathcal{H})$ for a Hilbert space $\mathcal{H}$ and regard $w = x \odot_B y$ as completely bounded map $b' \mapsto xb'y$ from B' into $\mathrm{B}(\mathcal{H})$. We recall [30, Lemma 3.2] that

$$x \odot_B y = 0 \Longleftrightarrow \exists \text{ a projection } P \in \mathrm{M}_\mathbb{J}(B) \text{ such that } xP = 0 \text{ and } Py = y. \tag{2.6}$$

Thus, $X \overset{eh}{\otimes}_B Y$ is defined as the space of all maps in $\mathrm{CB}(B', \mathrm{B}(\mathcal{H}))$ that can be represented in the form (2.5) with $x \in \mathrm{R}_\mathbb{J}(X)$ and $y \in \mathrm{C}_\mathbb{J}(Y)$ for some cardinal $\mathbb{J}$. (The two sums $\sum_{j\in\mathbb{J}} x_j x_j^*$ and $\sum_{j\in\mathbb{J}} y_j^* y_j$ are weak* convergent, so it suffices to take $\mathbb{J} = \dim \mathcal{H} \times \infty$ since a convergent sum of the form $\sum_{j\in\mathbb{J}} y_j \xi$ ($\xi \in \mathcal{H}$) has only countably many nonzero terms.) If $X \in {}_A\mathrm{SOM}_B$ and $Y \in {}_B\mathrm{SOM}_C$ then $X \overset{eh}{\otimes}_B Y \in {}_A\mathrm{SOM}_C$ and for each $w \in X \overset{eh}{\otimes}_B Y$

$$\|w\|_{\mathrm{cb}} = \inf\{\|x\|\|y\| : \ w = x \odot_B y, \ \ x \in \mathrm{R}_\mathbb{J}(X), \ y \in \mathrm{C}_\mathbb{J}(Y)\}. \tag{2.7}$$

For more see [30] and, for alternative approaches in the case $B = \mathbb{C}$, [19], [9]. We shall use the following basic property of the symbol $\odot_B$:

$$xb \odot_B y = x \odot_B by, \ \ (b \in \mathrm{M}_\mathbb{J}(B), \ x \in \mathrm{R}_\mathbb{J}(X), \ y \in \mathrm{C}_\mathbb{J}(Y)). \tag{2.8}$$

Remark 2.19. As a consequence of the fact that for Hilbert space vectors $(\xi_j) \in \mathrm{C}_\mathbb{J}(\mathcal{H})$ the sum $\sum_{j\in\mathbb{J}} \|\xi_j\|^2$ is convergent, the equalities

$$X \overset{eh}{\otimes}_A \mathcal{H} = X \overset{h}{\otimes}_A \mathcal{H} \ \text{ and } \ \mathcal{H}^* \overset{eh}{\otimes}_A X = \mathcal{H}^* \overset{h}{\otimes}_A X$$

hold for any Hilbert A-module $\mathcal{H}$ and operator module X.

3. Basic duality for normal bimodules

In this section A, B and C are von Neumann algebras and the bimodule duality is defined using faithful Hilbert modules $\mathcal{H}$, $\mathcal{K}$, l over A, B, C (resp.).

Definition 3.1. Given $X \in {}_A\mathrm{OM}_B$ and $Y \in {}_B\mathrm{OM}_C$, let $(X \overset{h}{\otimes}_B Y)^{\natural_{B\mathrm{nor}}}$ denote the subspace of the A, C-bimodule dual of $X \overset{h}{\otimes}_B Y$ consisting of all $\Omega \in (X \overset{h}{\otimes}_B Y)^{\natural}$ such that the map $B \ni b \mapsto \Omega(x \otimes_B by)$ is weak* continuous for all $x \in X$, $y \in Y$.

The first theorem in this section extends the Blecher, Smith duality from [9].

Theorem 3.2. *If $X \in {}_A\mathrm{NOM}_B$ and $Y \in {}_B\mathrm{NOM}_C$ then $(X \overset{h}{\otimes}_B Y)^{\natural_{B\mathrm{nor}}} = X^{\natural} \overset{eh}{\otimes}_{B'} Y^{\natural}$ completely isometrically as A', C'-bimodules.*

Proof. Consider the natural map $\iota : X^{\natural} \overset{eh}{\otimes}_{B'} Y^{\natural} \to (X \overset{h}{\otimes}_B Y)^{\natural_{B\mathrm{nor}}}$ defined by

$$\iota(\phi \odot_{B'} \psi)(x \otimes_B y) = \phi(x)\psi(y),$$

where $x \in X$, $y \in Y$, $\phi \in \mathrm{R}_{\mathbb{J}}(X^{\natural})$ and $\psi \in \mathrm{C}_{\mathbb{J}}(Y^{\natural})$. Note that $\phi(x) \in \mathrm{B}(\mathcal{K}^{\mathbb{J}}, \mathcal{H})$ and $\psi(y) \in \mathrm{B}(l, \mathcal{K}^{\mathbb{J}})$. It can be verified that ι is a well defined ((2.6) may help) completely contractive homomorphism of A', C'-bimodules. To show that ι is injective, suppose that $\phi \odot_{B'} \psi$ is in the kernel of ι. This means that $\phi(x)\psi(y) = 0$ for all $x \in X$ and $y \in Y$, which can be rewritten as (since ϕ and ψ are B-module maps)

$$\phi(X)B\psi(Y) = 0. \tag{3.1}$$

The projection p' in $\mathrm{M}_{\mathbb{J}}(\mathrm{B}(\mathcal{K})) = \mathrm{B}(\mathcal{K}^{\mathbb{J}})$ with range $[B\psi(Y)]$ is in $\mathrm{M}_{\mathbb{J}}(B')$ (since its range is invariant under B) and $p'\psi = \psi$. But (3.1) implies that $\phi p' = 0$, hence (using (2.8)) $\phi \odot_{B'} \psi = \phi \odot_{B'} p'\psi = \phi p' \odot_{B'} \psi = 0$.

Now it suffices to prove ι is a completely quotient map. Let

$$\Omega \in \mathrm{M}_n((X \overset{h}{\otimes}_B Y)^{\natural_{B\mathrm{nor}}}) \subseteq \mathrm{CB}_A(X \overset{h}{\otimes}_B Y, \mathrm{B}(l^n, \mathcal{H}^n))_C$$

be a complete contraction. Then from the factorization theorem ([20], [37, p. 251], [39]) it can be deduced (as in [30, Theorem 3.9]) that there exist a normal Hilbert B-module $\mathcal{G}$ and complete contractions $\phi \in \mathrm{CB}_A(X, \mathrm{B}(\mathcal{G}, \mathcal{H}^n))_B$ and $\psi \in \mathrm{CB}_B(Y, \mathrm{B}(l^n, \mathcal{G}))_C$ such that

$$\Omega(x \otimes_B y) = \phi(x)\psi(y) \ (x \in X, \ y \in Y).$$

Since each normal representation of B is contained in a direct sum of copies of the identity representation, we may assume that $\mathcal{G} = \mathcal{K}^{\mathbb{J}}$ for some cardinal $\mathbb{J}$. Then

$$\mathrm{CB}_A(X, \mathrm{B}(\mathcal{G}, \mathcal{H}^n))_B = \mathrm{M}_{n,\mathbb{J}}(\mathrm{CB}_A(X, \mathrm{B}(\mathcal{K}, \mathcal{H}))_B) = \mathrm{R}_{\mathbb{J}}(\mathrm{C}_n(X^{\natural}))$$

and

$$\mathrm{CB}_B(Y, \mathrm{B}(l^n, \mathcal{G}))_C = \mathrm{M}_{\mathbb{J},n}(\mathrm{CB}_B(Y, \mathrm{B}(l, \mathcal{K}))_C) = \mathrm{C}_{\mathbb{J}}(\mathrm{R}_n(Y^{\natural})),$$

hence $\phi \odot_B \psi$ is an element of $\mathrm{C}_n(X^{\natural}) \overset{eh}{\otimes}_B \mathrm{R}_n(Y^{\natural}) = \mathrm{M}_n(X^{\natural} \overset{eh}{\otimes}_B Y^{\natural})$ with $\|\phi \odot_B \psi\| \leq 1$ and $\iota_n(\phi \odot_B \psi) = \Omega$. □

A special case of Theorem 3.2 is the following result of Effros and Exel [13].

Corollary 3.3. [13] *$(\mathcal{K}^* \overset{h}{\otimes}_B \mathcal{K})^{\sharp} = B'$.*

Proof. We regard $\mathcal{K}$ as a $B, \mathbb{C}$-bimodule and $\mathcal{K}^*$ as a $\mathbb{C}, B$-bimodule. Since $\mathcal{K}^{\natural} = \mathrm{CB}_B(\mathcal{K}) = B'$ and $(\mathcal{K}^*)^{\natural} = B'$, we have $(\mathcal{K}^* \overset{h}{\otimes}_B \mathcal{K})^{\sharp} = B' \overset{eh}{\otimes}_{B'} B' = B'$. □

Definition 3.4. If $X \in {}_A\mathrm{OM}_B$, define the Banach B', A'-bimodule structure on $\mathcal{H}^* \overset{h}{\otimes}_A X \overset{h}{\otimes}_B \mathcal{K}$ by (using the conventions from Remark 2.7)

$$b'(\xi^* \otimes_A x \otimes_B \eta)a' = \xi^* a' \otimes_A x \otimes_B b'\eta.$$

In a C*-context part (i) of the following corollary was proved by Na [36].

Corollary 3.5. *For each $X \in {}_A\mathrm{NOM}_B$ the following natural maps are completely isometric isomorphisms of bimodules (regarded as equalities later on).*

(i) $\kappa : X^\natural \to (\mathcal{H}^* \overset{h}{\otimes}_A X \overset{h}{\otimes}_B \mathcal{K})^\sharp$, $\kappa(\phi)(\xi^* \otimes_A x \otimes_B \eta) = \langle \phi(x)\eta, \xi\rangle$. Here the A', B'-bimodule structure on $X^\natural$ is as in Definition 2.16, while on $(\mathcal{H}^* \overset{h}{\otimes}_A X \overset{h}{\otimes}_B \mathcal{K})^\sharp$ is dual (Definition 2.5) to that on $\mathcal{H}^* \overset{h}{\otimes}_A X \overset{h}{\otimes}_B \mathcal{K}$ (Definition 3.4).

(ii) $\iota : \mathcal{H}^* \overset{h}{\otimes}_{A'} X^\natural \overset{h}{\otimes}_{B'} \mathcal{K} \to X^{{}_A\sharp_B}$, $\iota(\xi^* \otimes_{A'} \phi \otimes_{B'} \eta) = \langle \phi(x)\eta, \xi\rangle$. Here the structure of B, A-bimodule on $\mathcal{H}^* \overset{h}{\otimes}_{A'} X^\natural \overset{h}{\otimes}_{B'} \mathcal{K}$ is as in Definition 3.4 (but with A and B replaced by A' and B', respectively), while $X^{{}_A\sharp_B}$ inherits its structure from $X^\sharp$ (Definitions 2.5 and 2.6).

(iii) $(X^{{}_A\sharp_B})^\sharp = X^{\natural\natural}$.

Proof. The verifications that κ and ι are bimodule homomorphisms are routine.

(i) That κ is a complete isometry follows from Theorem 3.2 and the associativity of the (extended) Haagerup tensor product. Namely, since $\mathcal{K}^\natural = B'$ and $(\mathcal{H}^*)^\natural = A'$, we have the following complete isometries (regarded as equalities):

$$(\mathcal{H}^* \overset{h}{\otimes}_A X \overset{h}{\otimes}_B \mathcal{K})^\sharp = (\mathcal{H}^* \overset{h}{\otimes}_A X)^\natural \overset{eh}{\otimes}_{B'} \mathcal{K}^\natural = (\mathcal{H}^*)^\natural \overset{eh}{\otimes}_{A'} X^\natural = X^\natural.$$

(ii) Similarly, regarding A as a $\mathbb{C}, A$-bimodule and B as a $B, \mathbb{C}$-bimodule, we have $A^\natural = \mathcal{H}^*$ and $B^\natural = \mathcal{K}$. Then

$$X^{{}_A\sharp_B} = (A \overset{h}{\otimes}_A X \overset{h}{\otimes}_B B)^{{}_A\sharp_B} = A^\natural \overset{eh}{\otimes}_{A'} X^\natural \overset{eh}{\otimes}_{B'} B^\natural = \mathcal{H}^* \overset{eh}{\otimes}_{A'} X^\natural \overset{eh}{\otimes}_{B'} \mathcal{K}.$$

(iii) A consequence of (i) and (ii): $(X^{{}_A\sharp_B})^\sharp = (\mathcal{H}^* \overset{h}{\otimes}_{A'} X^\natural \overset{h}{\otimes}_{B'} \mathcal{K})^\sharp = X^{\natural\natural}$. □

Immediately from Corollary 3.5(iii), $\mathrm{K}(\mathcal{K}, \mathcal{H})^{\natural\natural} = \mathrm{B}(\mathcal{K}, \mathcal{H})$, since all bounded linear functionals on the space $\mathrm{K}(\mathcal{K}, \mathcal{H})$ of compact operators are normal.

Corollary 3.6. *For each $X \in {}_A\mathrm{NOM}_B$ the natural homomorphism $X \to X^{\natural\natural}$ is completely isometric.*

Proof. Note that there is a contractive projection from $X^\sharp$ onto $X^{{}_A\sharp_B}$ [31, Proposition 4.4], hence (Corollary 3.5(iii)) $X^{\natural\natural} = (X^{{}_A\sharp_B})^\sharp \subseteq X^{\sharp\sharp}$. □

Observe that $X_\natural$ (Definition 2.17) is a strong A', B'-subbimodule in $X^\natural$ for each $X \in {}_A\mathrm{NDOM}_B$. The following is a counterpart to Theorem 2.18.

Theorem 3.7. *For each $X \in {}_A\mathrm{NDOM}_B$ the natural map $\iota : X \to (X_{\natural_{\mathrm{p}}})^{\natural_{\mathrm{p}}}$ is a completely isometric weak* homeomorphic isomorphism of A, B-bimodules.*

Proof. Set $Y = X_{\natural_{\mathrm{p}}}$. To prove that the natural A, B-bimodule complete contraction

$$\iota : X \to Y^{\natural_{\mathrm{p}}}, \quad \iota(x)(\phi) = \phi(x) \quad (\phi \in Y)$$

is completely isometric, let $x \in \mathrm{M}_n(X)$ with $\|x\| > 1$. By Theorem 2.15 applied to the normal dual $\mathrm{M}_n(A), \mathrm{M}_n(B)$-bimodule $\mathrm{M}_n(X)$ (with K the unit ball of $\mathrm{M}_n(X)$) there exist cyclic normal Hilbert modules $\tilde{\mathcal{G}}$ over $\mathrm{M}_n(A)$ and $\tilde{l}$ over $\mathrm{M}_n(B)$ and a weak* continuous completely contractive bimodule map $\tilde{\phi} : \mathrm{M}_n(X) \to \mathrm{B}(\tilde{l}, \tilde{\mathcal{G}})$ such that $\|\tilde{\phi}(x)\| > 1$. An elementary well known argument shows that $\tilde{\mathcal{H}} = \mathcal{H}^n$ and $\tilde{l} = l^n$ for some normal Hilbert modules $\mathcal{H}$ over A and $\mathcal{K}$ over B and (since $\tilde{\phi}$ is a homomorphism of $\mathrm{M}_n(A), \mathrm{M}_n(B)$-bimodules) $\tilde{\phi} = \phi_n$, where $\phi \in {}_{\mathrm{NCB}_A}(X, \mathrm{B}(l, \mathcal{G}))_B$ (that is, $\tilde{\phi}([x_{ij}]) = [\phi(x_{ij})]$ for all $[x_{ij}] \in \mathrm{M}_n(X)$). Since $\tilde{\mathcal{G}}$ and $\tilde{l}$ are cyclic over $\mathrm{M}_n(A)$ and $\mathrm{M}_n(B)$ (resp.), $\mathcal{G}$ and l are n-cyclic over A and B (resp.), which means that (up to a unitary equivalence) $\mathcal{G} \subseteq \mathcal{H}^n$ and $l \subseteq \mathcal{K}^n$, where $\mathcal{H}$ and $\mathcal{K}$ are the proper modules used in the definition of duality. Then ϕ may be regarded as an element of $\mathrm{NCB}_A(X, \mathrm{M}_n(\mathrm{B}(\mathcal{K}, \mathcal{H})))_B = \mathrm{M}_n(Y)$ and $\|\phi\|_{\mathrm{cb}} \leq 1$. Since $\|\phi_n(x)\| > 1$, it follows that $\|\iota(x)\| > 1$ and ι must be completely isometric.

Next note that ι is weak* continuous on the unit ball, hence a weak* homeomorphism onto the weak* closed subspace $\iota(X)$ in $Y^{\natural_{\mathrm{p}}}$ by the Krein - Smulian theorem. Indeed, if (x_j) is a bounded net in X weak* converging to an $x \in X$, then for each $\phi \in Y$ $(= X_{\natural_{\mathrm{p}}})$ the net $(\phi(x_j))$ converges to $\phi(x)$ in the weak* topology of $\mathrm{B}(\mathcal{K}, \mathcal{H})$, hence

$$\langle \iota(x_j), \xi^* \otimes_{A'} \phi \otimes_{B'} \eta \rangle = \langle \phi(x_j)\eta, \xi \rangle \to \langle \phi(x)\eta, \xi \rangle$$

for all $\xi \in \mathcal{H}$ and $\eta \in \mathcal{K}$. Since elements of the form $\xi^* \otimes_{A'} \phi \otimes_{B'} \eta$ generate the predual $\mathcal{H}^* \overset{h}{\otimes}_{A'} Y \overset{h}{\otimes}_{B'} \mathcal{K}$ of $Y^{\natural_{\mathrm{p}}}$, this proves that ι is weak* continuous.

Now we may identify X with $\iota(X)$ in $Y^{\natural_{\mathrm{p}}}$. If $X \neq Y^{\natural_{\mathrm{p}}}$, then by Theorem 2.15 there exists a nonzero $\phi \in (Y^{\natural_{\mathrm{p}}})_{\natural_{\mathrm{p}}}$ annihilating X. But, since Y is a strong A', B'-bimodule, $(Y^{\natural_{\mathrm{p}}})_{\natural_{\mathrm{p}}} = Y$ by Theorem 2.18. Thus $\phi \in Y$ and therefore $\phi(X) = 0$ implies $\phi = 0$ since $Y = X_{\natural_{\mathrm{p}}}$. This contradiction proves that $X = Y^{\natural_{\mathrm{p}}}$. □

Example 3.8. Theorem 3.7 does not hold for general (nonproper) duals. Let $A \subseteq \mathrm{B}(\mathcal{H})$, $X = \mathcal{H}$ regarded as an $A, \mathbb{C}$-bimodule, and let the duality be defined in terms of maps into $p'\mathcal{H}$, where $p \in A'$ is a projection. Then $\mathcal{H}^\natural = \mathrm{B}_A(\mathcal{H}, p'\mathcal{H}) = p'A'$ and $\iota : \mathcal{H} \to (\mathcal{H}^\natural)_\natural$ acts as $\iota(\xi)(p'a') = p'a'\xi$ $(a' \in A')$. If ι is isometric, then

$$\|\xi\| = \sup\{\|p'a'\xi\| : \ a' \in A', \ \|a'\| \leq 1\} \ \text{ for all } \xi \in \mathcal{H}. \tag{3.2}$$

As a concrete example, we may take $\mathcal{H} = \mathbb{C}^k \otimes \mathbb{C}^n$, $A = 1_k \otimes \mathrm{M}_n(\mathbb{C})$, $p' = e \otimes 1_n$, where $e \in \mathrm{M}_k(\mathbb{C})$ is a rank 1 projection, and $\xi = (\xi_1, \ldots, \xi_n) \in \mathcal{H} = (\mathbb{C}^k)^n$ a unit vector. Then each $a' \in A'$ is of the form $a' = b \otimes 1_n$ for some $b \in \mathrm{M}_k(\mathbb{C})$ and, denoting by ζ a unit vector in the range of e, (3.2) implies (by compactness) that for some $b \in \mathrm{M}_k(\mathbb{C})$ with $\|b\| = 1$ we have

$$1 = \|p'a'\xi\|^2 = \sum_{j=1}^{n} |\langle b\xi_j, \zeta \rangle|^2 \leq \sum_{j=1}^{n} \|b\xi_j\|^2 \leq \|\xi\|^2 = 1. \tag{3.3}$$

This is possible only if $|\langle b\xi_j, \zeta\rangle| = \|b\xi_j\|$ for all j, hence $b\xi_j = \alpha_j\zeta$ $(\alpha_j \in \mathbb{C})$. If $k \leq n$ and we choose ξ so that $\xi_1, \ldots \xi_n$ span $\mathbb{C}^k$, it follows that b has rank 1 and is of the form $b\nu = \langle \nu, \eta \rangle \zeta$ $(\nu \in \mathbb{C}^k)$ for a unit vector $\eta \in \mathbb{C}^k$. Then $\alpha_j = \langle \xi_j, \eta \rangle$ and $\sum_{j=1}^n |\langle \xi_j, \eta\rangle|^2 = \sum_{j=1}^n |\alpha_j|^2 = \sum_{j=1}^n \|b\xi_j\|^2 = 1$ by (3.3). But $1 = \sum_{j=1}^n |\langle \xi_j, \eta\rangle|^2 \leq \sum_{j=1}^n \|\xi_j\|^2 = 1$ implies that $\xi_j = \beta_j \eta$ $(\beta_j \in \mathbb{C})$, which is impossible (if $k \geq 2$) since the ξ_j's span $\mathbb{C}^k$.

Corollary 3.9. *$X = (\mathcal{H}^* \overset{h}{\otimes}_{A'} X_{\natural_{\mathrm{p}}} \overset{h}{\otimes}_{B'} \mathcal{K})^\sharp$ for each $X \in {}_A\mathrm{NDOM}_B$.*

Proof. By Theorem 3.7 $X = (X_{\natural_{\mathrm{p}}})^{\natural_{\mathrm{p}}}$; apply Corollary 3.5(i). □

The *bimodule adjoint* of $T \in \mathrm{B}_A(X,Y)_B$ is the A', B'-bimodule map

$$T^{\natural} : Y^{\natural} \to X^{\natural}, \quad T(\psi) = \psi \circ T \quad (\psi \in Y^{\natural}). \tag{3.4}$$

If $X^{\natural}$ and $Y^{\natural}$ are proper bimodule duals (Definition 2.16) then we write $T^{\natural_{\mathrm{p}}}$ instead of $T^{\natural}$. Using Corollary 3.5(i), $T^{\natural}$ is the usual completely bounded adjoint

$$T^{\natural} = T_h^{\sharp}, \text{ where } T_h = 1_{\mathcal{H}^*} \otimes_A T \otimes_B 1_{\mathcal{K}} : \mathcal{H}^* \overset{h}{\otimes}_A X \overset{h}{\otimes}_B \mathcal{K} \to \mathcal{H}^* \overset{h}{\otimes}_A Y \overset{h}{\otimes}_B \mathcal{K}. \tag{3.5}$$

Corollary 3.10. *If $X, Y \in {}_A\mathrm{SRM}_B$ and $T \in \mathrm{N}_{A'}(Y^{\natural_{\mathrm{p}}}, X^{\natural_{\mathrm{p}}})_{B'}$, then there exists a unique $S \in \mathrm{B}_A(X,Y)_B$ such that $T = S^{\natural_{\mathrm{p}}}$.*

Proof. For each $x \in X$, the map $Y^{\natural_{\mathrm{p}}} \ni \psi \mapsto T(\psi)(x)$ is in $(Y^{\natural_{\mathrm{p}}})_{\natural_{\mathrm{p}}}$, hence by Theorem 2.18 there exists a (unique) element $Sx \in Y$ such that $T(\psi)(x) = \psi(Sx)$. The rest of the proof is routine and will be omitted. □

Proposition 3.11. *Let $X, Y \in {}_A\mathrm{SOM}_B$ and $T \in \mathrm{CB}_A(X,Y)_B$. Then:*

(i) $\|T^{\natural}\|_{\mathrm{cb}} = \|T\|_{\mathrm{cb}}$;

(ii) T is a complete isometry if and only if $T^{\natural}$ is a completely quotient map.

(iii) $T^{\natural_{\mathrm{p}}}$ is a complete isometry if and only if for each $n \in \mathbb{N}$ the image $T(B_{\mathrm{M}_n(X)})$ of the unit ball of $\mathrm{M}_n(X)$ is dense in the unit ball $B_{\mathrm{M}_n(Y)}$ in the A, B-topology.

(iv) If $T^{\natural_{\mathrm{p}}}$ is a complete isometry and T is injective, then T is a completely isometric surjection.

Proof. It follows from (3.5), the classical properties of the adjoint operators and the fact that the Haagerup tensor product preserves completely isometric injections and completely quotient maps that $\|T^{\natural}\|_{\mathrm{cb}} \leq \|T\|_{\mathrm{cb}}$ and that T completely isometric (resp., completely quotient) implies that $T^{\natural}$ is completely quotient (resp., completely isometric). Since T is just the restriction of $T^{\natural\natural}$, it also follows that $T^{\natural}$ completely quotient implies T completely isometric and that $\|T\|_{\mathrm{cb}} \leq \|T^{\natural\natural}\|_{\mathrm{cb}} \leq \|T^{\natural}\|_{\mathrm{cb}}$. Applying Theorem 2.15, (to $\mathrm{M}_n(X)$ over $\mathrm{M}_n(A)$ and $\mathrm{M}_n(B)$), classical reasoning shows that $T^{\natural_{\mathrm{p}}}$ is a complete isometry if and only if $T(B_{\mathrm{M}_n(X)})$ is dense in $B_{\mathrm{M}_n(Y)}$ in the A, B-topology for each n. (But not necessarily in the norm topology, hence we can not conclude that T is open.)

It remains to prove (iv). Since T is injective, the same holds for T_h in (3.5). (If $T_j : X_j \to Y_j$ are injective bimodule maps then, using (2.6), so is $T_1 \otimes_B T_2 : X_1 \overset{eh}{\otimes}_B X_2 \to Y_1 \overset{eh}{\otimes}_B Y_2$.) Then, by classical duality and (3.5) $T^{\natural_{\mathrm{p}}}$ has dense range. On the other hand, since $T^{\natural_{\mathrm{p}}}$ is a weak* continuous isometry and the ball $B_{Y^{\natural_{\mathrm{p}}}}$ is weak* compact, $B_{T^{\natural_{\mathrm{p}}}(Y^{\natural_{\mathrm{p}}})} = T^{\natural_{\mathrm{p}}}(B_{Y^{\natural_{\mathrm{p}}}})$ must be weak* compact. Now the Krein - Smulian theorem shows that $T^{\natural_{\mathrm{p}}}(Y^{\natural_{\mathrm{p}}})$ is weak* closed, hence it follows that $T^{\natural_{\mathrm{p}}}$ is surjective. Thus $T^{\natural_{\mathrm{p}}}$ is a completely isometric weak* homeomorphism of the unit balls, hence $R := (T^{\natural_{\mathrm{p}}})^{-1}$ is weak* continuous by the Krein - Smulian theorem. By Corollary 3.10 there exists an $S \in \mathrm{CB}_A(Y,X)_B$ such that $R = S^{\natural_{\mathrm{p}}}$. From $S^{\natural_{\mathrm{p}}} = (T^{\natural_{\mathrm{p}}})^{-1}$ we conclude that $T = S^{-1}$; moreover, since S and T are complete contractions, both must be completely isometric. □

Proposition 3.12. *If $A = \mathrm{B}(\mathcal{H})$ and $B = \mathrm{B}(\mathcal{K})$, the functors*

$$F : {}_A\mathrm{SOM}_B \to \mathrm{OS}, \quad F(X) = \mathcal{H}^* \overset{h}{\otimes}_{\mathrm{B}(\mathcal{H})} X \overset{h}{\otimes}_{\mathrm{B}(\mathcal{K})} \mathcal{K}$$

and

$$G : \mathrm{OS} \to {}_A\mathrm{SOM}_B, \quad G(V) = \mathcal{H} \overset{eh}{\otimes} V \overset{eh}{\otimes} \mathcal{K}^*$$

are each other's inverse. Moreover, $G(V)^\natural = V^\sharp$ *for each* $V \in \mathrm{OS}$.

Proof. Since $\mathcal{H}^* \overset{h}{\otimes}_{\mathrm{B}(\mathcal{H})} \mathcal{H} = \mathbb{C}$ by Corollary 3.3, we have (using Remark 2.19)

$$F(G(V)) = (\mathcal{H}^* \overset{eh}{\otimes}_{\mathrm{B}(\mathcal{H})} \mathcal{H}) \overset{eh}{\otimes} V \overset{eh}{\otimes} (\mathcal{K}^* \overset{eh}{\otimes}_{\mathrm{B}(\mathcal{K})} \mathcal{K}) = V.$$

Similarly, since $\mathcal{H} \overset{eh}{\otimes} \mathcal{H}^* = \mathrm{B}(\mathcal{H})$ ([18], [5]),

$$G(F(X)) = (\mathcal{H} \overset{eh}{\otimes} \mathcal{H}^*) \overset{eh}{\otimes}_{\mathrm{B}(\mathcal{H})} X \overset{eh}{\otimes}_{\mathrm{B}(\mathcal{K})} (\mathcal{K} \overset{eh}{\otimes} \mathcal{K}^*) = X.$$

Further, by Corollary 3.5(i) $G(V)^\natural = (\mathcal{H}^* \overset{h}{\otimes}_{\mathrm{B}(\mathcal{H})} G(V) \overset{h}{\otimes}_{\mathrm{B}(\mathcal{K})} \mathcal{K})^\sharp = ((\mathcal{H}^* \overset{eh}{\otimes}_{\mathrm{B}(\mathcal{H})} \mathcal{H}) \overset{eh}{\otimes} V \overset{eh}{\otimes} (\mathcal{K}^* \overset{eh}{\otimes}_{\mathrm{B}(\mathcal{K})} \mathcal{K}))^\sharp = V^\sharp$. □

We remark without proof that the functor G does not preserve quotients.

Definition 3.13. A bimodule $X \in {}_A\mathrm{NOM}_B$ is *reflexive* (more precisely, A, B-*reflexive*) if the natural complete isometry $X \to X^{\natural\natural}$ is surjective.

Here is an extension of the classical characterization of reflexivity.

Corollary 3.14. *A bimodule* $X \in {}_A\mathrm{NOM}_B$ *is reflexive if and only if its unit ball* B_X *is compact in the topology induced by* $X^{{}_A\sharp_B}$.

Proof. By Corollary 3.5(i) $X^{\natural\natural} = (X^{{}_A\sharp_B})^\sharp$. By classical arguments the unit ball $B_{(X^{{}_A\sharp_B})^\sharp}$ is compact in the topology induced by $X^{{}_A\sharp_B}$, with B_X a dense subset. □

Example 3.15. If $A \subseteq \mathrm{B}(\mathcal{H})$ and $B \subseteq \mathrm{B}(\mathcal{K})$ are in the standard form, $\mathrm{B}(\mathcal{K}, \mathcal{H})$ is A, B-reflexive if and only if at least one of the algebras A or B is atomic and finite.

If, say B, is atomic and finite then so is B' (since $\mathcal{K}$ is standard) and by [35, Lemma 3.4]

$$\mathrm{B}(\mathcal{K}, \mathcal{H})^\natural = \mathrm{CB}_A(\mathrm{B}(\mathcal{K}, \mathcal{H}))_B = \mathrm{NCB}_A(\mathrm{B}(\mathcal{K}, \mathcal{H}))_B = A' \overset{eh}{\otimes} B'.$$

Since $\mathrm{M}_n(A' \overset{h}{\otimes} B')$ is dense in $\mathrm{M}_n(A' \overset{eh}{\otimes} B')$ in the A', B'-topology [35, Lemma 2.1], the same holds for the corresponding unit balls by [33] and it follows that $\mathrm{B}(\mathcal{K}, \mathcal{H})^{\natural\natural} = \mathrm{CB}_{A'}(A' \overset{h}{\otimes} B', \mathrm{B}(\mathcal{K}, \mathcal{H}))_{B'} = \mathrm{B}(\mathcal{K}, \mathcal{H})$.

By [14] $\mathrm{CB}_A(\mathrm{B}(\mathcal{K}, \mathcal{H}))_B = A' \overset{\sigma h}{\otimes} B' =: V$, and V contains $\mathrm{NCB}_A(\mathrm{B}(\mathcal{K}, \mathcal{H}))_B = A' \overset{eh}{\otimes} B' =: U$. If $\mathrm{B}(\mathcal{K}, \mathcal{H})^{\natural\natural} = \mathrm{B}(\mathcal{K}, \mathcal{H})$, the two strong A', B'-bimodules U and V have the same bimodule dual $\mathrm{B}(\mathcal{K}, \mathcal{H})$, hence $U = V$ by Theorem 2.18. It follows that $C \overset{\sigma h}{\otimes} D = C \overset{eh}{\otimes} D$ for all von Neumann algebras $C \subseteq A'$ and $D \subseteq B'$. If neither A' nor B' is atomic and finite, we can choose C and D both isomorphic to, say, $C = L_\infty[0, 1]$. But, with this choice, $C \overset{\sigma h}{\otimes} C \neq C \overset{eh}{\otimes} C$ since there exist non-normal completely bounded C-bimodule maps on $\mathrm{B}(L_2[0, 1])$.

Example 3.16. Consider $X = A \check{\otimes} V$ as an A-bimodule, where $V \in \mathrm{OS}$. If V is finite dimensional, then $X^\natural$ is algebraically isomorphic to $\mathrm{CB}(V, A') \simeq A' \otimes V$ since each A-bimodule map from X into $\mathrm{B}(\mathcal{H})$ maps $V \cong 1 \otimes V$ into A'. Similarly $X^{\natural\natural}$ is algebraically isomorphic to $A \otimes V = X$. Thus the natural map $X \to X^{\natural\natural}$ is surjective and X is reflexive.

If A is a factor, the same argument applies to every $X \in {}_A\mathrm{NOM}_A$ generated by a finite set $\mathcal{S}$ commuting with A since X is isomorphic to $A \otimes [\mathcal{S}]$ (see [26, p. 334]).

4. The normal part of an operator bimodule

Definition 4.1. Let $A \subseteq \mathrm{B}(\mathcal{H})$ and $B \subseteq \mathrm{B}(\mathcal{K})$ be C^*-algebras and $X \in {}_A\mathrm{OM}_B$. The norm closure of $\iota(X)$, where $\iota : X \to X^{\natural\natural}$ is the natural complete contraction, is called *the normal part* of X and is denoted by X_{n}.

Proposition 4.2. *If $X \in {}_A\mathrm{OM}_B$ and $Y \in {}_A\mathrm{NOM}_B$ then for each $T \in \mathrm{CB}_A(X,Y)_B$ there exists a unique $T_{\mathrm{n}} \in \mathrm{CB}_A(X_{\mathrm{n}},Y)_B$ such that $T_{\mathrm{n}}\iota = T$, where $\iota : X \to X_{\mathrm{n}}$ is the canonical map. By this property X_{n} is characterized as a normal operator A,B-bimodule up to a completely isometric isomorphism.*

Proof. If $\iota_Y : Y \to Y^{\natural\natural}$ is the canonical inclusion (completely isometric since Y is normal), then $\iota_Y T = T^{\natural\natural}\iota_X$, hence we may simply set $T_{\mathrm{n}} = T^{\natural\natural}|X_{\mathrm{n}}$. The rest is clear. □

Let $A \subseteq \mathrm{B}(\mathcal{H})$ *and* $B \subseteq \mathrm{B}(\mathcal{K})$ *be* C^**-algebras,* $\Phi : A \to \mathrm{B}(\tilde{\mathcal{H}})$, $\Psi : B \to \mathrm{B}(\tilde{\mathcal{K}})$ *the universal representations,* $\tilde{A} = \overline{\Phi(A)}$ *and* $\tilde{B} = \overline{\Psi(B)}$ *the von Neumann envelopes.*

Since Φ is the direct sum of all cyclic representations of A obtained from the GNS construction, each $\rho \in A^\sharp$ is of the form $\rho(a) = \langle \Phi(a)\eta, \xi\rangle$ for some vectors $\xi, \eta \in \tilde{\mathcal{H}}$, therefore $\rho\Phi^{-1}$ has a unique normal extension to $\tilde{A}$. It follows (that $\tilde{A} = A^{\sharp\sharp}$ and) that for each $T \in \mathrm{B}(A, \mathrm{B}(\mathcal{L}))$ the map $T\Phi^{-1}$ has a unique weak* continuous extension $\tilde{T} : \tilde{A} \to \mathrm{B}(\mathcal{L})$. In particular, with $T = i_A : A \to \mathrm{B}(\mathcal{H})$ the inclusion, $\tilde{i}_A : \tilde{A} \to \overline{A}$ is a normal $*$-homomorphism, hence

$$\ker \tilde{i}_A = P^\perp \tilde{A} \text{ and similarly } \ker \tilde{i}_B = Q^\perp \tilde{B} \tag{4.1}$$

for some central projections $P \in \tilde{A}$ and $Q \in \tilde{B}$. A map $T \in \mathrm{B}(A, \mathrm{B}(\mathcal{L}))$ is weak* continuous if and only if $T(a) = \tilde{T}(P\Phi(a))$ for all $a \in A$ [26, Section 10.1]. We shall sometimes regard A as a subalgebra of $\tilde{A}$ (via Φ, although Φ is in general not weak* continuous) and call $\tilde{T}$ the extension of T.

Now we are going to explain how a dual Banach A,B-bimodule is in a canonical way an $\tilde{A},\tilde{B}$-bimodule.

Definition 4.3. Given $X = V^\sharp \in {}_A\mathrm{DBM}_B$ (as in Definition 2.5), for each $x \in X$ and $v \in V$ let $\omega_{x,v} \in A^\sharp$ and $\rho_{x,v} \in B^\sharp$ be defined by

$$\omega_{x,v}(a) = \langle ax, v\rangle \quad \text{and} \quad \rho_{x,v}(b) = \langle xb, v\rangle$$

and let $\tilde{\omega}_{x,v}$ and $\tilde{\rho}_{x,v}$ be the weak* continuous extensions of $\omega_{x,v}$ and $\rho_{x,v}$ to $\tilde{A}$ and $\tilde{B}$, respectively. Then for $a \in \tilde{A}$, $b \in \tilde{B}$ and $x \in X$ define ax and xb by

$$\langle ax, v\rangle = \tilde{\omega}_{x,v}(a) \quad \text{and} \quad \langle xb, v\rangle = \tilde{\rho}_{x,v}(b). \tag{4.2}$$

This will be called *the canonical $\tilde{A},\tilde{B}$-bimodule* structure on X.

Relations (4.2) mean that if $a \in \tilde{A}$, $b \in \tilde{B}$ and (a_i), (b_j) are nets in A and B (resp.) such that $(\Phi(a_i))$ and $(\Psi(b_j))$ weak* converge to a and b (resp.), then

$$ax = \lim_i a_i x \quad \text{and} \quad xb = \lim_j x b_j \tag{4.3}$$

in the weak* topology of X.

Remark 4.4. On an operator space X each operator left A-module structure is given by a $*$-homomorphism π from A into the algebra $A_l(X)$ of left adjointable multipliers [6], which is a von Neumann algebra if X is a dual operator space by [7, Theorem 5.4]. The above structure of a left $\tilde{A}$-module then comes from the extension of π to a normal homomorphism $\tilde{\pi}:\tilde{A}\to A_l(X)$. The same for right modules and X is automatically an operator $\tilde{A},\tilde{B}$-bimodule by [37, Corollary 16.9].

If X is a general dual Banach bimodule, however, the relation

$$(ax)b = a(xb) \quad (a\in\tilde{A},\ b\in\tilde{B},\ x\in X) \tag{4.4}$$

requires a proof. We are interested in Banach bimodules since duals of operator bimodules are in general not operator bimodules.

Proposition 4.5. *(i) If $X\in {}_A\mathrm{DBM}_B$ then by (4.2) X is a Banach $\tilde{A},\tilde{B}$-bimodule. Moreover, if $X\in {}_A\mathrm{DOM}_B$ then X is a normal dual operator $\tilde{A},\tilde{B}$-bimodule.*

(ii) Each weak continuous A,B-bimodule map T between dual Banach A,B-bimodules is automatically an $\tilde{A},\tilde{B}$-bimodule map.*

Proof. (i) The relations $(a_1a_2)x = a_1(a_2x)$ and $x(b_1b_2) = (xb_1)b_2$ $(a_k\in\tilde{A},\ b_k\in\tilde{B})$ follow easily from (4.3). To prove (4.4), chose nets $(a_i)\subseteq A$ and $(b_j)\subseteq B$ so that $(\Phi(a_i))$ and $(\Psi(b_j))$ weak* converge to $a\in\tilde{A}$ and $b\in\tilde{B}$ (resp.). Then, since the right multiplication by b_j is weak* continuous on X,

$$(ax)b_j = (\lim_i a_ix)b_j = \lim_i(a_ixb_j) = \lim_i(a_i(xb_j)) = a(xb_j).$$

Therefore $(ax)b = \lim_j((ax)b_j) = \lim_j(a(xb_j))$ and we would like to show that this is equal to $a(xb)$ or, equivalently, that

$$\lim_j\langle a(xb_j),v\rangle = \langle a(xb),v\rangle = \tilde{\omega}_{xb,v}(a)$$

for each $v\in V = X_\sharp$. It suffices to note that (for $a\in\tilde{A}$) the functional $\tilde{B}\ni b\mapsto\tilde{\omega}_{xb,v}(a)$ is normal, which is a consequence of weak compactness of bounded operators from C*-algebras to preduals of von Neumann algebras [1]. Namely, the weak compactness of the operator $T: A\to B^\sharp$, $T(a)(b) = \theta(a,b)$, where $\theta(a,b) = \omega_{xb,v}(a) = \rho_{ax,v}(b)$, implies that the left and the right canonical extensions of θ to $\tilde{A}\times\tilde{B}$ agree [11, p. 12], meaning that $\tilde{\omega}_{xb,v}(a) = \tilde{\rho}_{ax,v}(b)$, normal in $b\in\tilde{B}$.

If $X\in {}_A\mathrm{DOM}_B$ then by Remark 4.4 and Theorem 2.9 X is a normal dual operator $\tilde{A},\tilde{B}$-bimodule.

(ii) This is a consequence of (4.3) and the weak* continuity of T. □

Remark 4.6. Given $X\in {}_A\mathrm{BM}_B$, $X^\sharp$ is a dual B,A-bimodule (Definition 2.5), hence $X^\sharp$ is also canonically a $\tilde{B},\tilde{A}$-bimodule. Now on $X^{\sharp\sharp}$ we have two $\tilde{A},\tilde{B}$-bimodule structures:

(i) The dual $\tilde{A},\tilde{B}$-bimodule structure as in Definition 2.5, that is $\langle aFb,\theta\rangle = \langle F,b\theta a\rangle$ $(a\in\tilde{A},\ b\in\tilde{B},\ \theta\in X^\sharp,\ F\in X^{\sharp\sharp})$; we denote this bimodule by $X_d^{\sharp\sharp}$.

(ii) The canonical $\tilde{A},\tilde{B}$-bimodule structure as in Definition 4.3, that is $aF = \lim_i a_iF$ and $Fb = \lim_j Fb_j$ in the weak* topology of $X^{\sharp\sharp}$ whenever $(\Phi(a_i))\to a$ and $(\Psi(b_j))\to b$ and as an A,B-bimodule $X^{\sharp\sharp}$ is dual to the B,A-bimodule $X^\sharp$.

If $X_d^{\sharp\sharp}$ is a normal $\tilde{A},\tilde{B}$-bimodule, then $X_d^{\sharp\sharp} = X^{\sharp\sharp}$ since $X_d^{\sharp\sharp}$ and $X^{\sharp\sharp}$ agree as A,B-bimodules.

Proposition 4.7. *If $X\in {}_A\mathrm{OM}_B$, then $X_d^{\sharp\sharp}$ is a normal dual operator $\tilde{A},\tilde{B}$-bimodule (hence $X_d^{\sharp\sharp} = X^{\sharp\sharp}$).*

Proof. There exist a Hilbert space l, representations $\pi: A\to\mathrm{B}(\mathcal{L})$ and $\sigma: B\to\mathrm{B}(\mathcal{L})$ and a completely isometric A,B-bimodule embedding $X\subseteq\mathrm{B}(\mathcal{L})$. Then $X_d^{\sharp\sharp}\subseteq$

$\mathrm{B}(\mathcal{L})_d^{\sharp\sharp} = \widetilde{\mathrm{B}(\mathcal{L})}$, hence it suffices to prove that $\widetilde{\mathrm{B}(\mathcal{L})}$ is a normal $\tilde{A}, \tilde{B}$-bimodule, where the bimodule structure on $\widetilde{\mathrm{B}(\mathcal{L})}$ is given by

$$\langle ax, \theta\rangle = \langle x, \theta a\rangle \text{ and } \langle xb, \theta\rangle = \langle x, b\theta\rangle \quad (a \in \tilde{A},\ b \in \tilde{B},\ \theta \in \mathrm{B}(\mathcal{L})^{\sharp},\ x \in \widetilde{\mathrm{B}(\mathcal{L})}).$$

Here $\langle x, \theta a\rangle$ means $\widetilde{\theta a}(x)$, where $\widetilde{\theta a}$ is the normal extension of the functional $\theta a \in \mathrm{B}(\mathcal{L})^{\sharp}$ to $\widetilde{\mathrm{B}(\mathcal{L})}$. But, since the multiplication $\tilde{A} \times \widetilde{\mathrm{B}(\mathcal{L})} \ni (a, x) \mapsto ax$ is separately weak* continuous in both variables (for ax is an abbreviation for the internal product $\pi^{\sharp\sharp}(a)x$ in $\widetilde{\mathrm{B}(\mathcal{L})}$ and $\pi^{\sharp\sharp} : \tilde{A} = A^{\sharp\sharp} \to \mathrm{B}(\mathcal{L})^{\sharp\sharp} = \widetilde{\mathrm{B}(\mathcal{L})}$ is normal), $\widetilde{\theta a}(x) = \tilde{\theta}(ax)$, where $\tilde{\theta}$ is the weak* continuous extension of $\theta \in \mathrm{B}(\mathcal{L})^{\sharp}$ to a functional on $\widetilde{\mathrm{B}(\mathcal{L})}$. It follows that $\langle x, \theta a\rangle = \tilde{\theta}(ax)$ and, since the map $\tilde{A} \ni a \mapsto \tilde{\theta}(ax)$ is weak* continuous, $\widetilde{\mathrm{B}(\mathcal{L})}$ is a normal left $\tilde{A}$-module. Similarly for $\tilde{B}$. □

Motivated by [26, Exercise 10.5.20], where the special case $X = A = B$ is considered, we observe the following:

Corollary 4.8. *If $X \in {}_A\mathrm{OM}_B$ then each norm closed B, A-subbimodule Y of $X^{\sharp}$ is automatically an $\tilde{B}, \tilde{A}$-subbimodule.*

Proof. To prove that $Ya \subseteq Y$ for each $a \in \tilde{A}$, it suffices to show that $F(Ya) = 0$ for each $F \in X^{\sharp\sharp}$ such that $F(Y) = 0$. Choosing a net (a_i) in A such that $(\Phi(a_i))$ weak* converges to a, since $X^{\sharp\sharp}$ is a normal $\tilde{A}$-module we have that for each $y \in Y$

$$\langle F, ya\rangle = \langle aF, y\rangle = \lim_i \langle a_i F, y\rangle = \lim_i \langle F, ya_i\rangle = 0.$$

□

Theorem 4.9. *Let A, B be von Neumann algebras and $X \in {}_A\mathrm{OM}_B$. Regard X as an A, B-subbimodule in $X^{\sharp\sharp}$ and let $P \in \tilde{A}$, $Q \in \tilde{B}$ be the central projections as in (4.1). Then $X^{\natural\natural} = PX^{\sharp\sharp}Q$ and X_{n} is the norm closure of PXQ in $X^{\sharp\sharp}$. For $x \in \mathrm{M}_n(X)$ (with $\iota : X \to X_{\mathrm{n}}$ the canonical map)*

$$\|\iota_n(x)\| = \inf\left(\sup_j \|a_j x b_j\|\right), \tag{4.5}$$

where the infimum is over all nets (a_j) and (b_j) in the unit balls of A and B (resp.) that weak converge to 1. It suffices to take for (a_j) and (b_j) the nets of projections.*

Proof. Since $X^{{}_A\sharp_B}$ consists of all $\rho \in X^{\sharp}$ such that the two maps $A \ni a \mapsto \rho(ax)$ and $B \ni b \mapsto \rho(xb)$ are normal and since a functional ω on A is normal if and only if $\rho = P\rho$ (and similarly for B), it is not hard to show that $X^{{}_A\sharp_B} = QX^{\sharp}P$. Since the $\tilde{A}, \tilde{B}$-bimodule $X^{\sharp\sharp}$ is dual to the $\tilde{B}, \tilde{A}$-bimodule $X^{\sharp}$ by Proposition 4.7 and Remark 4.6, this implies that $(X^{{}_A\sharp_B})^{\sharp} = PX^{\sharp\sharp}Q$, which by Corollary 3.5(i) means that $X^{\natural\natural} = PX^{\sharp\sharp}Q$, and X_{n} is the norm closure of PXQ.

If $(a_j) \subseteq B_A$ and $(b_j) \subseteq B_B$ are nets weak* converging to 1, then $\|\iota_n(x)\| \leq \sup_j \|a_j x b_j\|$ since X_{n} is normal. This proves the inequality $\leq$ in (4.5). To prove the reverse inequality, choose nets $(a_j) \subseteq B_A$ and $(b_j) \subseteq B_B$ so that $(\Phi(a_j))$ and $(\Psi(b_j))$ converge to P and Q, respectively. Since the normal extensions of Φ^{-1} and Ψ^{-1} map P and Q to 1_A and 1_B (resp.), (a_j) and (b_j) must converge to 1_A and 1_B. Since $\|\iota_n(x)\| = \|PxQ\|$ and $X^{\sharp\sharp}$ is a normal operator $\tilde{A}, \tilde{B}$-bimodule,

$$\|\iota_n(x)\| = \|PxQ\| = \sup_j \|\Phi(a_j)x\Psi(b_j)\| = \sup_j \|a_j x b_j\|.$$

We may replace a_j with the range projection $R(a_j) \in A$ since $a_j \leq R(a_j) \leq 1$. □

5. Central bimodules

Definition 5.1. A bimodule X over an Abelian operator algebra C is called *central* if $cx = xc$ for all $x \in X$ and $c \in C$. The classes of central C-bimodules among representable, operator, normal representable and normal operator bimodules are denoted by CRM_C, COM_C, CNRM_C and CNOM_C, respectively.

Remark 5.2. For an Abelian C*-algebra C we denote by Δ the spectrum of C and by C_t the kernel of a character $t \in \Delta$. If X is a central Banach C-bimodule, we consider the quotients $X(t) = X/[C_t X]$. A bimodule $X \in \mathrm{CRM}_C$ is equivalent to a Banach bundle over Δ [12], but we shall only need that for each $x \in X$ the function

$$\Delta \ni t \mapsto \|x(t)\|, \tag{5.1}$$

where $x(t)$ is the coset of x in $X(t)$, is upper semicontinuous [12] and that (see [22, p. 232] or [12])

$$\|x\| = \sup_{t\in\Delta} \|x(t)\|. \tag{5.2}$$

If $X \in \mathrm{COM}_C$, then the same holds for each $x \in \mathrm{M}_n(X)$, where $x(t)$ is the coset of x in $\mathrm{M}_n(X)/(C_t\mathrm{M}_n(X))$. We shall refer to the embedding

$$X \to \oplus_{t\in\Delta} X(t), \quad x \mapsto (x(t))_{t\in\Delta}$$

as the *canonical decomposition* of X.

Throughout the rest of the section C is an Abelian von Neumann algebra.

Lemma 5.3. *$X \in \mathrm{COM}_C$ is normal if and only if pX is a normal pC-bimodule for each σ-finite projection $p \in C$. If C is σ-finite, then X is normal if and only if*

$$\lim_j \|p_j x\| = \|x\| \tag{5.3}$$

for each $x \in \mathrm{M}_n(X)$ ($n \in \mathbb{N}$) and each sequence of projections $p_j \in pC$ increasing to 1*. Similarly for $X \in \mathrm{CRM}_C$, but with (5.3) required for $x \in X$ only.*

Proof. We may assume that C is σ-finite, for in general C is a direct sum of σ-finite subalgebras and X (being central) also decomposes in the corresponding ℓ_∞-direct sum. Then by Theorem 2.3 we have to prove that for each n, $x \in \mathrm{M}_n(X)$ and sequence (e_j) of projections in $\mathrm{M}_n(C)$ increasing to 1 the sequence $(\|e_j x\|)$ converges to $\|x\|$. Suppose the contrary, that for an x and (e_j)

$$\|e_j x\| \leq M \quad \text{for some constant} \quad M < \|x\|.$$

Let τ be the canonical normal central trace on $\mathrm{M}_n(C)$, the values of which on projections of $\mathrm{M}_n(C)$ are of the form $\frac{k}{n}p$, where $p \in C$ is a projection and $k \in \{0, 1, \ldots, n\}$. For each j set $\Delta_j = \{t \in \Delta : \ \tau(e_j)(t) = 1\}$, a clopen subset of Δ, and let $p_j \in C$ be the the characteristic function of Δ_j. Since the sequence (e_j) increases to 1 and τ is weak* continuous, $\bigcup_j \Delta_j$ is dense in Δ, hence the sequence (p_j) also increases to 1. For $t \in \Delta_j$, $e_j(t) \in \mathrm{M}_n(C)(t) = \mathrm{M}_n(\mathbb{C})$ is a projection with the normalized trace equal to 1, hence $e_j(t) = 1$. Thus, $e_j p_j = p_j$, which implies that $\|p_j x\| \leq \|e_j x\| \leq M < \|x\|$ for all j, a contradiction with (5.3). □

Proposition 5.4. *A bimodule $X \in \mathrm{COM}_C$ is normal if and only if for each $n \in \mathbb{N}$ and each $x \in \mathrm{M}_n(X)$ the function $\Delta \ni t \mapsto \|x(t)\|$ is continuous.*

Proof. If X is normal, then we may assume that $X \subseteq C'$, the commutant of C in $\mathrm{B}(\mathcal{H})$ for a normal Hilbert C-module $\mathcal{H}$, hence $\mathrm{M}_n(X)$ is contained in the commutant of C in $\mathrm{B}(\mathcal{H}^n)$ and the continuity of (5.1) follows from [22, p. 233].

For the converse, by Lemma 5.3 we may assume that C is σ-finite and we have to prove the condition (5.3). But this follows (using 5.2) from the assumed continuity since $\bigcup_j \Delta_j$ is dense in Δ, where Δ_j is as in the proof of Lemma 5.3. □

Proposition 5.5. *Let $X \in \mathrm{CNOM}_C$ be a strong bimodule and $Y \subseteq X$ a subbimodule. Then the quotient X/Y is normal if and only if Y is strong and in this case X/Y is also strong.*

Proof. It was observed in [32] that X/Y is normal only if Y is strong. For the converse, assuming that C is σ-finite and that the condition of Lemma 5.3 for normality is not satisfied, there exist an $\dot{x} \in \mathrm{M}_n(X/Y)$, a sequence of projections (p_j) in C increasing to 1 and a constant $M < \|\dot{x}\|$ such that $\|p_j\dot{x}\| < M$ for all j. Put $q_0 = p_0$ and $q_j = p_j - p_{j-1}$ if $j \geq 1$. Let $x \in \mathrm{M}_n(X)$ be any representative of the coset $\dot{x}$. By definition of the quotient norm for each j there exists an element $y_j = q_j y_j \in \mathrm{M}_n(Y)$ such that $\|q_j x - y_j\| < M$. Since the sequence (y_j) is bounded and Y is strong, the sum

$$y := \sum_{j=0}^{\infty} q_j y_j = \sum_{j=0}^{\infty} q_j y_j q_j$$

defines an element of Y. But the estimate

$$\|x - y\| = \|\sum_j q_j(x-y)q_j\| = \sup_j \|q_j(x-y)\| \leq \sup_j \|p_j(x - y_j)\| < M$$

implies that $\|\dot{x}\| < M$, which is in contradiction with the choice of M.

To verify that X/Y is a strong left C-module (hence a strong C-bimodule since it is central), let (p_j) be an orthogonal family of projections in C and $(\dot{x}_j)$ a family of elements in X/Y such that the sum $\sum_j \dot{x}_j^* \dot{x}_j$ converges in the strong operator topology in a $\mathrm{B}(\mathcal{H})$ containing X/Y as a normal operator C-bimodule. We can choose for each $\dot{x}_j$ a representative $x_j \in X$ so that the set $(x_j)_j$ is bounded, and then $x := \sum_j p_j x_j = \sum_j p_j x_j p_j \in X$. Since the quotient map $Q : X \to X/Y$ is a bounded C-bimodule map (hence C-continuous), it follows that $Q(x) = \sum_j p_j Q(x_j) = \sum_j p_j \dot{x}_j$, which proves that $\sum_j p_j \dot{x}_j \in X/Y$. □

For central bimodules we can now complete Proposition 3.11.

Corollary 5.6. *If $X, Y \in \mathrm{CNOM}_C$ are strong and $T \in \mathrm{CB}_C(X, Y)$, then T is completely isometric (resp., completely quotient) iff $T^{\natural_{\mathrm{P}}}$ is completely quotient (resp. completely isometric).*

Proof. By Proposition 3.11 it remains to prove that T is completely quotient if $T^{\natural_{\mathrm{P}}}$ is completely isometric. By Proposition 5.5 $X/\ker T$ is strong, hence, replacing T with the induced map $\tilde{T} : X/\ker T \to Y$ (note that $\tilde{T}^{\natural_{\mathrm{P}}}$ is the restriction of $T^{\natural_{\mathrm{P}}}$ to $(X/\ker T)^{\natural_{\mathrm{P}}} \subseteq X^{\natural_{\mathrm{P}}}$, hence completely isometric), we may assume that T is injective and the result then follows from Proposition 3.11(iv). □

Remark 5.7. If $X \in \mathrm{CNOM}_C$ then $X^{\natural_{\mathrm{P}}} = \mathrm{CB}_C(X, C)$ since each C-bimodule map $\phi : X \to \mathrm{B}(\mathcal{H})$ maps X into C' and $C' = C$ if $\mathcal{H}$ is proper.

Definition 5.8. For a function $f : \Delta \to \mathbb{R}$, let $\operatorname{essup} f$ be the infimum of all $c \in \mathbb{R}$ such that the set $\{t \in \Delta : \ f(t) > c\}$ is meager (contained in a countable union of closed sets with empty interiors).

The *essential direct sum*, $\mathrm{ess}\oplus_{t\in\Delta} X(t)$, of a family of Banach spaces $(X(t))_{t\in\Delta}$ is defined as the quotient of the ℓ_∞-direct sum $\oplus_{t\in\Delta}X(t)$ by the zero space of the seminorm $x \mapsto \operatorname{essup}\|x(t)\|$. Then $\mathrm{ess}\oplus_{t\in\Delta} X(t)$ with the norm $\dot{x} \mapsto \operatorname{essup}\|x(t)\|$ is a Banach space and we denote by $e : \oplus_{t\in\Delta}X(t) \to \mathrm{ess}\oplus_{t\in\Delta} X(t)$ the quotient map. If the $X(t)$'s are operator spaces, then $\mathrm{ess}\oplus_{t\in\Delta} X(t)$ is an operator space by

$$\mathrm{M}_n(\mathrm{ess}\oplus_{t\in\Delta} X(t)) := \mathrm{ess}\oplus_{t\in\Delta} \mathrm{M}_n(X(t)).$$

Theorem 5.9. *Given $X \in \mathrm{COM}_C$ with the canonical decomposition $\kappa : X \to \oplus_{t\in\Delta}X(t)$, X_{n} is the closure of $e\kappa(X)$ in $\mathrm{ess}\oplus_{t\in\Delta} X(t)$.*

Proof. First, to show that $e\kappa(X)$ is a normal operator C-module, by Lemma 5.3 we may assume that C is σ-finite and it suffices to prove that for each sequence of projections $p_j \in C$ increasing to 1 and each $x \in \mathrm{M}_n(X)$ the equality

$$\operatorname{essup}\|x(t)\| = \lim_j \operatorname{essup}\|p_j(t)x(t)\| \tag{5.4}$$

holds. With Δ_j the clopen subset of Δ corresponding to p_j, $\bigcup_j \Delta_j$ is dense in Δ. Since the function $\Delta \ni t \mapsto \|x(t)\|$ is upper semi-continuous (hence Borel), it agrees outside a meager set with a continuous function f on Δ by [26, p. 323]. Then $\operatorname{essup}\|x(t)\| = \sup f(t)$, $\operatorname{essup}\|p_j(t)x(t)\| = \sup p_j(t)f(t)$ and $\lim_j \sup_t p_j(t)f(t) = \sup f(t)$ by continuity (since $\bigcup_j \Delta_j$ is dense in Δ). This implies (5.4).

It remains to show that the closure of $e\kappa(X)$ has the universal property of X_{n} from Proposition 4.2. Let $Y \in {}_C\mathrm{NOM}_C$ and $T \in \mathrm{CB}_C(X,Y)$ with $\|T\|_{\mathrm{cb}} < 1$. We have to show that T can be factorized through $e\kappa(X)$. Replacing Y by the closure of $T(X)$, we may assume that Y is central. Let $x \in \mathrm{M}_n(X)$ and set $y = T_n(x)$. Since $\|T\|_{\mathrm{cb}} < 1$ and T is a C-module map, $\|y(t)\| \leq \|x(t)\|$ for each $t \in \Delta$. Set

$$c = \|(e\kappa)_n(x)\| = \operatorname{essup}_t \|x(t)\| \ \text{ and } \ V = \{t \in \Delta : \ \|y(t)\| > c\}.$$

Since Y is normal, the function $t \mapsto \|y(t)\|$ is continuous by Proposition 5.4, hence V is open. But for each $t \in V$ we have that $c < \|y(t)\| \leq \|x(t)\|$, hence V must be meager by the definition of essup, hence $V = \emptyset$ by Baire's theorem [42, p. 42]. Thus, $\|y(t)\| \leq c$ for all $t \in \Delta$, implying that $\|T_n(x)\| = \|y\| = \sup_t \|y(t)\| \leq c = \|(e\kappa)_n(x)\|$ and $T = S \circ (e\kappa)$ for a complete contraction $S : (e\kappa)(X) \to Y$. □

6. Maximal operator bimodules of a normal representable bimodule

In this section the duality is defined in terms of fixed proper modules $\mathcal{H}$ and $\mathcal{K}$ over von Neumann algebras A and B.

Since a proper module is locally cyclic and contains all cyclic normal Hilbert modules, for each $X \in {}_A\mathrm{NRM}_B$ the right side of

$$\|x\|_{{}_A\mathrm{m}_B} = \sup\{\|\phi_n(x)\| : \ \phi \in X^{\natural_{\mathrm{p}}} \ \|\phi\| \leq 1\} \ \ (x \in \mathrm{M}_n(X), \ n = 1, 2, \ldots) \tag{6.1}$$

defines operator A, B-bimodule norms (for $n = 1$ just the given norm on X by Theorem 2.15) dominated by, hence hence equal to the minimal norms (2.3). In particular, it follows from (6.1) that the minimal operator bimodule of X, denoted by $\mathrm{MIN}_A(X)_B$, is normal. In contrast, the maximal operator bimodule is not necessarily normal.

Definition 6.1. (i) For $X \in {}_A\mathrm{RM}_B$ define *maximal operator bimodule norms* by

$$\|x\|_{{}_A\mathrm{M}_B} = \sup \|T_n(x)\| \quad (x \in \mathrm{M}_n(X),\ n = 1, 2, \ldots), \tag{6.2}$$

where the sup is over all contractions $T \in \mathrm{CB}_A(X, \mathrm{M}_m(\mathrm{B}(\mathcal{K}, \mathcal{H})))_B$, with $m \in \mathbb{N}$ and $\mathcal{H}$, $\mathcal{K}$ the Hilbert spaces of the universal representations of A and B, respectively. Denote the operator bimodule so obtained $\mathrm{MAX}_A(X)_B$.

(ii) If $X \in {}_A\mathrm{NRM}_B$, the *maximal normal operator bimodule norms*, denoted by $\|x\|_{{}_A\mathrm{MN}_B}$, are defined by (6.2), but now $\mathcal{H}$ and $\mathcal{K}$ are (fixed) proper Hilbert modules over A and B. This operator bimodule is denoted by $\mathrm{MAXN}_A(X)_B$.

If (iii) $X \in {}_A\mathrm{NDRM}_B$, the *maximal normal dual operator bimodule norms*, denoted by $\|x\|_{{}_A\mathrm{MND}_B}$, are defined in the same way as $\|x\|_{{}_A\mathrm{MN}_B}$, except that in addition we require the maps T in (6.2) to be weak* continuous. Denote this operator bimodule by $\mathrm{MAXND}_A(X)_B$.

Clearly $\|x\|_{{}_A\mathrm{MND}_B} \leq \|x\|_{{}_A\mathrm{MN}_B} \leq \|x\|_{{}_A\mathrm{M}_B}$. These operator bimodules are characterized by the following properties.

Proposition 6.2. *(i) If $X \in {}_A\mathrm{RM}_B$, then $\|T\|_{\mathrm{cb}} = \|T\|$ for every $T \in \mathrm{B}_A(X, Y)_B$ ($Y \in {}_A\mathrm{OM}_B$), that is, $\mathrm{CB}_A(\mathrm{MAX}_A(X)_B, Y)_B = \mathrm{B}_A(X, Y)_B$.*

(ii) If $X \in {}_A\mathrm{NRM}_B$, then $\mathrm{MAXN}_A(X)_B$ is normal and as Banach spaces $\mathrm{CB}_A(\mathrm{MAXN}_A(X)_B, Y)_B = \mathrm{B}_A(X, Y)_B$ for each $Y \in {}_A\mathrm{NOM}_B$.

(iii) If $X \in {}_A\mathrm{NDRM}_B$, then $\mathrm{NCB}_A(\mathrm{MAXND}_A(X)_B, Y)_B = \mathrm{N}_A(X, Y)_B$ for each $Y \in {}_A\mathrm{NDOM}_B$ and $\mathrm{MAXND}_A(X)_B$ is a normal dual operator bimodule.

Proof. The Proposition is evident, except that in (iii) $\mathrm{MAXND}_A(X)_B$ is a dual operator bimodule. Let $V = X_\sharp$. From (2.2) the unit ball of $\mathrm{M}_n(\mathrm{MIN}_A(X)_B)$ is closed in the topology induced by $\mathrm{M}_n(V)$ for each n, hence $\mathrm{MIN}_A(X)_B$ is a dual operator space by [28, Proposition 3.1]. It follows that $\mathrm{MIN}_A(X)_B \in {}_A\mathrm{NDOM}_B$ and by Theorem 3.7 $X = (X_{\natural_\mathrm{p}})^{\natural_\mathrm{p}}$ (in particular) as normal representable bimodules. Then by Proposition 6.5(ii) below (applied to $X_{\natural_\mathrm{p}}$) $\mathrm{MAXND}_A(X)_B = \mathrm{MAXND}_A((X_{\natural_\mathrm{p}})^{\natural_\mathrm{p}})_B = (\mathrm{MIN}_{A'}(X_{\natural_\mathrm{p}})_{B'})^{\natural_\mathrm{p}}$, a normal dual operator bimodule. □

From this and the universality of normal part (Proposition 4.2) we conclude:

Corollary 6.3. $\mathrm{MAXN}_A(X)_B = (\mathrm{MAX}_A(X)_B)_\mathrm{n}$ *for each* $X \in {}_A\mathrm{NRM}_B$.

Example 6.4. In general $\mathrm{MAX}_A(X)_B \neq \mathrm{MAXN}_A(X)_B$ even if $A = B = C$ is Abelian and $X \in \mathrm{CNRM}_C$. To show this, let $U \subseteq V$ be Banach spaces such that the (completely contractive) inclusion of maximal operator spaces $\mathrm{MAX}(U) \to \mathrm{MAX}(V)$ is not completely isometric. (A concrete example of such spaces can be constructed by duality using [39, Section 9.1].) With Δ the spectrum of C and $t_0 \in \Delta$, let

$$X = \{f \in C(\Delta, V) :\ f(t_0) \in U\}.$$

We claim that for each $f \in \mathrm{M}_n(\mathrm{MAX}_C(X)_C)$

$$\|f\|_{{}_A\mathrm{M}_B} = \max\{\sup_{t \in \Delta} \|f(t)\|_{\mathrm{M}_n(\mathrm{MAX}(V))}, \|f(t_0)\|_{\mathrm{M}_n(\mathrm{MAX}(U))}\}. \tag{6.3}$$

To show this, it suffices to prove that, when the $\mathrm{M}_n(X)$ are equipped with the norms defined by the right side of (6.3), each contraction $T \in \mathrm{CB}_C(X, Y)_C$ into $Y \in {}_C\mathrm{OM}_C$, is completely contractive. Replacing Y with the closure of $T(X)$, we may assume that Y is central and therefore has the canonical decomposition

$Y \to \oplus_{t\in\Delta} Y(t)$ (Remark 5.2). Since T is a C-module map, T induces for each $t \in \Delta$ a contraction $T_t : X(t) \to Y(t)$. Since

$$X(t) = \begin{cases} V & \text{if } t \neq t_0 \\ U & \text{if } t = t_0 \end{cases}$$

carries the maximal operator space structure, T_t is a complete contraction, hence so is T (since $\|y\| = \sup_t \|y(t)\|$ for each $y \in \mathrm{M}_n(Y)$).

Since the inclusion $\mathrm{MAX}(X) \to \mathrm{MAX}(Y)$ is not completely isometric, there exists a $u \in \mathrm{M}_n(U)$ with $\|u\|_{\mathrm{M}_n(U)} > \|u\|_{\mathrm{M}_n(V)}$. Hence, if $f \in \mathrm{M}_n(X)$ is the constant $f(t) = u$, the function $t \mapsto \|f(t)\|$ is not continuous and $\mathrm{MAX}_C(X)_C$ is not normal by Proposition 5.4. On the other hand, $\mathrm{MAXN}_A(X)_B$ is normal.

Before showing an example for which $\mathrm{MAXND}_A(X)_B \neq \mathrm{MAXN}_A(X)_B$, we need to generalize operator space duality between MIN and MAX [4].

Proposition 6.5. *If* $X \in {}_A\mathrm{NRM}_B$ *then: (i)* $(\mathrm{MAXN}_A(X)_B)^{\natural_\mathrm{p}} = \mathrm{MIN}_{A'}(X^{\natural_\mathrm{p}})_{B'}$*;* *(ii)* $(\mathrm{MIN}_A(X)_B)^{\natural_\mathrm{p}} = \mathrm{MAXND}_{A'}(X^{\natural_\mathrm{p}})_{B'}$.

Proof. (i) Given $\phi = [\phi_{ij}] \in \mathrm{M}_n((\mathrm{MAXN}_A(X)_B)^{\natural_\mathrm{p}}) = \mathrm{B}_A(X, \mathrm{M}_n(\mathrm{B}(\mathcal{K},\mathcal{H})))_B$,

$$\|\phi\| = \sup\{\|[\phi_{ij}(x)]\| : \ x \in X, \ \|x\| \leq 1\}. \tag{6.4}$$

The norms (6.4) ($n \in \mathbb{N}$) introduce to $X^{\natural_\mathrm{p}}$ a structure of a normal operator A', B'-bimodule (for $n = 1$ just the given norm on $X^{\natural_\mathrm{p}}$) and are dominated by the minimal operator A', B'-bimodule norms by (6.1) applied to $X^{\natural_\mathrm{p}}$ (since $X \subseteq X^{\natural_\mathrm{p}\natural_\mathrm{p}}$), hence by minimality the two structures must coincide.

(ii) Given $\phi = [\phi_{ij}] \in \mathrm{M}_n((\mathrm{MIN}_A(X)_B)^{\natural_\mathrm{p}}) = \mathrm{CB}_A(\mathrm{MIN}_A(X)_B, \mathrm{M}_n(\mathrm{B}(\mathcal{K},\mathcal{H})))_B$,

$$\|\phi\| = \sup\{\|[\phi_{ij}(x_{kl})]\| : \ [x_{kl}] \in \mathrm{M}_s(X), \ \|[x_{kl}]\|_{{}_A\mathrm{m}_B} \leq 1, \ s \in \mathbb{N}\}. \tag{6.5}$$

Since $(\mathrm{MIN}_A(X)_B)^{\natural_\mathrm{p}}$ is a normal dual operator A', B'-bimodule (it is a dual space by Corollary 3.5(i)), it follows that $\|\phi\| \leq \|\phi\|_{{}_A\mathrm{MND}_B}$ by maximality of $\|\cdot\|_{{}_A\mathrm{MND}_B}$. For the reverse inequality, it suffices to show that

$$\|[T\phi_{ij}]\| \leq \|\phi\| \tag{6.6}$$

for each contraction $T \in \mathrm{NCB}_{A'}(X^{\natural_\mathrm{p}}, \mathrm{M}_m(\mathrm{B}(\mathcal{K},\mathcal{H})))_{B'}$ ($m \in \mathbb{N}$). First note that

$$\mathrm{M}_m(\mathrm{B}(\mathcal{K},\mathcal{H})) = (A \overset{eh}{\otimes} \mathrm{T}_m \overset{eh}{\otimes} B)^{\natural_\mathrm{p}}, \tag{6.7}$$

where T_m is the predual of $\mathrm{M}_m(\mathbb{C})$. Indeed, since $A \overset{h}{\otimes} \mathrm{T}_m \overset{h}{\otimes} B$ is dense in $A \overset{eh}{\otimes} \mathrm{T}_m \overset{eh}{\otimes} B$ in the A, B-topology (see [35, Lemma 2.1]), the same holds for the corresponding unit balls by [33, Theorem 3.5], hence the A, B-continuity of bounded A, B-bimodule maps implies that $(A \overset{eh}{\otimes} \mathrm{T}_m \overset{eh}{\otimes} B)^{\natural_\mathrm{p}} = (A \overset{h}{\otimes} \mathrm{T}_m \overset{h}{\otimes} B)^{\natural_\mathrm{p}}$ isometrically; in fact completely isometrically by applying similar arguments to the corresponding matrix spaces. Regarding A as $A, \mathbb{C}$-bimodule and B as $\mathbb{C}, B$-bimodule, it follows, say by Theorem 3.2, that

$$(A \overset{h}{\otimes} \mathrm{T}_m \overset{h}{\otimes} B)^{\natural_\mathrm{p}} = A^{\natural_\mathrm{p}} \overset{eh}{\otimes} \mathrm{T}_m^{\sharp} \overset{eh}{\otimes} B^{\natural_\mathrm{p}} = \mathcal{H} \overset{eh}{\otimes} \mathrm{M}_m(\mathbb{C}) \overset{eh}{\otimes} \mathcal{K}^* = \mathrm{M}_m(\mathrm{B}(\mathcal{K},\mathcal{H})).$$

Since T is weak* continuous, $T = S^{\natural}$ for some $S \in \mathrm{B}_A(A \overset{eh}{\otimes} \mathrm{T}_m \overset{eh}{\otimes} B, X)_B$ by Corollary 3.10, hence, using (6.7), the norm of $[T\phi_{ij}] \in \mathrm{M}_{mn}(\mathrm{B}(\mathcal{K},\mathcal{H})) = \mathrm{CB}_A(A \overset{eh}{\otimes}$

$\mathrm{T}_{mn} \overset{eh}{\otimes} B, \mathrm{B}(\mathcal{K},\mathcal{H}))_B$ is equal to

$$\|[T\phi_{ij}]\| = \sup \|[(T\phi_{ij})(v_{kl})]\| = \sup \|[\phi_{ij}(Sv_{kl})]\|, \tag{6.8}$$

where the supremum is over all $[v_{kl}] \in \mathrm{M}_r(A \overset{eh}{\otimes} \mathrm{T}_{mn} \overset{eh}{\otimes} B)$ with $\|[v_{kl}]\| \le 1$ and $r \in \mathbb{N}$. Since S is a contraction, S is a complete contraction into $\mathrm{MIN}_A(X)_B$, hence $\|[Sv_{kl}]\|_{A^{\mathrm{m}}B} \le 1$. Thus the right side of (6.8) is dominated by $\|\phi\|$ by (6.5), which proves (6.6). □

Corollary 6.6. *If C is an Abelian von Neumann algebra and $X \in \mathrm{CNRM}_C$, then* $(\mathrm{MIN}_C(X)_C)^{\natural_{\mathrm{p}}} = \mathrm{MAXN}_C(X^{\natural_{\mathrm{p}}})_C$*, hence* $\mathrm{MAXN}_C(X^{\natural_{\mathrm{p}}})_C = \mathrm{MAXND}_C(X^{\natural_{\mathrm{p}}})_C$.

Proof. To show that the natural complete contraction

$$\iota : \mathrm{MAXN}_C(X^{\natural_{\mathrm{p}}})_C \to (\mathrm{MIN}_C(X)_C)^{\natural_{\mathrm{p}}}$$

is completely isometric isomorphism, by Corollary 5.6 it suffices to show that $\iota^{\natural} : (\mathrm{MIN}_C(X)_C)^{\natural_{\mathrm{p}}\natural_{\mathrm{p}}} \to (\mathrm{MAXN}_C(X^{\natural_{\mathrm{p}}})_C)^{\natural_{\mathrm{p}}}$ is a completely isometric isomorphism. By Proposition 6.5(i) $(\mathrm{MAXN}_C(X^{\natural_{\mathrm{p}}})_C)^{\natural_{\mathrm{p}}} = \mathrm{MIN}_C(X^{\natural_{\mathrm{p}}\natural_{\mathrm{p}}})_C$ and $\iota^{\natural}$ is just the identity map from $(\mathrm{MIN}_C(X)_C)^{\natural_{\mathrm{p}}\natural_{\mathrm{p}}}$ onto $\mathrm{MIN}_C(X^{\natural_{\mathrm{p}}\natural_{\mathrm{p}}})_C$. Now, for some index set $\mathbb{I}$ there is a completely isometric C-bimodule embedding $\mathrm{MIN}_C(X)_C) \to \ell^{\mathbb{I}}_{\infty}(C)$ (this follows from Remark 5.7 and (6.1)), hence the proof is reduced to the case $X = \ell^{\mathbb{I}}_{\infty}(C)$. Since $(\ell^{\mathbb{I}}_{\infty}(C))^{\natural_{\mathrm{p}}\natural_{\mathrm{p}}} = (\ell^{\mathbb{I}}_{\infty}(C))^{C\sharp C})^{\sharp}$ is contained in the Abelian von Neumann algebra $(\ell^{\mathbb{I}}_{\infty}(C))^{\sharp\sharp}$, it carries the minimal operator space structure and $\iota^{\natural}$ must be completely isometric. The rest follows now from Proposition 6.5(ii). □

Remark 6.7. Although $\mathrm{MIN}_C(X)_C = \mathrm{MIN}(X)$ (from the above proof), $\mathrm{MAX}(C) \neq \mathrm{MAX}_C(C)_C = \mathrm{MIN}_C(C)_C$.

Example 6.8. In general $\mathrm{MIN}_A(X^{\natural_{\mathrm{p}}\natural_{\mathrm{p}}})_B \neq (\mathrm{MIN}_A(X)_B)^{\natural_{\mathrm{p}}\natural_{\mathrm{p}}}$, hence by Proposition 6.5(i) $(\mathrm{MIN}_A(X)_B)^{\natural_{\mathrm{p}}} \neq \mathrm{MAXN}_{A'}(X^{\natural_{\mathrm{p}}})_{B'}$.

Let A be the injective II_1 factor represented normally on a Hilbert space l such that l is not locally cyclic for A. Let $X = A\check{\otimes}A' \subseteq \mathrm{B}(l \otimes^2 l)$. By [41, Proposition 3.4] A ($\cong A\otimes 1$) is a norming subalgebra of X, which by (2.2) means that X carries the minimal operator A-bimodule structure. By Corollary 3.5 $X^{\natural_{\mathrm{p}}\natural_{\mathrm{p}}} = (X^{A\sharp A})^{\sharp} \subseteq X^{\sharp\sharp} = \tilde{X}$. Let $\mathcal{G}$ be the Hilbert space of the universal representation Φ of X. Since $A \cong A \otimes 1$ is a C^*-subalgebra of X, its universal von Neumann envelope $\tilde{A}$ can be regarded as a von Neumann subalgebra of $\tilde{X}$. Let P be the central projection in $\tilde{A}$ such that the weak* continuous extension α of $\Phi^{-1}|\Phi(A)$ to $\tilde{A}$ has kernel $P^{\perp}\tilde{A}$, so that α maps $P\tilde{A}$ isomorphically onto $A \otimes 1 \cong A$. Since A is a factor, $\mathrm{C}^*(A \cup A')$ is weak* dense in $\mathrm{B}(\mathcal{L})$, hence the representation $X = A\check{\otimes}A' \ni (a \otimes a') \mapsto aa'$ (bounded by injectivity [15]) is cyclic, therefore it can be regarded as a direct summand in Φ. So, we regard l as a subspace in $\mathcal{G}$ and denote by $e \in \tilde{X}'$ the projection onto l. Then $\Phi(X)e \cong C^*(A \cup A')$. If C_e is the central carrier of e in $\tilde{X}$, the map

$$\tilde{X}C_e \to \tilde{X}e, \quad x \mapsto xe$$

is an isomorphism of von Neumann algebras [26, p. 335], hence normal, and maps the C^*-subalgebra $\Phi(A \otimes 1)$ of $\tilde{X}$ onto $\Phi(A \otimes 1)e \cong A$. Since the representation $A \ni a \mapsto \Phi(a \otimes 1)|e\mathcal{G}$ is just the identity, it is normal, hence the representation $A \ni a \mapsto \Phi(a \otimes 1)|C_e\mathcal{G}$ is also normal. This implies that $C_e \le P$ (using [26, Theorem 10.1.13]), hence $\tilde{X}C_e \subseteq P\tilde{X}P = PX^{\sharp\sharp}P = X^{\natural_{\mathrm{p}}\natural_{\mathrm{p}}}$ by Theorem 4.9.

If the operator A-bimodule structure on $X^{\natural_{\rm p}\natural_{\rm p}}$ is minimal, the same holds for the subbimodule $\tilde{X}C_e$, hence also for the completely isometric A-bimodule $\tilde{X}e$. But $\tilde{X}e \cong {\rm B}(\mathcal{L})$, thus ${\rm B}(\mathcal{L})$ carries the minimal operator A-bimodule structure, hence by (2.2) A is a norming subalgebra of ${\rm B}(\mathcal{L})$. But this is a contradiction since by [41, Theorem 2.7] A is norming for ${\rm B}(\mathcal{L})$ only if l is locally cyclic for A.

Remark 6.9. By Proposition 6.5(i) and Corollary 3.5(i)

$$\mathrm{MIN}_{A'}(X^{\natural_{\rm p}})_{B'} = \left(\mathcal{H}^* \overset{h}{\otimes}_A \mathrm{MAXN}_A(X)_B \overset{h}{\otimes}_B \mathcal{K}\right)^\sharp =: V^\sharp;$$

in particular $X^{\natural_{\rm p}}$ is the Banach space dual of V. In general there is no operator space structure on V such that $Y := \mathrm{MAXN}_{A'}(X^{\natural_{\rm p}})_{B'}$ is the operator space dual of V. The presence of such structure means by Theorem 2.9 that Y is a dual normal operator A', B'-bimodule, hence by maximality it must coincide with $\mathrm{MAXND}_{A'}(X^{\natural_{\rm p}})_{B'}$, hence also with $(\mathrm{MIN}_A(X)_B)^{\natural_{\rm p}}$ by Proposition 6.5(ii). Thus, Example 6.8 gives an operator space Y which is the dual of a Banach space V, but not the operator space dual of V for any operator space structure on V. This phenomenon was discovered in [28] and a simple example is also in [16].

As noted also by Pop [40], the norm of an element $x \in \mathrm{M}_n(\mathrm{MAX}_A(X)_B)$ is given by

$$(6.9)\quad \|x\|_{{}_A\mathrm{M}_B} = \inf\{\|a\|\|d\|\|b\| :\ x = adb,\\ a \in \mathrm{M}_{n,k}(A),\ b \in \mathrm{M}_{k,n}(B),\ d \in \ell^k_\infty(X),\ k \in \mathbb{N}\},$$

where $\ell^k_\infty(X)$ denotes the diagonal $k \times k$ matrices with the entries in X. This can be proved similarly as Paulsen's maximal operator space formula [38].

Problem. If $X \in {}_A\mathrm{NRM}_B$, can the norm of an element $x \in \mathrm{M}_n(\mathrm{MAXN}_A(X)_B)$ be expressed simply by allowing in (6.9) k to take infinite values ?

For central bimodules the answer is affirmative.

Proposition 6.10. *If C is Abelian and $X \in \mathrm{CNRM}_C$, for $x \in \mathrm{M}_n(\mathrm{MAXN}_C(X)_C)$*

$$(6.10)\quad \|x\| = \inf\{\|a\|\|d\|\|b\| :\ x = adb\ ,\\ a \in \mathrm{M}_{n,\mathbb{J}}(C),\ b \in \mathrm{M}_{\mathbb{J},n}(C),\ d \in \ell^{\mathbb{J}}_\infty(X)\},$$

where $\mathbb{J}$ is a sufficiently large index set.

Proof. The only thing to be proved, which is not already present in [38], is that the operator bimodule structure introduced on X by norms (6.10) is normal. By Lemma 5.3 we may assume that C is σ-finite. If the normality condition (5.3) is not satisfied, there exist an $x \in \mathrm{M}_n(X)$, a sequence of orthogonal projections $q_j \in C$ with the sum 1 and a constant $M < \|x\|$ such that $\|q_j x\| < M$ for all j. For each j choose $a_j = q_j a \in \mathrm{M}_{n,\infty}(C)$, $b_j = b_j q_j \in \mathrm{M}_{\infty,n}(C)$ and $d_j \in \ell_\infty(X)$ such that

$$(6.11)\qquad q_j x = a_j d_j b_j,\ \|a_j\| \leq 1,\ \|b_j\| \leq 1 \text{ and } \|d_j\| < M.$$

Set

$$a = [a_1, a_2, \ldots] \in \mathrm{M}_{n,\infty}(C),\ b = [b_1, b_2, \ldots]^T \in \mathrm{M}_{\infty,n}(C) \text{ and } d = \oplus_j d_j \in \ell_\infty(X).$$

Then $x = \sum_j q_j x = \sum_j a_j d_j b_j = adb$ and (since q_j's are central) $\|a\| = \sup_j \|a_j\|$, $\|b\| = \sup_j \|b_j\|$, $\|d\| = \sup_j \|d_j\|$. Hence $\|x\| < M$, a contradiction. $\square$

7. The projective tensor product of central bimodules

Throughout this section C is a unital Abelian C^*-algebra and $X, Y \in \mathrm{CRM}_C$.

Let $X \overset{\gamma}{\otimes}_C Y$ be the quotient of $X \overset{\gamma}{\otimes} Y$ by the closed subspace generated by all elements of the form $xc \otimes y - x \otimes cy$ ($x \in X$, $y \in Y$, $c \in C$). First we shall prove that $X \overset{\gamma}{\otimes}_C Y$ is representable. (In classical terminology, this simplifies the definition of the tensor product of C-locally convex modules [25].)

As in Remark 5.2, we consider the canonical decompositions $X \to \oplus_{t\in\Delta} X(t)$ and $Y \to \oplus_{t\in\Delta} Y(t)$ along the spectrum Δ of C. For each $t \in \Delta$ the bilinear map

$$\kappa_t : X \times Y \to X(t) \overset{\gamma}{\otimes} Y(t), \quad \kappa_t(x, y) = x(t) \otimes y(t)$$

induces a contraction $\tilde{\kappa}_t : X \overset{\gamma}{\otimes}_C Y \to X(t) \overset{\gamma}{\otimes} Y(t)$. Since the kernel of $\tilde{\kappa}_t$ contains the submodule $C_t(X \overset{\gamma}{\otimes}_C Y)$ (where $C_t = \ker t$), $\tilde{\kappa}_t$ induces a contraction

$$\mu_t : (X \overset{\gamma}{\otimes}_C Y)(t) \to X(t) \overset{\gamma}{\otimes} Y(t). \tag{7.1}$$

On the other hand, the quotient map $X \otimes Y \to (X \overset{\gamma}{\otimes}_C Y)(t)$ annihilates $C_t X \otimes Y$ and $X \otimes C_t Y$, hence it induces a contraction $X(t) \overset{\gamma}{\otimes} Y(t) \to (X \overset{\gamma}{\otimes}_C Y)(t)$ by maximality of the cross norm γ, which is inverse to μ_t. Thus μ_t *is isometric.*

Theorem 7.1. *The contraction*

$$\kappa : X \overset{\gamma}{\otimes}_C Y \to \oplus_{t\in\Delta}(X(t) \overset{\gamma}{\otimes} Y(t)), \quad \kappa(x \otimes_C y) = (x(t) \otimes y(t))_{t\in\Delta} \tag{7.2}$$

is isometric, hence $X \overset{\gamma}{\otimes}_C Y$ is representable.

Set $Z = X \overset{\gamma}{\otimes}_C Y$. Since the C-bimodule $\oplus_{t\in\Delta} Z(t)$ is representable and $Z(t)$ can be identified with $X(t) \overset{\gamma}{\otimes} Y(t)$ (via μ_t), it suffices to prove that the map (7.2) is isometric. We denote the norm on $X \overset{\gamma}{\otimes}_C Y$ by γ_C. Since for each $w \in X \overset{\gamma}{\otimes}_C Y$,

$$\gamma_C(w) = \sup\{|\theta(w)| : \ \theta \in (X \overset{\gamma}{\otimes}_C Y)^\sharp, \ \|\theta\| \le 1\},$$

to prove Theorem 7.1 it suffices to show that

$$|\theta(w)| \le \sup_{t\in\Delta} \|w(t)\| \tag{7.3}$$

for each θ in the unit ball of $X \overset{\gamma}{\otimes}_C Y$, where $w(t) = \tilde{\kappa}_t(w) \in X(t) \otimes Y(t)$.

Given $\theta \in (X \overset{\gamma}{\otimes}_C Y)^\sharp$ (regarded as a bilinear form) and an open subset Λ of Δ, let us define that $\theta|\Lambda = 0$ iff $\theta(x, cy) = 0$ for all $c \in C = C(\Delta)$ with the support contained in Λ and all $x \in X$, $y \in Y$. If (Λ_j) is a family of open subsets of Δ with the union Λ and if $\theta|\Lambda_j = 0$ for all j, then a standard partition of unity argument shows that $\theta|\Lambda = 0$. It follows that there exists the largest open subset Λ of Δ such that $\theta|\Lambda = 0$; then $\Delta \setminus \Lambda$ is called the *support of* θ, denoted by $\operatorname{supp}\theta$.

Lemma 7.2. *If θ is an extreme point in the unit ball of $(X \overset{\gamma}{\otimes}_C Y)^\sharp$ then $\operatorname{supp}\theta$ is a singleton.*

Proof. We can extend θ to a contractive bilinear form on $X^{\sharp\sharp} \times Y^{\sharp\sharp}$, denoted by θ again, such that the maps

$$X^{\sharp\sharp} \ni F \mapsto \theta(F, y) \ \text{ and } \ Y^{\sharp\sharp} \ni G \mapsto \theta(x, G) \ \text{ are weak* continuous} \tag{7.4}$$

for $x \in X$ and $y \in Y$ (see [11, p. 12] if necessary). Since X and Y are representable, we may regard $X^{\sharp\sharp}$ and $Y^{\sharp\sharp}$ as normal dual operator bimodules over $\tilde{C} = C^{\sharp\sharp}$ by Proposition 4.7. In particular, we may define for each bounded Borel function f on Δ a bilinear form $f\theta$ on $X \times Y$ by

$$(f\theta)(x,y) = \theta(x, fy),$$

which satisfies

$$(cf)\theta = c(f\theta) \quad (c \in C) \tag{7.5}$$

and (by (7.4))

$$\theta(xc, y) = \theta(x, cy) \quad (c \in \tilde{C},\ x \in X,\ y \in Y). \tag{7.6}$$

Suppose that there exist two different points $t_1, t_2 \in \operatorname{supp}\theta$. Choose open neighborhoods Δ_1 and Δ_2 of t_1 and t_2 (resp.) such that $\theta|\Delta_1 \neq 0$ and $\theta|\Delta_2 \neq 0$ and let χ be the characteristic function of Δ_1. Then $\chi\theta \neq 0$ and $(1-\chi)\theta \neq 0$. (Indeed, $\chi\theta = 0$ implies for all $c \in C$ with support in Δ_1 that $c\theta = (c\chi)\theta = c(\chi\theta) = 0$ by (7.5)), hence $\theta(x, cy) = (c\theta)(x,y) = 0$ for all x, y, thus $\theta|\Delta_1 = 0$.) Further,

$$\|\chi\theta\| + \|(1-\chi)\theta\| = \|\theta\| = 1. \tag{7.7}$$

Indeed, given $x, u \in X$ and $y, v \in Y$, for suitable $\alpha, \beta \in \mathbb{C}$ of modules 1 we have

$$\begin{aligned} |(\chi\theta)(x,y)| + |((1-\chi)\theta)(u,v)| &= \alpha(\chi\theta)(x,y) + \beta((1-\chi)\theta)(u,v) \\ &= \theta(x\chi, \alpha\chi y) + \theta(u(1-\chi), \beta(1-\chi)v) \\ &= \theta(x\chi + u(1-\chi), \alpha\chi y + \beta(1-\chi)v) \\ &\leq \|x\chi + u(1-\chi)\|\|\alpha\chi y + \beta(1-\chi)v\| \\ &\leq \max\{\|x\|, \|u\|\}\max\{\|y\|, \|v\|\}, \end{aligned}$$

where we have used (7.6). This implies that $\|\chi\theta\| + \|(1-\chi)\theta\| \leq 1$ $(= \|\theta\|)$, while the reverse inequality is immediate from $\theta = \chi\theta + (1-\chi)\theta$.

Setting $s = \|\chi\theta\|$, it follows that θ is the convex combination $\theta = s(s^{-1}\chi\theta) + (1-s)((1-s)^{-1}(1-\chi)\theta)$, where $s^{-1}\chi\theta$ and (by (7.7)) $(1-s)^{-1}(1-\chi)\theta$ are in the unit ball of $(X \overset{\gamma}{\otimes}_C Y)^\sharp$. This is a contradiction since θ is extreme. □

Proof of Theorem 7.1. As we have already noted, it suffices to prove (7.3), where we may assume that θ is an extreme point in the unit ball of $X \overset{\gamma}{\otimes}_C Y$. Then by Lemma 7.2 $\operatorname{supp}\theta = \{t\}$ for some $t \in \Delta$. This implies that $\theta(XC_t, Y) = 0 = \theta(X, C_tY)$ since each $c \in C_t$ can be approximated by functions with supports in $\Delta \setminus \{t\}$. Consequently θ can be factored through $X(t) \overset{\gamma}{\otimes} Y(t)$: there exists a contraction $\theta_t \in (X(t) \overset{\gamma}{\otimes} Y(t))^\sharp$ such that $\theta = \theta_t \circ \tilde{\kappa}_t$. It follows that $|\theta(w)| \leq \|w(t)\|$ for each $w \in X \overset{\gamma}{\otimes}_C Y$. □

Corollary 7.3. *For each* $w \in X \otimes_C Y$

$$\gamma_C(w) = \inf\{\|\sum_{j=1}^n c_j\| :\ w = \sum_{j=1}^n c_j x_j \otimes_C y_j,\ c_j \in C^+,\ x_j \in B_X,\ y_j \in B_Y\}. \tag{7.8}$$

Proof. Since $X \overset{\gamma}{\otimes}_C Y \in \mathrm{CRM}_C$ by Theorem 7.1, the norm of $w \in X \overset{\gamma}{\otimes}_C Y$ is

$$\gamma_C(w) = \sup\{\|\phi(w)\| :\ \phi \in \mathrm{B}_C(X \overset{\gamma}{\otimes}_C Y, \tilde{C}),\ \|\phi\| \leq 1\},$$

where $\tilde{C}$ is the universal von Neumann envelope of C in the standard form. This follows from Corollary 3.6 and Remark 5.7 applied to the normal $\tilde{C}$-bimodule $(X \overset{\gamma}{\otimes}_C Y)^{\sharp\sharp}$ (Proposition 4.7). For w of the form $w = \sum_{j=1}^n c_j x_j \otimes_C y_j$, where $c_j \in C^+$, $\|x_j\| \leq 1$, $\|y_j\| \leq 1$, and a contraction $\phi \in \mathrm{B}_C(X \overset{\gamma}{\otimes}_C Y, \tilde{C})$ we have

$$\begin{aligned} &\|\phi(w)\| = \| \textstyle\sum_{j=1}^n c_j^{1/2} \phi(x_j \otimes_C y_j) c_j^{1/2} \| \\ &\leq \|[c_1^{1/2}, \dots, c_n^{1/2}]\| \max_j \|\phi(x_j \otimes_C y_j)\| \| \begin{bmatrix} c_1^{1/2} \\ \vdots \\ c_n^{1/2} \end{bmatrix} \| \\ &\leq \| \textstyle\sum_{j=1}^n c_j \| \max_j \|x_j \otimes_C y_j\| \leq \| \textstyle\sum_{j=1}^n c_j \|. \end{aligned}$$

This implies that $\gamma_C(w)$ is dominated by the right side of (7.8). From definition,

$$\gamma_C(w) = \inf\{\sum_{j=1}^n \lambda_j : \ w = \sum_{j=1}^n \lambda_j x_j \otimes_C y_j, \ \lambda_j \in \mathbb{R}^+, \ x_j \in B_X, \ y_j \in B_Y, \ n \in \mathbb{N}\},$$

which clearly dominates the right side of (7.8) since $\mathbb{C} \subseteq C$. □

Example 7.4. If C is an Abelian von Neumann algebra and $X, Y \in \mathrm{CNRM}_C$, then $X \overset{\gamma}{\otimes}_C Y$ is not necessarily normal representable. To show this, we modify an idea from [27, Example 3.1]. Let $U_0 \subseteq U$ and V be Banach spaces such that the contraction $U_0 \overset{\gamma}{\otimes} V \to U \overset{\gamma}{\otimes} V$ is not isometric. Choose $t_0 \in \Delta$ and set $X = \{f \in C(\Delta, U) : \ f(t_0) \in U_0\}$, $Y = C(\Delta, V)$. Then

$$X(t) = \begin{cases} U & \text{if } t \neq t_0 \\ U_0 & \text{if } t = t_0 \end{cases} \quad \text{and} \quad Y(t) = V \text{ for all } t \in \Delta.$$

Choose $w = \sum_{j=1}^n u_j \otimes v_j \in U_0 \otimes V$ so that $\|w\|_{U \overset{\gamma}{\otimes} V} < \|w\|_{U_0 \overset{\gamma}{\otimes} V}$, denote by $\tilde{u}_j$ and $\tilde{v}_j$ the constant functions $\tilde{u}_j(t) = u_j$ and $\tilde{v}_j(t) = v_j$ and set $\tilde{w} = \sum_{j=1}^n \tilde{u}_j \otimes_C \tilde{v}_j$. Then the function $t \mapsto \|\tilde{w}(t)\|$, where $\tilde{w}(t) = \sum_{j=1}^n \tilde{u}_j(t) \otimes \tilde{v}_j(t) \in X(t) \overset{\gamma}{\otimes} Y(t)$, is not continuous since $\|\tilde{w}(t_0)\| = \|w\|_{U_0 \overset{\gamma}{\otimes} V} > \|w\|_{U \overset{\gamma}{\otimes} V} = \|w(t)\|$ if $t \neq t_0$. By Proposition 5.4 (and Theorem 7.1) this implies that $X \overset{\gamma}{\otimes}_C Y$ is not normal.

Definition 7.5. If $X, Y \in \mathrm{CNRM}_C$, let $X \overset{\nu}{\otimes}_C Y$ be the completion of $X \otimes_C Y$ with the norm

$$\nu_C(w) = \sup \|\phi(w)\|, \quad (w \in X \otimes_C Y),$$

where the supremum is over all C-bilinear contractions ϕ from $X \times Y$ into normal representable C-bimodules.

Dominating the Haagerup norm on $\mathrm{MIN}(X) \otimes_C \mathrm{MIN}(Y)$, ν_C is indeed a norm.

Proposition 7.6. *If $X, Y \in \mathrm{CNRM}_C$, then $X \overset{\nu}{\otimes}_C Y \in \mathrm{CNRM}_C$ and for each bounded C-bilinear map $\psi : X \times Y \to Z \in \mathrm{CNRM}_C$ there exists a unique $\tilde{\psi} \in \mathrm{B}_C(X \overset{\nu}{\otimes}_C Y, Z)$ such that $\tilde{\psi}(x \otimes_C y) = \psi(x, y)$ for all $x \in X$, $y \in Y$, and $\|\tilde{\psi}\| = \|\psi\|$.*

Proof. That $X \overset{\nu}{\otimes}_C Y$ is a normal C-bimodule follows from Lemma 5.3, the rest follows from the definition of the norm ν_C. □

The *normal part* X_{n} *of a bimodule* $X \in {}_A\mathrm{RM}_B$ is defined in the same way as for operator bimodules (Definition 4.1).

Proposition 7.7. *(i)* $X \overset{\nu}{\otimes}_C Y = (X \overset{\gamma}{\otimes}_C Y)_{\rm n}$*, hence the canonical map*

$$X \overset{\nu}{\otimes}_C Y \to \text{ess} \oplus_{t\in\Delta} \Big(X(t) \overset{\gamma}{\otimes} Y(t)\Big)$$

is isometric.

(ii) $\nu_C(\sum_{j=1}^n x_j \otimes_C y_j) = \sup \| \sum_{j=1}^n \theta(x_j, y_j)\|$*, where the supremum is over all* C*-bilinear contractions from* $X \times Y$ *to* C*.*

(iii) The same as (ii), but with the supremum over all $\mathbb{C}$*-bilinear* C*-balanced contractions* $\theta : X \times Y \to \mathbb{C}$ *such that the map* $C \ni c \mapsto \theta(x, cy)$ *is weak* continuous for all* $x \in X$*,* $y \in Y$*.*

Proof. (i) This follows from the universal properties of $X \overset{\nu}{\otimes}_C Y$ and the normal part (Propositions 7.6, 4.2) and Theorem 5.9 since $(X \overset{\gamma}{\otimes}_C Y)(t) = X(t) \overset{\gamma}{\otimes} Y(t)$.

(ii) This is a consequence of the fact that the norm of an element w in a bimodule $Z \in \text{CNRM}_C$ is equal to $\sup\{\|\phi(w)\| : \ \phi \in Z^{\natural_{\rm p}}, \ \|\phi\| \leq 1\}$ (Theorem 2.15 or Corollary 3.6) and Remark 5.7.

(iii) For each $w = \sum_{j=1}^n x_j \otimes_C y_j \in X \otimes_C Y$ set

$$\tilde{\nu}_C(w) = \sup \| \sum_{j=1}^n \theta(x_j, y_j)\|,$$

where the supremum is over all θ as in (iii). Since for each $\rho \in C_\sharp$ of norm 1 and each C-bilinear contraction $\psi : X \times Y \to C$ the map $\theta = \rho \circ \psi$ is of required type, it follows that $\tilde{\nu}_C(w) \geq \nu_C(w)$. To prove the reverse inequality, regard a C-balanced contraction $\theta : X \times Y \to \mathbb{C}$ as a linear functional on $V := X \overset{\gamma}{\otimes}_C Y$. Since $V \in \text{CRM}_C$ by Theorem 7.1, we may equip V by, say, the minimal operator C-bimodule structure. If the functionals $c \mapsto \theta_{x,y}(c) = \theta(x, cy)$ are normal, then $\theta(w) = \lim_j \theta(c_j w)$ for each $w \in X \overset{\gamma}{\otimes}_C Y$ and net of projections $c_j \in C$ converging to 1. Thus by Theorem 4.9 $\tilde{\nu}_C(w) \leq \|\iota(w)\|$; but $\|\iota(w)\| = \nu_C(w)$ by (i). □

Remark 7.8. If $(x_j)_{j\in\mathbb{J}} \subseteq B_X$, $(y_j)_{j\in\mathbb{J}} \subseteq B_Y$ and $(c_j)_{j\subseteq\mathbb{J}} \subseteq C^+$ are such that $\sum_{j\in\mathbb{J}} c_j$ weak* converges, then the sum $\sum_{j\in\mathbb{J}} c_j x_j \otimes_C y_j$ weak* converges in every $\text{B}(\mathcal{L})$ containing $X \overset{\nu}{\otimes}_C Y$ as a normal C-subbimodule since the sum is just the product of bounded operator matrices

$$\sum_{j\in\mathbb{J}} c_j x_j \otimes_C y_j = [c_j]_{j\in\mathbb{J}}^{1/2} \text{diag}(x_j \otimes_C y_j)(c_j^{1/2})_{j\in\mathbb{J}}. \tag{7.9}$$

Theorem 7.9. *Given* $X, Y \in \text{CNRM}_C$*, let* $X \overset{\tilde{\nu}}{\otimes} Y$ *be the smallest strong* C*-bimodule containing* $X \overset{\nu}{\otimes}_C Y$*. Then every* $w \in X \overset{\tilde{\nu}}{\otimes} Y$ *can be represented as*

$$w = \sum_{j\in\mathbb{J}} c_j x_j \otimes_C y_j, \ \ x_j \in B_X, \ y_j \in B_Y, \ c_j \in C^+, \ \sum_{j\in\mathbb{J}} c_j \textit{ weak* convergent.} \tag{7.10}$$

The norm of w *is equal to* $\inf \| \sum_{j\in\mathbb{J}} c_j\|$ *over all such representations.*

Proof. For $w \in X \overset{\nu}{\otimes}_C Y$ set $g(w) = \inf \| \sum_{j\in\mathbb{J}} c_j\|$, where the infimum is over all representations of w as in (7.10). The inequality $\nu_C(w) \leq g(w)$ is proved by essentially the same computation as in the proof of Corollary 7.3. The reverse inequality follows from the maximality of ν_C since $X \otimes_C Y$ equipped with the norm g is a representable normal C-bimodule. The verification of normality, based on Lemma 5.3, is similar to the proof of Proposition 6.10 and will be omitted.

To prove that $X \overset{\nu}{\tilde{\otimes}} Y$ consists of elements of the form (7.10), we may assume (by a direct sum decomposition) that C is σ finite. Then the index set $\mathbb{J}$ in (7.10) may be taken to be countable. Given w as in (7.10), it follows by the Egoroff theorem [44, p. 85] that there exists an orthogonal sequence of projections $p_k \in C$ with the sum 1 such that the sum $\sum_{j\in\mathbb{J}} c_j p_k$ is norm convergent for each k. Then the sum $w_k := \sum_{j\in\mathbb{J}} c_j p_k x_j \otimes_C y_j$ is also norm convergent (to see this, write w_k in the form similar to (7.9)), hence $w_k \in X \overset{\nu}{\otimes}_C Y$ and $w = \sum_k w_k p_k \in X \overset{\nu}{\tilde{\otimes}} Y$. Conversely, for each $w \in X \overset{\nu}{\tilde{\otimes}} Y$ there exists an orthogonal sequence of projections $p_k \in C$ such that $wp_k \in X \overset{\nu}{\otimes}_C Y$ by [31, Proposition 2.2]. By the first paragraph of the proof $wp_k = \sum_j c_{jk} x_{jk} \otimes_C y_{jk}$ for some elements $x_{jk} \in B_X$, $y_{jk} \in B_Y$ and $c_{jk} = c_{jk} p_k \in C^+$ such that $\|\sum_j c_{jk}\| < \|wp_k\| + \varepsilon$, where $\varepsilon > 0$. Then $\|\sum_{j,k} c_{jk}\| \leq \|w\| + \varepsilon$ and $w = \sum_{j,k} c_{jk} x_{jk} \otimes_C y_{jk}$. This proves that $\inf \|\sum_{j=1}^n c_j\| \leq \|w\|$; the reverse inequality is clear from (7.9). ☐

Corollary 7.10. *If $X_0 \subseteq X$ and $Y_0 \subseteq Y$ in CNRM_C are strong, then the canonical map $X \overset{\nu}{\tilde{\otimes}} Y \to (X/X_0) \overset{\nu}{\tilde{\otimes}} (Y/Y_0)$ maps the open unit ball onto the open unit ball.*

8. Normality of the injective operator bimodule tensor product

If $X \in {}_A\mathrm{RM}_B$ and $Y \in {}_B\mathrm{RM}_C$, the *injective tensor seminorm* [2] on $X \otimes_B Y$ is

$$\Lambda^B_{A,C}(\sum_{j=1}^n x_j \otimes_B y_j) = \sup \|\sum_{j=1}^n \phi(x_j)\psi(y_j)\|, \tag{8.1}$$

where the supremum is over all contractions $\phi \in \mathrm{B}_A(X, \mathrm{B}(\mathcal{K}, \mathcal{H}))_B$ and $\psi \in \mathrm{B}_B(Y, \mathrm{B}(l, \mathcal{K}))_C$ with $\mathcal{H}$, $\mathcal{K}$ and l varying over all cyclic Hilbert modules over A, B and C (resp.).

If A, B and C are von Neumann algebras and $X \in {}_A\mathrm{NRM}_B$, $Y \in {}_B\mathrm{NRM}_C$, we define a norm $\lambda^B_{A,C} \leq \Lambda^B_{A,C}$ on $X \otimes_B Y$ by the same formula (8.1), but requiring in addition that $\mathcal{H}$, $\mathcal{K}$ and l are normal. (To show that $\lambda^B_{A,C}$ is definite, suppose that $w = x \odot_B y = \sum_{j=1}^n x_j \otimes y_j \in X \otimes_B Y$ is such that $\sum_{j=1}^n \phi(x_j)\psi(y_j) = 0$ for all ϕ and ψ. Assuming that $A \subseteq \mathcal{H}_A$, $B \subseteq \mathcal{H}_B$, $C \subseteq \mathcal{H}_C$, $X \subseteq \mathrm{B}(\mathcal{H}_B, \mathcal{H}_A)$ and $Y \subseteq \mathrm{B}(\mathcal{H}_C, \mathcal{H}_B)$ and decomposing $\mathcal{H}_A$, $\mathcal{H}_B$ and $\mathcal{H}_C$ into direct sums of cyclic submodules, it follows that $xB'y = 0$, which means that $x \odot_B y = 0$.)

We would like to show that $\Lambda^B_{A,C} = \lambda^B_{A,C}$, but first we observe below that $\Lambda^B_{A,C}$ and $\lambda^B_{A,C}$ are independent of A and C. We simplify the notation by $\lambda_B := \lambda^B_{\mathbb{C},\mathbb{C}}$ and $\Lambda_B := \Lambda^B_{\mathbb{C},\mathbb{C}}$. Note that the extension theorem for completely bounded bimodule maps [45] (together with Theorem 2.4) implies that both norms $\Lambda^B_{A,C}$ and $\lambda^B_{A,C}$ are preserved under isometric embeddings of bimodules.

Proposition 8.1. *The seminorms $\Lambda^B_{A,C}$ and $\lambda^B_{A,C}$ do not depend on A and C.*

Proof. We shall only prove the statement about $\Lambda^B_{A,C}$, the proof for $\lambda^B_{A,C}$ is similar. Choose $\varepsilon > 0$. Given $w = \sum_{j=1}^n x_j \otimes_B y_j \in X \otimes_B Y$ and contractions

$\phi \in \mathrm{B}_A(X, \mathrm{B}(\mathcal{K}, \mathcal{H}))_B$, $\psi \in \mathrm{B}_B(Y, \mathrm{B}(l, \mathcal{K}))_C$ as in (8.1), we choose unit vectors $\xi \in \mathcal{H}$ and $\eta \in l$ such that

$$|\langle \sum_{j=1}^n \phi(x_j)\psi(y_j)\eta, \xi\rangle| > \| \sum_{j=1}^n \phi(x_j \psi(y_j)\| - \varepsilon.$$

Then

$$\alpha : X \to \mathcal{K}^*, \ \alpha(x) = (\phi(x)^*\xi)^* \quad \text{and} \quad \beta : Y \to \mathcal{K}, \ \beta(y) = \psi(y)\eta$$

are B-module contractions such that

$$|\sum_{j=1}^n \langle \beta(y_j), \alpha(x_j)^*\rangle| > \| \sum_{j=1}^n \phi(x_j)\psi(y_j)\| - \varepsilon.$$

This implies that $\Lambda_B(w) \geq \Lambda_{A,C}^B(w)$.

For the reverse inequality, let $\pi : B \to \mathrm{B}(\mathcal{K})$ be a cyclic representation and let $\alpha \in \mathrm{B}(X, \mathcal{K}^*)_B$, $\beta \in \mathrm{B}_B(Y, \mathcal{K})$ be contractions such that

$$|\sum_{j=1}^n \langle \beta(y_j), \alpha(x_j)^*\rangle| > \Lambda_B(w) - \varepsilon. \tag{8.2}$$

Since $\Lambda_{A,C}^B$ is preserved by inclusions we may assume that X and Y are C*-algebras containing $A \cup B$ and $B \cup C$ (resp.). Then, since α and β can be regarded as complete contractions by Theorem 2.4, it follows by [37, p. 102] that there exist Hilbert spaces $\mathcal{H}$ and l, $*$-representations $\Phi : X \to \mathrm{B}(\mathcal{H})$ and $\Psi : Y \to \mathrm{B}(\mathcal{L})$, unit vectors $\xi \in \mathcal{H}$ and $\eta \in l$ and contractions $S \in \mathrm{B}(\mathcal{K}, \mathcal{H})$, $T \in \mathrm{B}(l, \mathcal{K})$ such that

$$\alpha(x) = \xi^*\Phi(x)S \quad \text{and} \quad \beta(y) = T\Psi(y)\eta. \tag{8.3}$$

Further, we may assume that $[\Phi(X)\xi] = \mathcal{H}$ and $[\Psi(Y)\eta] = l$. Then it follows from (8.3) (since α and β are B-module maps) that

$$\Phi(b)S = S\pi(b) \quad \text{and} \quad T\Psi(b) = \pi(b)T \quad (b \in B). \tag{8.4}$$

Replace $\mathcal{H}$ with the subspace $\mathcal{H}_1 = [\Phi(A)\xi]$ and l with $l_1 = [\Psi(C)\eta]$ and define

$$\phi : X \to \mathrm{B}(\mathcal{K}, \mathcal{H}_1), \ \phi(x) = P\Phi(x)S \quad \text{and} \quad \psi : Y \to \mathrm{B}(l_1, \mathcal{K}), \ \psi(y) = T\Psi(y)|l_1,$$

where $P \in \mathrm{B}(\mathcal{H})$ is the orthogonal projection onto $\mathcal{H}_1$. Then by (8.3)

$$\alpha(x) = \xi^*\phi(x) \ (x \in X) \quad \text{and} \quad \beta(y) = \psi(y)\eta \ (y \in Y), \tag{8.5}$$

$\mathcal{H}_1$, $\mathcal{K}$ and l_1 are cyclic modules over A, B and C (resp.) and (8.4) (together with the fact that $\mathcal{H}_1$ and l_1 are invariant under $\Phi(A)$ and $\Psi(C)$, resp.) implies that ϕ and ψ are bimodule maps. Since from (8.5) and (8.2) we have that

$$\| \sum_{j=1}^n \phi(x_j)\psi(y_j)\| \geq |\langle \sum_{j=1}^n \phi(x_j)\psi(y_j)\eta, \xi\rangle| = |\sum_{j=1}^n \langle \beta(y_j), \alpha(x_j)^*\rangle| > \Lambda_B(w) - \varepsilon,$$

it follows that $\Lambda_{A,C}^B(w) \geq \Lambda_B(w)$. □

Lemma 8.2. *If K is a $B, \mathbb{C}$-absolutely convex (Definition 2.11) weak* compact subset of a von Neumann algebra B, then the set $L = \{x^*x : \ x \in K\}$ is convex and weak* compact.*

Proof. Given $x, y \in K$ and $t \in [0,1]$, consider the polar decomposition

$$\begin{bmatrix} \sqrt{t}x \\ \sqrt{1-t}y \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix} z,$$

where $z = \sqrt{tx^*x + (1-t)y^*y}$ and $[u, v]^T$ is the partial isometric part. Since

$$z = [u^* \ v^*] \begin{bmatrix} \sqrt{t}x \\ \sqrt{1-t}y \end{bmatrix} = u^*x\sqrt{t} + v^*y\sqrt{1-t}$$

and K is $B, \mathbb{C}$-absolutely convex, $z \in K$. It follows that $tx^*x+(1-t)y^*y = z^*z \in L$, proving that L is convex.

Since K (hence also L) is bounded, it suffices now to prove that L is closed in the strong operator topology (SOT). Let y be in the closure of L and (x_j) a net in K such that $(x_j^*x_j)$ converges to y in the SOT. Since the function $x \mapsto \sqrt{x}$ is SOT continuous on bounded subsets of B^+, the net $(|x_j|)$ converges to $\sqrt{y}$. Since K is $B, \mathbb{C}$-absolutely convex, the polar decomposition shows that $|x_j| \in K$. Since K is weak* closed, it follows that $\sqrt{y} \in K$, hence $y \in L$. □

Theorem 8.3. *For all* $X \in \mathrm{NRM}_B$ *and* $Y \in {}_B\mathrm{NRM}$, $\Lambda_B = \lambda_B$ *on* $X \otimes_B Y$.

Proof. The theorem will be proved first for free modules by translating the problem to states on B and approximating states by normal states. Then elements of general modules will be approximated by elements of free modules.

First assume that X and Y are free with basis $\{x_1, \ldots, x_n\}$ and $\{y_1, \ldots, y_n\}$, respectively. More precisely, set

$$x = [x_1, \ldots, x_n], \quad y = [y_1, \ldots, y_n]^T$$

and assume that the two maps

$$f : \mathrm{C}_n(B) \to X, \ f(b) = xb \ \text{ and } \ g : \mathrm{R}_n(B) \to Y, \ g(b) = by$$

are invertible (with bounded inverses by the open mapping theorem). Set

$$\mathcal{S} = \{b \in \mathrm{C}_n(B) : \ \|xb\| \leq 1\}, \quad \mathcal{T} = \{b \in \mathrm{R}_n(B) : \ \|by\| \leq 1\}$$

and

$$\alpha = \sup\{\|b\| : \ b \in \mathcal{S} \cup \mathcal{T}\}.$$

Let $0 < \varepsilon < 1$. Choose $w \in X \otimes_B Y$ and note that w can be written as

$$w = \sum_{i,j=1}^{n} x_i \otimes_B d_{ij} y_j \quad (d_{ij} \in B).$$

By the definition of Λ_B there exist a cyclic representation $\pi : B \to \mathrm{B}(\mathcal{K})$ and contractions $\phi \in \mathrm{B}(X, \mathcal{K}^*)_B$, $\psi \in \mathrm{B}_B(Y, \mathcal{K})$ such that

$$|\sum_{i,j=1}^{n} \langle \pi(d_{ij})\psi(y_j), \phi(x_i)^* \rangle| > \Lambda_B(w) - \varepsilon. \tag{8.6}$$

Let $\xi_0 \in \mathcal{K}$ be a unit cyclic vector for $\pi(B)$, ρ the state $\rho(b) = \langle \pi(b)\xi_0, \xi_0 \rangle$ on B, and choose $a_i, c_i \in B$ so that

$$\|\phi(x_i)^* - \pi(a_i^*)\xi_0\| < \varepsilon \ \text{ and } \ \|\psi(y_i) - \pi(c_i)\xi_0\| < \varepsilon \quad (i = 1, \ldots, n). \tag{8.7}$$

For $b = [b_{ij}] \in \mathrm{M}_{m,n}(B)$ denote the matrix $[\pi(b_{ij})]$ simply by $\pi(b)$. Set

$$\xi = [\phi(x_1)^*, \ldots, \phi(x_n)^*]^T \ \ (\in \mathcal{K}^n), \quad \eta = [\psi(y_1), \ldots, \psi(y_n)]^T \ \ (\in \mathcal{K}^n), \tag{8.8}$$

$$a = [a_1, \ldots, a_n] \text{ and } c = [c_1, \ldots, c_n]^T.$$

Then from (8.7)

$$\|\xi - \pi(a)^*\xi_0\| < \varepsilon\sqrt{n} \text{ and } \|\eta - \pi(c)\xi_0\| < \varepsilon\sqrt{n}. \tag{8.9}$$

Since ϕ and ψ are contractive B-module mappings,

$$\|\pi(b)^*\xi\| \leq \|xb\| \ (b \in \mathrm{C}_n(B)) \text{ and } \|\pi(b)\eta\| \leq \|by\| \ (b \in \mathrm{R}_n(B)). \tag{8.10}$$

Thus, if $b \in \mathcal{S}$, then

$$\begin{aligned} \rho(abb^*a^*) &= \|\pi(b^*a^*)\xi_0\|^2 \\ &\leq (\|\pi(b)^*\xi\| + \|\pi(b)^*(\pi(a)^*\xi_0 - \xi)\|)^2 \\ &\leq (\|xb\| + \|\pi(b)\|\varepsilon\sqrt{n})^2 \text{ (by (8.10) and (8.9)} \\ &\leq (1 + \alpha\varepsilon\sqrt{n})^2 \text{ (by definition of } \mathcal{S} \text{ and } \alpha) \\ &= :\beta. \end{aligned}$$

Similar arguments are valid for $b \in \mathcal{T}$, hence

$$\rho(abb^*a^*) \leq \beta \ (b \in \mathcal{S}) \quad \text{and} \quad \rho(c^*b^*bc) \leq \beta \ (b \in \mathcal{T}). \tag{8.11}$$

Set

$$K_1 = \{b^*a^* : \ b \in \mathcal{S}\}, \quad K_2 = \{bc : \ b \in \mathcal{T}\},$$

$$L_1 = \{v^*v : \ v \in K_1\}, \quad L_2 = \{v^*v : \ v \in K_2\}.$$

Since X and Y are normal and f, g invertible, $\mathcal{S}$ and $\mathcal{T}$ are weak* compact, hence such are also K_1 and K_2. To verify that, say $\mathcal{T}$ (hence also K_2), is $B,\mathbb{C}$-absolutely convex, let $b_j \in \mathcal{T}$ $(j = 1, \ldots, n)$ and let $\lambda_j \in \mathbb{C}$ and $d_j \in B$ satisfy $\sum |\lambda_j|^2 \leq 1$ and $\sum d_j d_j^* \leq 1$. Then to show that $\sum(d_j b_j \lambda_j)$ is in $\mathcal{T}$, just note that $\|(\sum d_j b_j \lambda_j)y\| = \|\sum d_j(b_j y)\lambda_j\| \leq 1$.

Now it follows by Lemma 8.2 that L_1 and L_2 are convex weak* compact subsets of B, hence the same holds for the convex hull $\mathrm{co}(L_1 \cup L_2)$ and

$$L = \mathrm{co}(L_1 \cup L_2) - B^+$$

is weak* closed. Set

$$L^\circ = \{\theta \in B^\sharp : \ \mathrm{Re}(\theta(v)) \leq 1 \ \forall v \in L\} \text{ and } L_\circ = L^\circ \cap B_\sharp.$$

Since L is weak* closed and convex, $L_\circ$ is weak* dense in L° by a variant of the bipolar theorem. From (8.11), $\rho \in \beta(L_1^\circ \cap L_2^\circ) = \beta(\mathrm{co}(L_1 \cup L_2))^\circ$, hence (since ρ is positive) $\rho \in \beta L^\circ$. Since $L_\circ$ is weak* dense in L°, there exists an $\omega_0 \in \beta L_\circ$ such that

$$|(\omega_0 - \rho)(\sum_{i,j=1}^{n} a_i d_{ij} c_j)| < \varepsilon \text{ and } |(\omega_0 - \rho)(1)| < \varepsilon. \tag{8.12}$$

Since $L \supseteq -B^+$ and $\omega_0 \in \beta L_\circ$, ω_0 is positive, hence $\omega = \omega_0/\omega_0(1)$ is a state. From $\|\omega - \omega_0\| = \|(1 - \omega_0(1))\omega\| = |1 - \omega_0(1)| < \varepsilon$ and (8.12) we have that

$$|(\omega - \rho)(\sum_{i,j=1}^{n} a_i d_{ij} c_j)| < D\varepsilon, \tag{8.13}$$

where $D = 1 + \|\sum_{i,j=1}^{n} a_i d_{ij} c_j\|$. Let $\sigma : B \to \mathrm{B}(\mathcal{H})$ be the normal representation constructed from ω by the GNS construction and let $\eta_0 \in \mathcal{H}$ be the corresponding unit cyclic vector. From (8.13), (8.7) and (8.6) we deduce that

$$\begin{array}{rcl} |\sum_{i,j=1}^n \langle\sigma(a_i d_{ij} c_j)\eta_0, \eta_0\rangle| & > & |\sum_{i,j=1}^n \langle\pi(a_i d_{ij} c_j)\xi_0, \xi_0\rangle - D\varepsilon \\ & > & |\sum_{i,j=1}^n \langle\pi(d_{ij})\psi(y_j), \phi(x_i)^*\rangle| - D\varepsilon \\ & & -n^2\varepsilon \max_{i,j} \|d_{ij}\|(\|x\| + \|y\| + \varepsilon) \\ & > & \Lambda_B(w) - r(\varepsilon), \end{array} \tag{8.14}$$

where $r(\varepsilon)$ tends to 0 as $\varepsilon \to 0$.

Define $\Phi_0 \in \mathrm{B}(X, \mathcal{H}^*)_B$ and $\Psi_0 \in \mathrm{B}_B(Y, \mathcal{H})$ by

$$\Phi_0(\sum_{j=1}^n x_j b_j) = (\sum_{j=1}^n \sigma(b_j^* a_j^*)\eta_0)^*, \quad \Psi_0(\sum_{j=1}^n b_j y_j) = \sum_{j=1}^n \sigma(b_j c_j)\eta_0 \quad (b_j \in B). \tag{8.15}$$

Since $\omega_0 \in \beta L_\circ$, $\omega = \omega_0/\omega_0(1)$ and $\|\omega - \omega_0\| < \varepsilon$, we have that $\omega \in \omega_0(1)^{-1}\beta L_\circ \subseteq (1-\varepsilon)^{-1}\beta L_\circ$, hence it follows from (8.15) and the definition of the set L that

$$\|\Phi_0(xb)\|^2 = \|\sigma(b^*a^*)\eta_0\|^2 = \omega(abb^*a^*) \leq (1-\varepsilon)^{-1}\beta \quad (b \in \mathcal{S})$$

and similarly

$$\|\Psi_0(by)\|^2 \leq (1-\varepsilon)^{-1}\beta \quad (b \in \mathcal{T}).$$

Thus, with $\delta = (1-\varepsilon)^{-1/2}\beta^{1/2} = (1-\varepsilon)^{-1/2}(1 + \alpha\varepsilon\sqrt{n})$, we have (recalling the definitions of $\mathcal{S}$ and $\mathcal{T}$) that $\|\Phi_0\| \leq \delta$ and $\|\Psi_0\| \leq \delta$. From (8.15), $\Phi_0(x_j) = (\sigma(a_j^*)\eta_0)^*$ and $\Psi_0(y_j) = \sigma(c_j)\eta_0$, hence we may rewrite (8.14) as

$$|\sum_{i,j=1}^n \langle\sigma(d_{ij})\Psi_0(y_j), \Phi_0(x_i)^*\rangle| > \Lambda_B(w) - r(\varepsilon).$$

Finally, setting $\Phi = \frac{1}{\delta}\Phi_0$ and $\Psi = \frac{1}{\delta}\Psi_0$, we have a normal cyclic Hilbert module $\mathcal{H}$ and contractions $\Phi \in \mathrm{B}(X, \mathcal{H}^*)_B$, $\Psi \in \mathrm{B}_B(Y, \mathcal{H})$ such that $|\sum\langle\sigma(d_{ij})\Psi(y_j), \Phi(x_i)^*\rangle|$ approaches $\Lambda_B(w)$ as ε tends to 0 since $r(\varepsilon) \to 0$ and $\delta \to 1$. Thus $\Lambda_B(w) = \lambda_B(w)$.

In general, when X and Y are not free, let $w = \sum_{j=1}^n x_j \otimes_B y_j \in X \otimes_B Y$ and

$$X_1 = X \oplus \mathrm{R}_n(B) \quad \text{and} \quad Y_1 = Y \oplus \mathrm{C}_n(B).$$

Since both norms Λ_B and λ_B respect isometric embeddings, it suffices to prove that $\Lambda_B(w) \leq \lambda_B(w)$ in $X_1 \otimes_B Y_1$. For each real $t > 0$ put

$$w(t) = \sum_{j=1}^n (x_j, te_j^T) \otimes_B (y_j, te_j),$$

where $e_j = (0, \ldots, 1, \ldots, 0) \in \mathrm{C}_n(\mathbb{C}) \subseteq \mathrm{C}_n(B)$. Since the elements $x_j(t) := (x_j, te_j^T)$ $(j = 1, \ldots, n)$ generate a free module in the above sense and similarly the $y_j(t) := (y_j, te_j)$, it follows that $\Lambda_B(w(t)) = \lambda_B(w(t))$. But, as t tends to 0, $\Lambda_B(w(t))$ tends to $\Lambda_B(w)$ (since $\Lambda_B(w(t) - w) \leq t\sum_{j=1}^n(\|x_j\| + \|y_j\| + t)$) and $\lambda_B(w(t))$ tends to $\lambda_B(w)$, hence $\Lambda_B(w) = \lambda_B(w)$. □

By Theorem 8.3 and Proposition 8.1 the injective norm is given by (8.1) where $\mathcal{H}$, $\mathcal{K}$ and l are normal, hence we conclude:

Corollary 8.4. *If $X \in {}_A\mathrm{NRM}_B$ and $Y \in {}_B\mathrm{NRM}_C$, then $X \overset{\Lambda}{\otimes}_B Y \in {}_A\mathrm{NRM}_C$.*

9. The dual of the extended Haagerup tensor product of bimodules

Effros and Ruan [19] defined the *normal Haagerup tensor product* of dual operator spaces by $U^\sharp \overset{\sigma h}{\otimes} V^\sharp := (U \overset{eh}{\otimes} V)^\sharp$. Similarly (recall Theorem 3.7):

Definition 9.1. For $X \in {}_A\mathrm{NDOM}_B$ and $Y \in {}_B\mathrm{NDOM}_C$ let

$$X \overset{\sigma h}{\otimes}_B Y = (X_{\natural_{\mathrm{p}}} \overset{eh}{\otimes}_{B'} Y_{\natural_{\mathrm{p}}})^{\natural_{\mathrm{p}}},$$

where $X_{\natural_{\mathrm{p}}} \overset{eh}{\otimes}_{B'} Y_{\natural_{\mathrm{p}}}$ is regarded as an A', C'-bimodule.

Theorem 9.2. $X \overset{\sigma h}{\otimes}_B Y = (X \overset{\sigma h}{\otimes} Y)/N$, *where N is the weak* closed subspace of $X \overset{\sigma h}{\otimes} Y$ generated by all elements of the form $xb \otimes y - x \otimes by$ ($x \in X,\ y \in Y,\ b \in B$).*

Proof. Let $\mathcal{H}$, $\mathcal{K}$ and l be proper Hilbert modules over A, B and C (resp.) in terms of which the duals are defined. By Corollary 3.5(i)

$$X \overset{\sigma h}{\otimes}_B Y = (X_{\natural_{\mathrm{p}}} \overset{eh}{\otimes}_{B'} Y_{\natural_{\mathrm{p}}})^{\natural_{\mathrm{p}}} = (\mathcal{H}^* \overset{eh}{\otimes}_{A'} X_{\natural_{\mathrm{p}}} \overset{eh}{\otimes}_{B'} Y_{\natural_{\mathrm{p}}} \overset{eh}{\otimes}_{C'} l)^\sharp.$$

By [18], [5] $\mathcal{K} \overset{eh}{\otimes} \mathcal{K}^* = (\mathcal{K}^* \overset{h}{\otimes} \mathcal{K})^\sharp = \mathrm{B}(\mathcal{K})$, hence $B' \subseteq \mathcal{K} \overset{eh}{\otimes} \mathcal{K}^*$ and it follows that

$$U := \mathcal{H}^* \overset{eh}{\otimes}_{A'} X_{\natural_{\mathrm{p}}} \overset{eh}{\otimes}_{B'} Y_{\natural_{\mathrm{p}}} \overset{eh}{\otimes}_{C'} l = \mathcal{H}^* \overset{eh}{\otimes}_{A'} X_{\natural_{\mathrm{p}}} \overset{eh}{\otimes}_{B'} B' \overset{eh}{\otimes}_{B'} Y_{\natural_{\mathrm{p}}} \overset{eh}{\otimes}_{C'} l$$

is an operator subspace of

$$\begin{aligned} V: &= \mathcal{H}^* \overset{eh}{\otimes}_{A'} X_{\natural_{\mathrm{p}}} \overset{eh}{\otimes}_{B'} \mathrm{B}(\mathcal{K}) \overset{eh}{\otimes}_{B'} Y_{\natural_{\mathrm{p}}} \overset{eh}{\otimes}_{C'} l \\ &= \mathcal{H}^* \overset{eh}{\otimes}_{A'} X_{\natural_{\mathrm{p}}} \overset{eh}{\otimes}_{B'} \mathcal{K} \overset{eh}{\otimes} \mathcal{K}^* \overset{eh}{\otimes}_{B'} Y_{\natural_{\mathrm{p}}} \overset{eh}{\otimes}_{C'} l. \end{aligned}$$

Note that $V = X_\sharp \overset{eh}{\otimes} Y_\sharp$ by Corollary 3.9. The adjoint of the inclusion $U \to V$ is the weak* continuous completely quotient map

$$q : X \overset{\sigma h}{\otimes} Y = V^\sharp \to U^\sharp = X \overset{\sigma h}{\otimes}_B Y$$

with $\ker q = U^\perp$, the annihilator of U in $V^\sharp$. It remains to prove that $U^\perp = N$ or equivalently, since N is weak* closed, that $U = N_\perp\ (\subseteq V)$.

A general element v of V has the form $v = \xi^* \odot_{A'} \phi \odot_{B'} T \odot_{B'} \psi \odot_{C'} \eta$, where $\xi \in \mathrm{C}_{\mathbb{J}}(\mathcal{H}),\ \eta \in \mathrm{C}_{\mathbb{J}}(l),\ \phi = [\phi_{ij}] \in \mathrm{M}_{\mathbb{J}}(X_{\natural_{\mathrm{p}}}),\ \psi = [\psi_{ij}] \in \mathrm{M}_{\mathbb{J}}(Y_{\natural_{\mathrm{p}}}),\ T \in \mathrm{M}_{\mathbb{J}}(\mathrm{B}(\mathcal{K}))$ for some cardinal $\mathbb{J}$. The condition $v \in N_\perp$ means that

$$\langle v, xb \otimes y - x \otimes by \rangle = 0 \ \text{ for all } x \in X,\ y \in Y,\ b \in B$$

and can be written as $\langle (\phi(xb)T\psi(y) - \phi(x)T\psi(by))\eta, \xi \rangle = 0$ or

$$\langle \phi(X)(bT - Tb)\psi(Y)\eta, \xi \rangle = 0. \tag{9.1}$$

Since $[\psi(Y)\eta]$ is a B-submodule of $\mathrm{C}_{\mathbb{J}}(\mathcal{K}) = \mathcal{K}^{\mathbb{J}}$, we have that $[\psi(Y)\eta] = q'\mathcal{K}^{\mathbb{J}}$ for a projection $q' \in \mathrm{M}_{\mathbb{J}}(B')$. Similarly $[\phi(X)^*\xi] = p'\mathcal{K}^{\mathbb{J}}$ for some projection $p' \in \mathrm{M}_{\mathbb{J}}(B')$ and (9.1) may be rewritten as $p'(bT - Tb)q' = 0$ ($b \in B$) or

$$p'Tq' \in \mathrm{M}_{\mathbb{J}}(B'). \tag{9.2}$$

Let $e' \in \mathrm{M}_{\mathbb{J}}(A')$ and $f' \in \mathrm{M}_{\mathbb{J}}(C')$ be the projections with ranges $[A\xi]$ and $[C\eta]$ (resp.). From $q'\psi(y)\eta = \psi(y)\eta$ ($y \in Y$) we have that $q'^\perp[\psi(Y)C\eta] = 0$ (since ψ is a C-module map), hence $q'^\perp\psi(Y)f' = 0$. This means that

$$q'^\perp \psi f' = 0 \ \text{ and similarly } \ e'\phi p'^\perp = 0. \tag{9.3}$$

Finally, it follows that

$$\begin{aligned} v &= \xi^* \odot_{A'} \phi \odot_{B'} T \odot_{B'} \psi \odot_{C'} \eta \\ &= (e'\xi)^* \odot_{A'} \phi \odot_{B'} T \odot_{B'} \psi \odot_{B'} f'\eta \\ &= \xi^* \odot_{A'} e'\phi \odot_{B'} T \odot_{B'} \psi f' \odot_{C'} \eta \\ &= \xi^* \odot_{A'} e'\phi p' \odot_{B'} T \odot_{B'} q'\psi f' \odot_{C'} \eta \ \ \text{(by (9.3))} \\ &= \xi^* \odot_{A'} e'\phi \odot_{B'} p'Tq' \odot_{B'} \psi f' \odot_{C'} \eta \\ &\in U \ \text{(by (9.2))}. \end{aligned}$$

This computation proves that $U = N_\perp$. □

We shall need a generalization of a result of Blecher, Smith [9].

Proposition 9.3. *Given normal $*$-homomorphisms $A \to R$, $A \to T$, $B \to S$ and $B \to T$ of von Neumann algebras (introducing the bimodule structures to the spaces appearing in the formula below) acting on Hilbert spaces $\mathcal{H}_A, \ldots, \mathcal{H}_T$, we have*

$$\mathrm{B}_A(\mathcal{H}_T, \mathcal{H}_R) \overset{eh}{\otimes}_{T'} \mathrm{B}_B(\mathcal{H}_S, \mathcal{H}_T) = \mathrm{NCB}_A(T, \mathrm{B}(\mathcal{H}_S, \mathcal{H}_R))_B$$

completely isometrically by

$$(a' \odot_{T'} b')(t) = a'tb' \quad (t \in T,\ a' \in \mathrm{R}_{\mathbb{J}}(\mathrm{B}_A(\mathcal{H}_T, \mathcal{H}_R)),\ b' \in \mathrm{C}_{\mathbb{J}}(\mathrm{B}_B(\mathcal{H}_S, \mathcal{H}_T)).$$

Proof. In the case $A = \mathbb{C} = B$ the proof consists of the following computation:

$$\begin{aligned} \mathrm{B}(\mathcal{H}_T, \mathcal{H}_R) \overset{eh}{\otimes}_{T'} \mathrm{B}(\mathcal{H}_S, \mathcal{H}_T) &= (\mathcal{H}_R \overset{eh}{\otimes} \mathcal{H}_T^*) \overset{eh}{\otimes}_{T'} (\mathcal{H}_T \overset{eh}{\otimes} \mathcal{H}_S^*) \\ &= \mathcal{H}_R \overset{eh}{\otimes} (\mathcal{H}_T^* \overset{eh}{\otimes}_{T'} \mathcal{H}_T) \overset{eh}{\otimes} \mathcal{H}_S^* \\ &= \mathcal{H}_R \overset{eh}{\otimes} T_\sharp \overset{eh}{\otimes} \mathcal{H}_S^* \ \ \text{(by Corollary 3.3)} \\ &\cong \mathrm{B}(\mathcal{H}_S, \mathcal{H}_R) \overline{\otimes} T_\sharp \\ &= \mathrm{NCB}(T, \mathrm{B}(\mathcal{H}_S, \mathcal{H}_R)). \end{aligned}$$

In general, we have only to show that each $\theta \in \mathrm{NCB}_A(T, \mathrm{B}(\mathcal{H}_S, \mathcal{H}_R))_B$, just proved to be of the form $\theta = a' \odot_{T'} b'$ for some $a' \in \mathrm{R}_{\mathbb{J}}(\mathrm{B}(\mathcal{H}_T, \mathcal{H}_R))$ and $b' \in \mathrm{C}_{\mathbb{J}}(\mathrm{B}(\mathcal{H}_S, \mathcal{H}_T))$, has this form with the requirement that $a' \in \mathrm{R}_{\mathbb{J}}(\mathrm{B}_A(\mathcal{H}_T, \mathcal{H}_R))$ and $b' \in \mathrm{C}_{\mathbb{J}}(\mathrm{B}_B(\mathcal{H}_S, \mathcal{H}_T))$; for this see the proof of [29, Theorem 1.2]. □

For bimodules $U \in {}_A\mathrm{OM}_B$ and $V \in {}_B\mathrm{OM}_A$ we denote by $U_A\hat{\otimes}_B V$ the quotient of the maximal operator space tensor product $U \hat{\otimes} V$ by the closed subspace N generated by $\{aub \otimes v - u \otimes bva : \ a \in A,\ b \in B,\ u \in U,\ v \in V\}$. Consider the natural completely isometric isomorphism [8]

$$\iota : \mathrm{CB}(U, V^\sharp) \to (U \hat{\otimes} V)^\sharp, \ \ \iota(\phi)(u \otimes v) = \phi(u)(v)$$

and note that $\iota(\phi)$ annihilates N if and only if $\phi \in \mathrm{CB}_A(U, V^\sharp)_B$, where $V^\sharp$ is the dual A, B-bimodule of V (Definition 2.5). Hence

$$\mathrm{CB}_A(U, V^\sharp)_B = (U_A \hat{\otimes}_B V)^\sharp. \tag{9.4}$$

The following is a generalization of a result of Effros and Kishimoto [14].

Theorem 9.4. *Let $B \to A \subseteq \mathrm{B}(\mathcal{H})$ and $B \to C \subseteq \mathrm{B}(\mathcal{L})$ be normal $*$-homomorphisms of von Neumann algebras (so that A and C are B-bimodules). Then*

$$A \overset{\sigma h}{\otimes}_B C = \mathrm{CB}_{A'}(\mathrm{B}_B(l, \mathcal{H}), \mathrm{B}(l, \mathcal{H}))_{C'}$$

by the completely isometric weak homeomorphism that sends $a \otimes_B c$ to the operator $x \mapsto axc$ ($x \in \mathrm{B}_B(l, \mathcal{H})$).*

Proof. Let $\mathcal{K}$ be a proper Hilbert B-module. Regarding A as a $\mathbb{C}, B$-bimodule and C as a $B, \mathbb{C}$-bimodule, as special cases of Proposition 9.3 we have

$$A_{\natural_{\mathrm{p}}} = \mathrm{NCB}(A, \mathcal{K}^*)_B = \mathcal{H}^* \overset{eh}{\otimes}_{A'} \mathrm{B}_B(\mathcal{K}, \mathcal{H}),\ C_{\natural_{\mathrm{p}}} = \mathrm{NCB}_B(C, \mathcal{K}) = \mathrm{B}_B(l, \mathcal{K}) \overset{eh}{\otimes}_{C'} l,$$

hence

$$A \overset{\sigma h}{\otimes}_B C = (A_{\natural_{\mathrm{p}}} \overset{eh}{\otimes}_{B'} C_{\natural_{\mathrm{p}}})^\sharp = \left(\mathcal{H}^* \overset{eh}{\otimes}_{A'} \mathrm{B}_B(\mathcal{K}, \mathcal{H}) \overset{eh}{\otimes}_{B'} \mathrm{B}_B(l, \mathcal{K}) \overset{eh}{\otimes}_{C'} l\right)^\sharp.$$

Since by Proposition 9.3

$$\mathrm{B}_B(\mathcal{K}, \mathcal{H}) \overset{eh}{\otimes}_{B'} \mathrm{B}_B(l, \mathcal{K}) = \mathrm{NCB}_B(B, \mathrm{B}(l, \mathcal{H}))_B = \mathrm{B}_B(l, \mathcal{H}),$$

it follows (by using Remark 2.19, the commutativity and associativity of $\hat{\otimes}$ and the identities $\mathcal{H}^* \overset{h}{\otimes} V = \mathcal{H}^* \hat{\otimes} V$, $V \overset{h}{\otimes} l = V \hat{\otimes} l$) that

$$\begin{aligned} A \overset{\sigma h}{\otimes}_B C &= (\mathcal{H}^* \overset{eh}{\otimes}_{A'} \mathrm{B}_B(l, \mathcal{H}) \overset{eh}{\otimes}_{C'} l)^\sharp \\ &= (\mathcal{H}^* \hat{\otimes}_{A'} \mathrm{B}_B(l, \mathcal{H}) \hat{\otimes}_{C'} l)^\sharp \\ &\cong (\mathrm{B}_B(l, \mathcal{H})\ {}_{A'}\hat{\otimes}_{C'} (l \hat{\otimes} \mathcal{H}^*))^\sharp \\ &= \mathrm{CB}_{A'}(\mathrm{B}_B(l, \mathcal{H}), \mathrm{B}(l, \mathcal{H}))_{C'} \text{ (by (9.4))}. \end{aligned}$$

□

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