Infinitely many one-regular Cayley graphs on dihedral groups of any prescribed valency

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Abstract

A graph is one-regular if its automorphism group acts regularly on the arc set. In this paper, we construct a new infinite family of one-regular Cayley graphs of any prescribed valency. In fact, for any two positive integers \( \ell, k \geq 2 \) except for \((\ell, k) \in \{(2, 3), (2, 4)\}\), the Cayley graph \( \text{Cay}(D_n, S) \) on dihedral groups \( D_n = \langle a, b \mid a^n = b^2 = (ab)^2 = 1 \rangle \) with \( S = \{a^{1+\ell+\cdots+\ell^r} b \mid 0 \leq r \leq k - 1 \} \) and \( n = \sum_{j=0}^{k-1} \ell^j \) is one-regular. All of these graphs have cyclic vertex stabilizers and girth 6. As a continuation of Marušič and Pisanski’s classification of cubic one-regular Cayley graphs on dihedral groups in [D. Marušič, T. Pisanski, Symmetries of hexagonal graphs on the torus, Croat. Chemica Acta 73 (2000) 969–981], the 5-valent one-regular Cayley graphs on dihedral groups are classified. Also, with only finitely many possible exceptions, all of one-regular Cayley graphs on dihedral groups of any prescribed prime valency are constructed. © 2007 Elsevier Inc. All rights reserved.

Keywords: One-regular graph; Cayley graph; Dihedral group

1. Introduction

In this paper, we consider undirected finite connected graphs without loops or multiple edges. For a graph \( G \), every edge of \( G \) gives rise to a pair of opposite arcs. By \( V(G) \), \( E(G) \), \( D(G) \) and \( \text{Aut}(G) \), we denote the vertex set, the edge set, the arc set and the automorphism group of \( G \), respectively. For a given vertex \( v \in V(G) \), we denote by \( N_i(v) \) the set of all vertices at distance \( i \)

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1. Introduction

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from $v$. A graph $G$ is said to be vertex-transitive, edge-transitive and arc-transitive if $\text{Aut}(G)$ acts transitively on the vertex set, the edge set and the arc set of $G$, respectively. A graph $G$ is one-regular if $\text{Aut}(G)$ acts regularly on $D(G)$, that is, if it is arc-transitive and the stabilizer of an arc in $\text{Aut}(G)$ is trivial.

Given a group $\Gamma$ and a generating set $S$ of $\Gamma$ such that $S = S^{-1}$ and $1 \notin S$, the Cayley graph $\text{Cay}(\Gamma, S)$ on $\Gamma$ relative to $S$ has the vertex set $\Gamma$ and the edge set $\{\{g, gs\} \mid g \in \Gamma, s \in S\}$.

The left regular representation $L_{\Gamma} = \{L_g \mid g \in \Gamma\}$ of $\Gamma$, defined by $xL_g = g^{-1}x$, can also be considered as a vertex-regular subgroup of $\text{Aut}(\text{Cay}(\Gamma, S))$. A Cayley graph $\text{Cay}(\Gamma, S)$ is said to be normal if $L_{\Gamma}$ is a normal subgroup of $\text{Aut}(\text{Cay}(\Gamma, S))$.

The dihedral group $D_n$, $n \geq 3$, is a group of order $2n$ given by the presentation $D_n = \langle a, b \mid a^n = b^2 = (ab)^2 = 1 \rangle$.

This paper deals with connected one-regular graphs. The first interesting case is of valency 3 because connected one-regular graphs of valency 2 are just cycles. The first example of one-regular cubic graph was given by Frucht [8]. Much later, many constructions of valency 3 and 4 were given in [2,5–7,16,19,20]. Regarding any even valency, there is an infinite family of one-regular graphs from the classification of symmetric graphs of order a prime by C.Y. Chao [3] and independently by B. Alspach [1], and regarding any valency, there is an infinite family of one-regular graphs from the classification of weakly symmetric graphs of order twice a prime by Y. Cheng and J. Oxley [4]. Recently, the first and third authors [13] constructed an infinite family of one-regular Cayley graphs on dihedral groups of any even valency with cyclic vertex stabilizer, including an infinite family of one-regular graphs of order neither a prime nor twice a prime. Infinitely many one-regular Cayley graphs on dihedral groups of valency 4 and 6 with girth 6 were constructed by J.M. Oh and K.W. Hwang in [21].

In this paper, we construct an infinite family of one-regular normal Cayley graphs on dihedral groups of any prescribed valency and with cyclic stabilizer and girth 6. The main results are stated as the following two theorems.

**Theorem A.** For every pair of integers $\ell, k \geq 2$ except $(\ell, k) = (2, 3)$ and $(2, 4)$, the Cayley graph

$$\text{Cay}(D_n, S) \quad \text{with} \quad n = \sum_{i=0}^{k-1} \ell^i \quad \text{and} \quad S = \{a^{1+\ell+\cdots+\ell^t}b \mid 0 \leq t \leq k - 1\}$$

is one-regular.

For an integer $k \geq 3$, the classification of $k$-valent one regular Cayley graphs on dihedral groups has been done for only $k = 3$. D. Marušič and T. Pisanski [18] classified the arc-transitive cubic Cayley graphs on dihedral groups. All of such arc-transitive graphs are the graphs constructed in this paper except for two graphs, namely, the cube $Q_3 \cong \text{Cay}(D_4, \{b, ab, a^2b\})$ and the Möbius–Kantor graph $\text{Cay}(D_8, \{b, ab, a^3b\})$. Among them, one-regular graphs are those graphs with $n \geq 11$. The classification of one-regular 5-valent Cayley graphs on dihedral groups up to isomorphism is given as follows.

**Theorem B.** A Cayley graph on a dihedral group $D_n$ of valency 5 is one-regular if and only if it is isomorphic to one of

$$\text{Cay}(D_n, \{b, ab, a^{\ell+1}b, a^{\ell^2+\ell+1}b, a^{\ell^3+\ell^2+\ell+1}b\})$$
for pairs \((n, \ell)\) such that \(n \geq 31\) and \(\ell^4 + \ell^3 + \ell^2 + \ell + 1 \equiv 0 \pmod{n}\). Furthermore, all of these graphs have girth 6.

For circulant graphs, the classification of arc-transitive circulant graphs (hence also of one-regular circulant graphs) was given independently by I. Kovacs [12] and by C.H. Li [14]. Regarding one-regular infinite graphs, A. Malnič et al. [15] constructed an infinite family of such graphs, which steps into an important territory of symmetry in infinite graphs.

This paper is organized as follows. In Section 2, we introduce some known results related to normal Cayley graphs on dihedral groups with cyclic vertex stabilizer. In Section 4, it is shown that for any odd prime \(p\), only finitely many additional \(p\)-valent one-regular Cayley graphs on dihedral groups are possible beyond our construction. Theorems A and B are proved in Sections 3 and 5, respectively.

2. Preliminaries and notation

For a Cayley graph \(\text{Cay}(\Gamma, S)\), let \(\text{Aut}(\Gamma, S) = \{\alpha \in \text{Aut}(\Gamma) \mid S^\alpha = S\}\) denote the group of automorphisms of the group \(\Gamma\) which fix \(S\) setwise. Then, \(\text{Aut}(\Gamma, S)\) is clearly a subgroup of the stabilizer of the identity element 1 in \(\text{Aut}(\text{Cay}(\Gamma, S))\).

M.Y. Xu characterized normal Cayley graphs as follows.

**Proposition 2.1.** (See [22].) A Cayley graph \(\text{Cay}(\Gamma, S)\) is normal if and only if \(\text{Aut}(\Gamma, S)\) is the stabilizer of the identity element 1 in \(\text{Aut}(\text{Cay}(\Gamma, S))\).

The automorphism group \(\text{Aut}(D_n)\) of the dihedral group \(D_n\) can be described as

\[
\{\sigma_{i,j} \mid a^{a_{i,j}} = a^i, \ b^{a_{i,j}} = a^jb \text{ and } i, j = 0, 1, \ldots, n-1 \text{ such that } (i, n) = 1\}.
\]

In this paper, for any finite set \(X\), we denote its cardinality by \(|X|\). The next proposition gives a necessary condition for a Cayley graph on a dihedral group to be one-regular and normal with cyclic vertex stabilizer.

**Proposition 2.2.** Let \(k \geq 3\) be an integer and let \(\text{Cay}(D_n, S)\) be a connected Cayley graph on a dihedral group \(D_n\) with \(|S| = k\). Then, \(\text{Aut}(D_n, S)\) contains a cyclic subgroup of order \(k\) which acts transitively on \(S = N_1(1)\) if and only if \(\text{Cay}(D_n, S)\) is isomorphic to a Cayley graph of the form

\[
\text{Cay}(D_n, S) \quad \text{with} \quad S = \{a^{1+t+\cdots+t^\ell} b \mid 0 \leq t \leq k - 1\}
\]

for an integer \(\ell\) satisfying \(\sum_{j=0}^{k-1} j^\ell \equiv 0 \pmod{n}\). Hence, all of such graphs are arc-transitive.

**Proof.** Let \(G = \text{Cay}(D_n, S)\) be a \(k\)-valent connected Cayley graph on a dihedral group \(D_n\) such that \(\text{Aut}(D_n, S)\) contains a cyclic subgroup of order \(k\) which acts transitively on \(S = N_1(1)\). Let \(\alpha \in \text{Aut}(D_n, S)\) be a generator of such a cyclic subgroup. Suppose that \(S\) contains \(a^t\) for some \(t\). Then, \(S = \{(a^t)^r \mid 0 \leq r \leq k - 1\}\) and \(S \subseteq \langle a \rangle\). This contradicts the connectivity of \(G\). Hence, up to isomorphism one may assume that \(S = \{b, a^{t_1}b, a^{t_2}b, \ldots, a^{t_k}b\}\), and that \(b^a = a^{t_1}b, (a^{t_1}b)^a = a^{t_2}b, \ldots, (a^{t_k-1}b)^a = a^{t_k}b\) and \((a^{t_k-1}b)^a = b\). Since \(a\) becomes an automorphism of the cyclic subgroup \(\langle a \rangle\), \(\langle a^r \rangle = \langle a^t \rangle\) for all \(r = 0, 1, \ldots, n - 1\). Because for any \(j = 1, 2, \ldots, k - 1\),

\[
(a^{-i})^a = (ba^{j+1}b)^a = a^{i-j}ba^{i+1}b = a^{i-j+1},
\]
where \( i_k = 0 \), one can obtain \( \langle a^{i_k-1} \rangle = \langle a^i \rangle \) and
\[
a^{-i_k-j} = (a^{i_k-i_j-k+1})^{a^{-1}} = (a^i)^{a^{-1}} (a^{-i_k-j+1})^{a^{-1}} = (a^i)^{a^{-1}} (a^{i_k-2})^{a^{-2}}
\]
\[
= \cdots = (a^i)^{a^{-1}} (a^i)^{a^{-2}} \cdots (a^i)^{a^{-j+1}} (a^i)^{a^{-j+1}} = (a^i).
\]
It implies that \( a^r \in (a^i) \) for any \( r = 0, 1, \ldots, k-1 \) and \( \langle b, a^{i_1}b, \ldots, a^{i_k-1}b \rangle \subseteq (a^i) \cup (a^i) ).
Because \( G \) is connected, \( (n, i_1) = 1 \). So, up to isomorphism, one may assume that \( i_1 = 1 \), which means that \( S = \{ b, ab, a^{i_2}b, \ldots, a^{i_k-1}b \} \), and
\[
b^a = ab, \quad (ab)^a = a^2b, \quad \ldots, \quad (a^{i_k-2}b)^a = a^{i_k-1}b \quad \text{and} \quad (a^{i_k-1}b)^a = b.
\]
Then, \( a^i = (abb)^a = (ab)^a b^a = a^{i-1} \) and for any \( j = 2, 3, \ldots, k-1 \),
\[
a^{i_j} b = (a^{i_j-1}b)^a = (a^{i-1}) b^a = a^{i-1} b ab = a^{i(j-1)} b j-1 b.
\]
Let \( \ell = i_2 - 1 \). Then, for any \( r = 2, 3, \ldots, k-1 \), we have
\[
i_r = (i_2 - 1) i_{r-1} + 1 = \ell i_{r-1} + 1 = \ell (\ell i_{r-2} + 1) + 1
\]
\[
= \ell^2 i_{r-2} + \ell + 1 = \cdots = \ell^{r-2} i_2 + \ell^{r-3} + \cdots + 1 = \sum_{j=0}^{r-1} \ell^j
\]
and \( \sum_{j=0}^{k-1} \ell^j \equiv 0 \pmod n \).
For sufficiency, consider a Cayley graph
\[
G = \text{Cay}(D_n, \{a^{1+\ell+\cdots+\ell t} b \mid 0 \leq t \leq k-1 \})
\]
with \( \sum_{j=0}^{k-1} \ell^j \equiv 0 \pmod n \). Define a mapping \( \alpha : D_n \to D_n \) by \( a \mapsto a^\ell, \ b \mapsto ab \). Note that \( (\ell, n) = 1 \). Then, it is easy to see that \( \alpha \in \text{Aut}(D_n, S) \), and that \( \langle \alpha \rangle \) is a cyclic group of order \( k \) which acts transitively on \( S = \{ a^{1+\ell+\cdots+\ell t} b \mid 0 \leq t \leq k-1 \} \).

Clearly, the Cayley graph \( \text{Cay}(D_n, \{a^{1+\ell+\cdots+\ell t} b \mid 0 \leq t \leq k-1 \}) \) for a pair \( (n, \ell) \) satisfying \( \sum_{j=0}^{k-1} \ell^j \equiv 0 \pmod n \) is a bipartite graph with partite sets \( \langle a \rangle \) and \( \langle a \rangle b \), and every element in \( \langle a \rangle b \) is an involution.

**Remark.** In the proof of Proposition 2.2, the cyclic group \( Z := \langle a \rangle \) generated by \( a \) is the subgroup of the stabilizer of the identity, and one can see that \( a^{-1} L_{D_n} \alpha \leq L_{D_n} \) by a direct calculation. Hence, the semidirect product \( L_{D_n} \rtimes Z \) is a group of order \( k \) which acts regularly on the arc set of \( G \). By [9, Theorem 1], the graphs in Proposition 2.2 admit regular embeddings into closed orientable surfaces. Moreover, because \( L_{D_n} \rtimes Z \) contains the normal subgroup \( L_{D_n} \) which acts transitively on the vertex set of \( G \), these regular embeddings are balanced Cayley maps. (See [10].)

Throughout this paper, we adopt the following notation.

**Notation.**

1. For any two positive integers \( \ell, t \geq 1 \), let \( \ell[t] := \ell t^{-1} + \ell t^{-2} + \cdots + 1 \) and \( \ell[0] = 0 \).
2. Let \( T \) be the set of triples \( (n, \ell, k) \) of positive integers in which \( \ell < n \) and \( k \) is the smallest positive integer such that
\[
\ell[k] = \ell^{k-1} + \ell^{k-2} + \cdots + 1 \equiv 0 \pmod n.
\]
(3) For an \((n, \ell, k) \in \mathcal{T}\), let \(\text{Cay}(D_n; \ell, k) := \text{Cay}(D_n, S)\), where \(S = \{a^{1+\ell+\cdots+\ell t} | 0 \leq t \leq k - 1\} = \{a^{\ell t}b | 0 \leq t \leq k - 1\}\). Note that \(a^{1+\ell+\cdots+\ell^{k-1}}b = a^{\ell k}b = b\).

(4) Let \(\mathcal{O}\) be the set of triples \((n, \ell, k) \in \mathcal{T}\) satisfying the following two conditions:

(i) For any \(r, s, t, u \geq 0, r \leq s, t, u < k - 1\), \(\ell [r] + \ell [s] = \ell [r] + \ell [u] \pmod n\) if and only if either \((r, s) = (t, u)\) or \((r, s) = (u, t)\).

(ii) For any sequence of numbers \(0 \leq i_0, i_1, i_2, i_3, i_4, i_5 \leq k - 1\) such that \(i_j \neq i_{j+1}\) and \(i_5 \neq i_0\), \(\ell [i_0] + \ell [i_2] + \ell [i_4] \equiv \ell [i_1] + \ell [i_3] + \ell [i_5] \pmod n\) if and only if the numbers \(i_0, i_1, i_2, i_3, i_4\) and \(i_2, i_5\) are all distinct.

Note that the condition 4(i) implies that the graph \(\text{Cay}(D_n; \ell, k)\) has no 4-cycles and the condition 4(ii) implies that the graph contains 6-cycles only of a special type for every \((n, \ell, k) \in \mathcal{O}\), and hence the girth of the graph \(\text{Cay}(D_n; \ell, k)\) is 6. For any triple \((n, \ell, k) \in \mathcal{T}\), any \(g \in D_n\) and for any three distinct integers \(r, s, t \in \{0, 1, \ldots, k - 1\}\), consider the sequence \((g, g a^{\ell r}b, g a^{\ell r}b a^{\ell s}b)\) starting from \(g\) and successively connecting with generators \(a^{\ell r}b, a^{\ell s}b, a^{\ell t}b\) in cyclic order. It gives rise to a 6-cycle in the graph \(\text{Cay}(D_n; \ell, k)\) for each \(k \geq 3\). So, for any \((n, \ell, k) \in \mathcal{T}\) with \(k \geq 3\), the girth of \(\text{Cay}(D_n; \ell, k)\) is at most 6. In fact, the condition 4(ii) in the definition of \(\mathcal{O}\) is an arithmetic (necessary and sufficient) condition to guarantee that all 6-cycles of \(\text{Cay}(D_n; \ell, k)\) are of the type mentioned above. In particular, for any \(g \in D_n\) and for any two distinct integers \(r, s \in \{0, 1, \ldots, k - 1\}\), the 3-path

\[ (g, g a^{\ell r}b, g a^{\ell r}b a^{\ell s}b) = g a^{\ell r - \ell s}b, g a^{\ell r - \ell s + \ell r}b, g a^{\ell r - \ell s}b \]

does not belong to any 6-cycle. Furthermore, the two conditions 4(i)–(ii) in the definition of \(\mathcal{O}\) are equivalent to the condition

\[ |N_2(1)| + |N_3(1)| = k(k - 1) + \frac{k(k - 1)(k - 2)}{2} + k(k - 1) = \frac{k(k - 1)(k + 2)}{2} \tag{1} \]

in the Cayley graph \(\text{Cay}(D_n; \ell, k)\).

In summary, we have the following lemma.

**Lemma 2.3.** Let \((n, \ell, k) \in \mathcal{O}\) be any triple and let \(r \neq s\) and \(s \neq t\) be any three integers in \(\{0, 1, \ldots, k - 1\}\). Then for any \(g \in D_n\), the 3-path

\[ (g, g a^{\ell r}b, g a^{\ell r}b a^{\ell s}b) = g a^{\ell r - \ell s}b, g a^{\ell r - \ell s + \ell r}b \]

can be extended uniquely to a 6-cycle in \(\text{Cay}(D_n; \ell, k)\) if \(r \neq t\), and it cannot be extended to any 6-cycle if \(r = t\).

3. A proof of Theorem A

In this section, with the same notation \(\mathcal{T}\), \(\text{Cay}(D_n; \ell, k)\) and \(\mathcal{O}\) as in Section 2, we show that for every \((n, \ell, k) \in \mathcal{O}\), the graph \(\text{Cay}(D_n; \ell, k)\) is one-regular and that for any two positive integers \(\ell, k \geq 2\) except for \((\ell, k) \in \{(2, 3), (2, 4)\}\), \(\text{Cay}(D_{\ell k}; \ell, k)\) is also one-regular even if the triple \((\ell k, \ell, k)\) does not belong to the set \(\mathcal{O}\).

For any subset \(\{g_1, g_2, \ldots, g_i\}\) of \(D_n\) and for any Cayley graph \(\text{Cay}(D_n, S)\), let \(\text{Aut}(\text{Cay}(D_n, S))_{\{g_1, g_2, \ldots, g_i\}}\) denote the pointwise stabilizer of \(\{g_1, g_2, \ldots, g_i\}\) in the automorphism group \(\text{Aut}(\text{Cay}(D_n, S))\). Note that the stabilizer \(\text{Aut}(\text{Cay}(D_n, S))_1\) can be considered to act on the set \(S = N_1(1)\).
Lemma 3.1. Let \((n, \ell, k) \in \mathcal{O}\).

1. If \(\alpha \in \text{Aut}(\text{Cay}(D_n; \ell, k))_{(1,b)}\) fixes \(a^\ell b\) for some \(1 \leq t \leq k - 1\) then \(\alpha\) is the identity.
2. If \(\alpha \in \text{Aut}(\text{Cay}(D_n; \ell, k))_{(1,b)}\) contains a cycle \((a^\ell b, a^{\ell+1} b, \ldots, a^{\ell+t} b)\) as a permutation on \(N_1(1) = S\) for some \(1 \leq s \leq k - 1\), then \(\alpha L_{a^{-m}} \alpha^{-1} = L_{a^{-m}}\) with \(m = \sum_{j=1}^s \ell \{i_j\}\), where \(L_{a^{-m}} : g \mapsto a^m g\) is a left translation belonging to \(\text{Aut}(\text{Cay}(D_n; \ell, k))\).

Proof. To prove (1), let \(\alpha \in \text{Aut}(\text{Cay}(D_n; \ell, k))_{(1,b)}\) fix \(a^\ell b\) for some \(1 \leq t \leq k - 1\). Then \(\alpha\) clearly fixes the 2-path \((b, 1, a^\ell b)\). Among its extended 3-paths in \(\text{Cay}(D_n; \ell, k)\), only the path \((b, 1, a^\ell b, a^{\ell+1} b, \ldots)\) cannot belong to a 6-cycle, while each of the others \((b, 1, a^\ell b, a^{\ell+2} b, a^{\ell+3} b, \ldots)\) with \(0 \neq t \neq 2\) belongs to a 6-cycle, by Lemma 2.2. Hence, \(\alpha\) should fix the 3-path \((b, 1, a^\ell b, a^{\ell+1} b)\) pointwise. By repeating the same process to its tail \((1, a^\ell b, a^{\ell+1} b)\) with Lemma 2.3 again, one can show that \(\alpha\) should fix the 4-path \((b, 1, a^\ell b, a^{\ell+1} b, a^{\ell+2} b)\) pointwise. Continuing this process to the last 2-path, one can show that \(\alpha\) fixes pointwise the path

\[
(b, 1, a^\ell b, a^{\ell+1} b, a^{\ell+2} b, a^{\ell+3} b, \ldots)
\]

which is formed by multiplying the generators \(b\) and \(a^\ell b\) alternately to the tail. In particular, \(\alpha\) fixes both \(a^{\ell j} b\) and \(b\) for all \(j \geq 1\), which implies that \(\alpha\) fixes both \(a^{\ell j}\) and \(a^{\ell+2} b\) for every multiple \(r\) of \(\gcd(\ell, n)\), the greatest common divisor of \(\ell, n\).

If \(t = 1\) then \(\ell = 1\). So, \(\alpha\) fixes all vertices of \(\text{Cay}(D_n; \ell, k)\), that is, \(\alpha\) is the identity. Let \(t \in [2, 3, \ldots, k - 1]\). Noting that \(\alpha\) fixes a vertex \(a^{\ell j}\) for every multiple \(r\) of \(\ell\), one can see that \(\alpha\) fixes the two end vertices of the 2-path \((1, a^{\ell j} b, a^{\ell(j+1)} b)\) with \(0 \neq t \neq 1\). Hence, it fixes its central vertex \(a^{\ell j} b = ab\) because the graph \(\text{Cay}(D_n; \ell, k)\) does not have 4-cycles. Therefore, \(\alpha\) is the identity automorphism of \(\text{Cay}(D_n; \ell, k)\).

To prove (2), let \(\alpha \in \text{Aut}(\text{Cay}(D_n; \ell, k))_{(1,b)}\) satisfy the hypothesis. In this proof, subscripts are done modulo \(s\). If \(\alpha\) fixes a vertex \(a^{\ell j} b\), namely, \(s = 1\) then \(\alpha\) is the identity in \(\text{Aut}(\text{Cay}(D_n; \ell, k))\) by (1). So, it holds that \(\alpha L_{a^{-\ell j} b} \alpha^{-1} = L_{a^{-\ell j} b}\).

Let \(s \geq 2\). Note that for each \(j = 1, 2, \ldots, s\), the 2-path \((b, 1, a^{\ell j} b)\) can be extended to the 3-path in a unique way which does not belong to any 6-cycle by Lemma 2.3. In fact, it should be

\[
(b, 1, a^{\ell j} b, a^{\ell(j+1)} b)\]

Since their third vertices \(a^{\ell(j+1)} b\) permute cyclically by \(\alpha\) as a cycle \((a^{\ell j} b, a^{\ell(j+1)} b, a^{\ell(j+2)} b, \ldots)\) by the hypothesis, their forth vertices \(a^{\ell(j+1)} b, a^{\ell(j+2)} b, \ldots, a^{\ell j} b\) should be also cyclically permuted in the same order by \(\alpha\). Moreover, for each \(j = 1, \ldots, s\),

\[
N_1(a^{\ell j}) \cap N_1(a^{\ell(j+1)}) = \{a^{\ell j + \ell(j+1)} b\}
\]

So, \(\alpha\) sends \(a^{\ell j + \ell(j+1)} b\) to \(a^{\ell j + \ell(j+2)} b\), that is, \(\alpha\) maps the 4-path

\[
(b, 1, a^{\ell j} b, a^{\ell j} b, a^{\ell j+\ell(j+1)} b)
\]

to the 4-path

\[
(b, 1, a^{\ell(j+1)} b, a^{\ell(j+1)} b, a^{\ell(j+1)+\ell(j+2)} b).
\]

Applying the same process to the 2-path \((a^{\ell j} b, a^{\ell j} b, a^{\ell(j+1)} b, a^{\ell(j+2)} b)\) for each \(j = 1, 2, \ldots, s\), one can show that it can be extended to a 3-path in a unique way which cannot belong to any 6-cycle. In fact, it is

\[
(a^{\ell j} b, a^{\ell j} b, a^{\ell(j+1)} b, a^{\ell(j+1)} b, a^{\ell(j+2)} b = a^{\ell j} + \ell(j+1)} b).
\]
Hence, $\alpha$ should send $\alpha^{\ell[i_j]+\ell[i_{j+1}]}$ to $\alpha^{\ell[i_{j+1}]+\ell[i_{j+2}]}$ for each $j$. Repeating the same process, that is, for each 2-path appeared as the tail of the paths constructed in a previous step, find its unique 3-path extension which cannot belong to any 6-cycle and the intersection of the neighborhoods for each 2-path appeared as the tail of the paths constructed in a previous step, find its unique 3-path extension which cannot belong to any 6-cycle and the intersection of the neighborhoods is, for each 2-path appeared as the tail of the paths constructed in a previous step, find its unique 3-path extension which cannot belong to any 6-cycle, and the intersection of the neighborhoods should be an arbitrary automorphism in $\text{Aut}(\text{Cay}(D_n, S))$. Since it is already known that the graph $\text{Cay}(D_n, S)$ is arc-transitive, it suffices to show that the stabilizer $\text{Aut}(\text{Cay}(D_n, S))_{(1,b)}$ is trivial in order to obtain one-regularity.

Let $\alpha$ be an arbitrary automorphism in $\text{Aut}(\text{Cay}(D_n, S))_{(1,b)}$ and let $\alpha$ have the following cycle decomposition

$$
\alpha\mid_{N(1)} = (b) (\alpha^{\ell[i_{1,1}]}b \quad \alpha^{\ell[i_{1,2}]}b \quad \ldots \quad \alpha^{\ell[i_{1,r}]}b) (\alpha^{\ell[i_{2,1}]}b \quad \alpha^{\ell[i_{2,2}]}b \quad \ldots \quad \alpha^{\ell[i_{2,r}]}b) \quad \ldots \\
(\alpha^{\ell[i_{r,1}]}b \quad \alpha^{\ell[i_{r,2}]}b \quad \ldots \quad \alpha^{\ell[i_{r,r}]}b).
$$

Remark. If $\alpha L_{a^{-m}\alpha^{-1}} = L_{a^{-m}}$ then $\alpha L_{a^{-im}\alpha^{-1}} = (\alpha L_{a^{-m}\alpha^{-1}})^i = L_{a^{-im}}$. So, we have

$$a^{im} = 1^{L_{a^{-im}}} = 1^{\alpha L_{a^{-im}\alpha^{-1}}} = 1^{L_{a^{-im}\alpha^{-1}}} = (a^{im})^{-1}$$

and

$$a^{im}b = b^{L_{a^{-im}}} = b^{\alpha L_{a^{-im}\alpha^{-1}}} = b^{L_{a^{-im}\alpha^{-1}}} = (a^{im}b)^{-1}.$$

It means that $\alpha$ fixes both $a^{im}$ and $a^{im}b$ for every integer $i$. Therefore, $\alpha$ fixes both $a^{r}$ and $a^{r}b$ for every multiple $r$ of $\gcd(m, n)$.

**Theorem 3.2.** For every triple $(n, \ell, k) \in \mathcal{O}$ with $k > 2$, the Cayley graph

$$\text{Cay}(D_n, S) \quad \text{with} \quad S = \{a^{1+\ell+\ldots+\ell^t}b \mid 0 \leq t \leq k - 1\}$$

is one-regular and its girth is 6.

**Proof.** From the condition 4(i) in the definition of $\mathcal{O}$, we know that the girth of $\text{Cay}(D_n; \ell, k)$ is 6.

Since it is already known that the graph $\text{Cay}(D_n, S) = \text{Cay}(D_n; \ell, k)$ is arc-transitive, it suffices to show that the stabilizer $\text{Aut}(\text{Cay}(D_n; \ell, k))_{(1,b)}$ is trivial in order to obtain one-regularity.
Then, for any \(j = 1, 2, \ldots, r\), \(\alpha L_{a^{-m_j}}\alpha^{-1} = L_{a^{-m_j}}\) by Lemma 3.1(2), where \(m_j = \sum_{t=1}^{s_j} \ell[i_{j,t}].\) It implies that for any integer \(i\),

\[
\alpha L_{a^{-\sum_{j=1}^{r} m_j}}\alpha^{-1} = (\alpha L_{a^{-\sum_{j=1}^{r} m_j}}\alpha^{-1})^i = \left(\sum_{j=1}^{r} \alpha L_{a^{-m_j}}\alpha^{-1}\right)^i = \left(\sum_{j=1}^{r} L_{a^{-m_j}}\right)^i
\]

By taking its values at 1 and \(b\), one can see that \(\alpha\) fixes both \(a^{im}\) and \(a^{im}b\) for each \(i\), where \(m = \sum_{i=1}^{r} m_i\). It means that \(\alpha\) fixes both \(a^{m'}\) and \(a^{m'}b\) for every multiple \(m'\) of \(\gcd(m, n)\). Because

\[
n - (\ell - 1)m \equiv \sum_{i=0}^{k-1} \ell^i - (\ell - 1) \sum_{i=1}^{k-1} \ell[i] = \sum_{j=0}^{k-1} \ell^j - \sum_{i=1}^{k-1} (\ell^i - 1) = k \pmod{n},
\]

\(k\) is a multiple of \(\gcd(m, n)\) and \(\alpha\) fixes both \(a^{m''}\) and \(a^{m''}b\) for every multiple \(m''\) of \(k\).

**Case 1:** Let there exist \(t \in \{1, 2, \ldots, k - 1\}\) such that \(\ell[t]\) is a multiple of \(k\). In this case, \(\alpha\) fixes \(a^{\ell[t]}b\). By Lemma 3.1(1), \(\alpha\) is the identity in \(\text{Aut}(\text{Cay}(D_n; \ell, k))\).

**Case 2:** Let there exist two different integers \(t_1, t_2 \in \{1, 2, \ldots, k - 1\}\) such that \(\ell[t_1] \equiv \ell[t_2] \pmod{k}\). Then, \(\ell[t_1] - \ell[t_2]\) is a multiple of \(k\). Hence, \(\alpha\) fixes \(a^{\ell[t_1] - \ell[t_2]}b\). Because

\[
N_1(1) \cap N_1(a^{\ell[t_1] - \ell[t_2]}) = \{a^{\ell[t_1]}b\},
\]

\(\alpha\) fixes \(a^{\ell[t_1]}b\). By Lemma 3.1(1), \(\alpha\) is the identity.

**Case 3:** As the final case, let there exist neither \(t \in \{1, 2, \ldots, k - 1\}\) such that \(\ell[t]\) is a multiple of \(k\) nor \(t_1, t_2 \in \{1, 2, \ldots, k - 1\}\) such that \(\ell[t_1] \equiv \ell[t_2] \pmod{k}\). Then, there exist \(t_1\) and \(t_2\) in the set \(\{1, 2, \ldots, k - 1\}\) such that \(\ell[t_1] \equiv 1 \pmod{k}\) and \(\ell[t_2] \equiv 2 \pmod{k}\). It means that \(2 \ell[t_1] - \ell[t_2]\) is a multiple of \(k\) and \(\alpha\) fixes \(a^{2(\ell[t_1] - \ell[t_2])b}\). Because there does not exist any 6-cycle containing the 3-path \((1, a^{\ell[t_1]}b, a^{\ell[t_1] - \ell[t_2]}, a^{2(\ell[t_1] - \ell[t_2])b})\) by Lemma 2.3, \(N_1(1) \cap N_2(a^{2(\ell[t_1] - \ell[t_2])b}) = \{a^{\ell[t_1]}b\}\). So, \(\alpha\) fixes \(a^{\ell[t_1]}b\) and by Lemma 3.1(1) again, \(\alpha\) is the identity.

**Lemma 3.3.** For any two positive integers \(k, \ell \geq 3\), the triple \((\ell[k], \ell, k)\) belongs to \(\mathcal{O}\).

**Proof.** Let \(n = \ell[k] = \sum_{j=0}^{k-1} \ell^j\). Then, clearly the triple \((n, \ell, k)\) belongs to the set \(\mathcal{T}\). First, note that for each \(t = 2, 3, \ldots, k - 1\), we have \(3 \ell[t - 1] = 3 \sum_{j=0}^{t-2} \ell^j < \sum_{j=0}^{t-1} \ell^j = \ell[t]\) because \(\ell \geq 3\). Now, we shall show that the triple \((\ell[k], \ell, k)\) satisfies the conditions 4(i)–(ii) in the definition of \(\mathcal{O}\).

First, for any given integers \(r, s, t, u \pmod{k - 1}\), let us assume that \(\ell[r] + \ell[s] = \ell[t] + \ell[u]\) (mod \(n\)). Then \(\ell[r] = \ell[s] = \ell[t] + \ell[u]\) because both numbers \(\ell[r] + \ell[s]\) and \(\ell[t] + \ell[u]\) are less than \(n\). Without any loss of generality, one can assume that \(r\) is the largest integer among \(r, s, t, u\). If \(t < r\) and \(u < r\) then \(\ell[r] > 3\ell[r - 1] \geq \ell[t] + \ell[u] - \ell[s]\), which is contradictory to the assumption \(\ell[r] = \ell[t] + \ell[u] - \ell[s]\). It implies that \((r = t\) and \(s = u)\) or \((r = u\) and \(s = t)\). The converse is quite clear: if \((r = t\) and \(s = u)\) or \((r = u\) and \(s = t)\) then \(\ell[r] + \ell[s] = \ell[t] + \ell[u]\) (mod \(n\)).

For any sequence of numbers \(i_0, i_1, i_2, i_3, i_4, i_5 \in \{0, 1, 2, \ldots, k - 1\}\) such that \(i_j \neq i_{j+1}\) for all \(j = 0, 1, 2, 3, 4\) and \(i_5 \neq i_0\), let \(\ell[i_0] + \ell[i_2] + \ell[i_4] = \ell[i_1] + \ell[i_3] + \ell[i_5] \pmod{n}\). Then,
because $-n < \ell[i_0] + \ell[i_2] + \ell[i_4] - \ell[i_1] = \ell[i_3] - \ell[i_5] < n$, it holds that $\ell[i_0] + \ell[i_2] + \ell[i_4] = \ell[i_1] + \ell[i_3] + \ell[i_5]$. Without any loss of generality, one can assume that $i_0$ be the largest integer among $i_0, i_1, i_2, i_3, i_4, i_5$. If all the numbers $i_1, i_3$ and $i_5$ are less than $i_0$, then $\ell[i_0] > 3\ell[i_0 - 1] \geq \ell[i_1] + \ell[i_3] + \ell[i_5]$, a contradiction. So, $i_3$ should be equal to $i_0$ because $i_5 \neq i_0 \neq i_1$, and then $\ell[i_2] + \ell[i_4] = \ell[i_1] + \ell[i_5]$. By the same reason as shown in the previous paragraph, $(i_2 = i_1$ and $i_4 = i_5)$ or $(i_2 = i_5$ and $i_4 = i_1)$. Since $i_j \neq i_{j+1}$ for any $j = 0, 1, 2, 3, 4$ and $i_5 \neq i_0$, it holds that $i_2 = i_5$ and $i_4 = i_1$. So, the three numbers $i_0, i_1$ and $i_2$ are all distinct and $i_0 = i_3, i_1 = i_4$ and $i_2 = i_5$. The converse is also clear: if the three numbers $i_0, i_1$ and $i_2$ are all distinct and $i_0 = i_3, i_1 = i_4$ and $i_2 = i_5$ then $\ell[i_0] + \ell[i_2] + \ell[i_4] = \ell[i_1] + \ell[i_3] + \ell[i_5]$ (mod $n$). \hfill $\square$

In contrast to the case $k, \ell \geq 3$, when $\ell = 2$ and $k \geq 3$, the triple $(2[k], 2, k)$ does not belong to the set $O$ because there exists a 6-cycle

$$(1, b, b \cdot ab = a^{-1}, a^{-1}b, a^{-1}b \cdot ab = a^{-2}, ab)$$

containing the 3-path $(1, b, b \cdot ab = a^{-1}, a^{-1}b)$. So, we need to find another method to show that the graph $\text{Cay}(D_{2[k]}, 2, k)$ is one-regular for any $k \geq 5$. Note that $2[r] = 2^r - 1$ for all $t$.

Lemma 3.4. Let $k \geq 5$ and let $0 \leq r \neq s \leq k - 1$ be integers with $r \neq 0$. Then, the 4-path

$$(b, 1, a^{2[r]}b, a^{2[r] - 2[s]}b, a^{2[r] - 2[s] + 2[t]}b)$$

in the Cayley graph $\text{Cay}(D_{2[k]}, 2, k)$ formed by successive generators $b, a^{2[r]}b, a^{2[s]}b, a^{2[t]}b$ can be extended to a 6-cycle for any $t \neq 0$ if $(r, s) = (1, 0)$. However, if $(r, s) \neq (1, 0)$ then there exists $t \neq s$ such that the 4-path cannot be extended to a 6-cycle.

Proof. Let $(r, s) = (1, 0)$. Then, for every $1 \leq t \leq k - 1$, the given 4-path becomes $(b, 1, ab, a, a^{1+2[t]}b)$. It is extended to a 6-cycle

$$(b, 1, ab, a, a^{1+2[t]}b, a^{1+2[t] - 2[t+1]}b),$$

because $1 + 2[r] - 2[t+1] = 1 + 2^r - 1 - 2^{t+1} + 1 = -(2^t - 1) = -2[r]$.

Now, let $(r, s) \neq (1, 0)$. To prove it by contradiction, suppose that for every $t \neq s$, the 4-path $(b, 1, a^{2[r]}b, a^{2[r] - 2[s]}b, a^{2[r] - 2[s] + 2[t]}b)$ can be extended to a 6-cycle, say

$$(b, 1, a^{2[r]}b, a^{2[r] - 2[s]}b, a^{2[r] - 2[s] + 2[t]}b, a^{2[r] - 2[s] + 2[t] - 2[t+1]}b),$$

for some $0 \leq t_1 \neq t_2 \leq k - 1$ with $t \neq t_1$ and $t_2 \neq 0$. Then, $2[r] + 2[t] + 2[t_2]$ is equal to either $2[s] + 2[t_1]$ or $2^k - 1 + 2[s] + 2[t_1]$ because $-(2^k - 1) < 2[r] + 2[t] + 2[t_2] - 2[s] - 2[t_1] < 2(2^k - 1)$. That is, it holds that either

$$2^r + 2^t + 2^{t_2} = 2^r + 2^{t_1} + 1 \quad \text{or} \quad 2^r + 2^t + 2^{t_2} = 2^r + 2^{t_1} + 2^k. \quad (2)$$

Case 1: $r \leq k - 2$ and $s = 0$.

In this case, $r$ is neither 0 nor 1 by assumption. Take $t = r + 1$. If $r < k - 2$ or $t_2 \leq k - 2$ then $2^r + 2^r + 1 + 2^{t_2} \leq 2^k$ and it can be easily checked that $2^r + 2^r + 1 + 2^{t_2} = 3 \cdot 2^r + 2^{t_2} \neq 2^{t_1} + 2$ for any $0 \leq t_1 \neq t_2 \leq k - 1$ with $r + 1 \neq t_1$ and $t_2 \neq 0$. If $r = k - 2$ and $t_2 = k - 1$ then $2^{k-2} + 2^{k-1} + 2^{k-1} = 2^k + 2^{k-2} > 2^{t_1} + 2$ and $2^{k-2} \neq 2^{t_1} + 1$ for any $t_1 \in \{0, 1, 2, \ldots, k - 1\}$. In both cases, Eq. (2) does not hold. It is a contradiction.
Case 2: $r \leq k - 2$ and $1 \leq s \leq k - 2$.
Take $t = s + 1$. Then, $2^r + 2^{s+1} + 2^{t+2} - 2^s = 2^r + 2^{s+1} + 2^{t+2} \leq 2^k$, and $2^r + 2^{s+1} + 2^{t+2} \neq 2^{t+1} + 1$ for any $0 \leq t_1 \neq t_2 \leq k - 1$ with $s + 1 \neq t_1$ and $t_2 \neq 0$. So, Eq. (2) does not hold.

Case 3: $r \leq k - 2$ and $s = k - 1$.
Take $t = 2$. Then, $2^r + 4 + 2^t < 2^{k-1} + 2^k$, and $2^r + 4 + 2^t \neq 2^{k-1} + 2^k + 1$ for any $0 \leq t_1 \neq t_2 \leq k - 1$ with $2 \neq t_1$ and $t_2 \neq 0$. Again, Eq. (2) does not hold.

Case 4: $r = k - 1$ and $s \leq k - 4$.
Take $t = s + 2$. Then, there exists $0 \leq t_1 \neq t_2 \leq k - 1$ such that $s + 2 \neq t_1$, $t_2 \neq 0$ and $2^{k-1} + 2^{s+2} + 2^t = 2^s + 2^{t_1} + 2^k$ because $2^{k-1} + 2^{s+2} + 2^t > 2^s + 2^{t_1} + 2^k$ for any $t_1, t_2$. Since $2^{s+2} \leq 2^{k-2}$, $t_2$ should be $k - 1$. It means that $3 \cdot 2^t = 2^{t_1}$. It is a contradiction.

Case 5: $r = k - 1$ and $s = k - 2$ or $k - 3$.
Take $t = s - 1$. Then, there exists $0 \leq t_1 \neq t_2 \leq k - 1$ such that $s - 1 \neq t_1$, $t_2 \neq 0$ and $2^{k-1} + 2^{s-1} + 2^t = 2^s + 2^{t_1} + 2^k$ because $2^{k-1} + 2^{s-1} + 2^t < 2^s + 2^{t_1} + 2^k$ for any $t_1, t_2$. In this situation, $t_1$ should be $k - 1$ because $2^t \leq 2^{k-2}$. It means that $2^t = 2^{k-1} + 1$. A contradiction.

In summary, for $(r, s) \neq (1, 0)$, there exists $t$ ($\neq s$) such that the 4-path

$$(b, 1, a^{2[r]}b, a^{2[r]-2[s]}a^{2[r]-2[s]+2[r]}b)$$

cannot be extended to any 6-cycle in the graph $\text{Cay}(D_{2[k]}; 2, k)$. $\square$

Note that Lemma 3.4 implies that if an automorphism $\alpha \in \text{Aut}(\text{Cay}(D_{2[k]}; 2, k))$ fixes both $1$ and $b$ then $\alpha$ also fixes both $ab$ and $a$.

**Proof of Theorem A.** If $k = 2$, the graph $\text{Cay}(D_{\ell[k]}; \ell, 2)$ is a $2(\ell + 1)$-cycle and it is one-regular.

Now, let $k \geq 3$. First, let $\ell = 2$, $k \geq 5$ and let $\alpha$ be an arbitrary automorphism in the stabilizer $\text{Aut}(\text{Cay}(D_{2[k]}; 2, k))_{(1,b)}$. By Lemma 3.4, $\alpha$ fixes both $ab$ and $a$. Because $L_{-a}\alpha L_a$ fixes both $1$ and $b$, it also fixes both $ab$ and $a$ by the same reason. It follows that $\alpha$ fixes both $a^2b$ and $a^2$. Repeating the same process, one can show that $\alpha$ fixes all the vertices of $\text{Cay}(D_{2[k]}; 2, k)$. Hence, $\alpha$ should be the identity. It implies that $\text{Cay}(D_{2[k]}; 2, k)$ is one-regular. If $\ell \geq 3$ as the remaining case, $(\ell[k], \ell, k) \in O$ by Lemma 3.3, and then $\text{Cay}(D_{\ell[k]}; \ell, k)$ is one-regular by Theorem 3.2. $\square$

The graph $\text{Cay}(D_{2[3]}; 2, 3)$ is not one-regular because

$$(1)(b)(ab, a^3b)(a^6)(a, a^2)(a^5, a^3)(a^2, a^6)(a^4, a^5)$$

is a nontrivial automorphism of $\text{Cay}(D_{2[3]}; 2, 3)_{(1,b)}$. Also the graph $\text{Cay}(D_{2[4]}; 2, 4)$ is not one-regular because

$$(1)(b)(ab, a^3b, a^7b)(a, a^{11}, a^6)(a^2, a^4, a^{13})(a^3, a^7, a^9)(a^8, a^{12}, a^{14})$$

$$(a^{12}b, a^{11}b, a^{13}b)(a^6b, a^{14}b, a^9b)(a^6b, a^8b, a^{12}b)(a^5b)(a^{10}b)(a^5)(a^{10})$$

is a nontrivial automorphism of $\text{Cay}(D_{2[4]}; 2, 4)_{(1,b)}$.

**Remark.** (1) If $\ell = 1$ in Theorem A, then $\ell[t] = 1[t] = t$ for any $t \geq 0$. So, a triple $(n, 1, k)$ belongs to $\mathcal{T}$ if and only if $n = k$. In this case, the graph $\text{Cay}(D_{k}; 1, k)$ is the complete bipartite graph $K_{k,k}$ which is not one-regular for $k \geq 3$. 
(2) In [13], an infinite family of one-regular graphs of any even valency was constructed as Cayley graphs on dihedral groups, and all of them are of girth 4. However, all one-regular Cayley graphs on dihedral groups constructed in this paper are of girth 6. Hence, these two infinite families do not have any overlap.

4. One-regular Cayley graphs on dihedral groups of any prime valency

In this section, we shall show that for each odd prime \( p \), there exist at most finitely many \( p \)-valent one-regular Cayley graphs on dihedral groups which do not appear in our construction. To do this, we introduce the next proposition, and show that the set \( \{(n, \ell) \mid (n, \ell, p) \in T - O\} \) is finite for each \( p \).

**Proposition 4.1.** (See [11].) Every prime valent one-regular Cayley graph on a dihedral group is normal.

Let \( \mathbb{Z}, \mathbb{Q}, \mathbb{Z}[x] \) and \( \mathbb{Q}[x] \) denote the set of all integers, of all rational numbers, of all polynomials in an indeterminate \( x \) with coefficients in \( \mathbb{Z} \) and in \( \mathbb{Q} \), respectively.

**Lemma 4.2.** For any odd prime \( p \), the set \( \{(n, \ell) \mid (n, \ell, p) \in T - O\} \) is finite.

**Proof.** Let \( p \) be an odd prime and let \( (n, \ell, p) \in T - O \). Then, the triple \((n, \ell, p)\) does not satisfy at least one of the two conditions in the definition of the set \( O \). Therefore, there exist \( r, s, u, v \in \{0, 1, \ldots, p - 1\} \) such that \( r \neq s \neq u \neq v \) (repetition allowed) and \( \ell[r] + \ell[s] \equiv \ell[u] + \ell[v] \pmod n \) or there exist \( i_0, i_1, i_2, i_3, i_4, i_5 \in \{0, 1, \ldots, p - 1\} \) such that \( i_0, i_2, i_4 \neq i_1, i_3, i_5 \) (repetition allowed) and \( \ell[i_0] + \ell[i_2] + \ell[i_4] \equiv \ell[i_1] + \ell[i_3] + \ell[i_5] \pmod n \). Either of these equations can be written as \( g(\ell) \equiv 0 \pmod n \) for a polynomial \( g(x) = \sum_{j=0}^{t} a_j x^j \in \mathbb{Z}[x] \) such that \( 1 \leq t \leq p - 2, |a_j| \leq 3, a_j \neq 0 \).

Since there exist only finitely many polynomials \( g(x) = \sum_{j=0}^{t} a_j x^j \in \mathbb{Z}[x] \) such that \( 1 \leq t \leq p - 2, -3 \leq a_j \leq 3 \) and \( a_j \neq 0 \), it is enough to show that there exist at most finitely many pairs \((n, \ell)\) such that \((n, \ell, p) \in T \) and \( g(\ell) \equiv 0 \pmod n \) for each of such polynomials \( g(x) \). Let \( g(x) \) be such a polynomial and let \( (n, \ell, p) \in T \) satisfy \( g(\ell) \equiv 0 \pmod n \). Let \( f(x) := \sum_{j=0}^{p-1} x^j \). Then, by the division algorithm for \( \mathbb{Q}[x] \), there are unique polynomials \( q(x) \) and \( r(x) \) in \( \mathbb{Q}[x] \) such that \( f(x) = g(x)q(x) + r(x) \), where the degree of \( r(x) \) is less than the degree of \( g(x) \). Since \( f(x) \) is irreducible over \( \mathbb{Q}, r(x) \neq 0 \). By clearing the denominators of the coefficients in \( q(x) \) and \( r(x) \), one can get \( d_1 f(x) = g(x)q_1(x) + r_1(x) \) for \( d_1 \in \mathbb{Z}, r_1(x) \neq 0 \) and \( q_1(x), r_1(x) \in \mathbb{Z}[x] \), where the degrees of \( q_1(x) \) and \( r_1(x) \) are the degrees of \( q(x) \) and \( r(x) \), respectively. Since \( f(\ell) \equiv 0 \pmod n \) and \( g(\ell) \equiv 0 \pmod n \), we get \( r_1(\ell) \equiv 0 \pmod n \). If \( r_1(x) \) is a constant polynomial, then \( n \) is a divisor of the number \( r_1(x) \). Otherwise, by applying the same method to \( f(x) \) and \( r_1(x) \), one can get \( d_2 f(x) = r_1(x)q_2(x) + r_2(x) \) for \( d_2 \in \mathbb{Z}, r_2(x) \neq 0 \) and \( q_2(x), r_2(x) \in \mathbb{Z}[x] \), where the degree of \( r_2(x) \) is less than that of \( r_1(x) \). Also, it holds that \( r_2(\ell) \equiv 0 \pmod n \). By continuing this process, one can conclude that \( n \) is a divisor of a fixed nonzero integer, and hence there exist at most finitely many pairs \((n, \ell)\) such that \((n, \ell, p) \in T \) and \( g(\ell) \equiv 0 \pmod n \). □

It follows from Lemma 4.2 that for any odd prime \( p \), there exists a constant \( M \) which depends on \( p \) such that if \( n \geq M \) and \((n, \ell, p) \in T \) then \((n, \ell, p) \in O \).
Note that every one-regular graph of prime valency has a cyclic vertex stabilizer. Hence, by Proposition 2.2, Theorem 3.2 and Lemmas 4.1–4.2, we have the following theorem.

**Theorem 4.3.** Let \( p \) be an odd prime number. Then, any \( p \)-valent one-regular Cayley graph on a dihedral group \( D_n \) is isomorphic to one of \( \text{Cay}(D_n; \ell, p) \) for \( (n, \ell, p) \in \mathcal{O} \) except at most finitely many ones.

5. A proof of Theorem B

Finally, we classify the one-regular 5-valent Cayley graphs on dihedral groups up to isomorphism. They will be the graphs in Proposition 2.2 with \( k = 5 \) and their girths are all 6.

By Propositions 2.1–2.2 and 4.1, every 5-valent one-regular Cayley graph on a dihedral group \( D_n \) is one of the Cayley graphs \( \text{Cay}(D_n, \{a^{\sum_{i=0}^{\ell} k_i b} \mid 0 \leq i \leq 4\}) \) for an integer \( \ell \) satisfying \( \ell^4 + \ell^3 + \ell^2 + \ell + 1 \equiv 0 \) (mod \( n \)) up to isomorphism.

For any two positive integers \( n \) and \( \ell \) such that \( \ell[5] = \ell^4 + \ell^3 + \ell^2 + \ell + 1 \equiv 0 \) (mod \( n \)), it is easy to see that the triple \((n, \ell, 5)\) belongs to the set \( \mathcal{O} \) if and only if \(|N_2(1) \cup N_3(1)| = 70\) by Eq. (1). By a method similar to the proof of Lemma 4.2, one can prove the following lemma.

**Lemma 5.1.** For any triple \((n, \ell, 5) \in \mathcal{T}, \) if \(|N_2(1) \cup N_3(1)| \neq 70\) in the graph \( \text{Cay}(D_n; \ell, 5) \), namely, the triple \((n, \ell, 5)\) does not belong to \( \mathcal{O} \) then \((n, \ell)\) is one of the following pairs:

\((5, 1), (11, 9), (11, 15), (11, 4), (11, 3), (31, 2), (31, 16), (31, 4), (31, 8), (41, 37), (41, 10), (41, 16), (41, 18), (55, 16), (55, 26), (55, 31), (55, 36), (61, 9), (61, 20), (61, 34)\) or \((61, 58)\).

Now, we aim to check whether the graph \( \text{Cay}(D_n; \ell, 5) \) is one-regular or not for each pair \((n, \ell)\) listed in Lemma 5.1. To do this for the pairs \((n, \ell)\) with prime \( n \), we first review some results in the paper [4]. For any odd prime \( p \) and a positive integer \( k \) dividing \( p - 1 \), let \( H(p, k) \) denote the unique subgroup of the multiplicative group \( \mathbb{Z}_p^* \) of order \( k \). And, let \( A = \{i \mid i \in \mathbb{Z}_p\} \) and \( A' = \{i' \mid i' \in \mathbb{Z}_p\} \), as another copy of \( \mathbb{Z}_p \). Define a new graph \( G(2p, k) \) to have the vertex set \( A \cup A' \) and the edge set \( \{(x, y') \mid x, y \in \mathbb{Z}_p \text{ and } y - x \in H(p, k)\} \). Then, we have the following result.

**Lemma 5.2.** (See [4].) For any odd prime \( p \) and a positive integer \( k \) dividing \( p - 1 \), the graph \( G(2p, k) \) is not one-regular if \((p, k) = (7, 3)\) or \((11, 5)\). Otherwise, \( G(2p, k) \) is one-regular.

Let \( p \) be an odd prime and let \((p, \ell, k)\) be a triple in \( \mathcal{T} \) with \( k < p \). Then, \( k \) should divide \( p - 1 \) because \( \ell \neq 1 \) (mod \( p \)), \((\ell - 1)(\ell^{k-1} + \ell^{k-2} + \cdots + 1) = \ell^k - 1 \equiv 0 \) (mod \( p \)) and the multiplicative group \( \mathbb{Z}_p^* \) is cyclic of order \( p - 1 \). And, the graph \( G = \text{Cay}(D_p; \ell, k) = \text{Cay}(D_p, \{b = a^{\ell[0]}b, a^{\ell[1]}b, a^{\ell[2]}b, \ldots, a^{\ell[k-1]}b\}) \) is isomorphic to the graph \( G' = \text{Cay}(D_p, \{ab, a^\ell b, a^{\ell^2} b, \ldots, a^{\ell^{k-1}} b\}) \) because \( \ell \neq 1 \) (mod \( p \)) and the group automorphism \( \alpha \) of \( D_p \) defined by \( a^\alpha = a^{\ell-1} \) and \( b^\alpha = ab \) induces a graph isomorphism from \( G \) to \( G' \). Furthermore, the bijection \( \beta : D_n \rightarrow A \cup A' \) defined by \((a^i)^\beta = i \) and \((a^i b)^\beta = i' \) for any \( i \in \mathbb{Z}_p \) induces the graph isomorphism from \( G' \) to the
graph \( G(2p, k) \). Hence, \( \text{Cay}(D_p; \ell, k) \) is isomorphic to \( G(2p, k) \) for any triple \((p, \ell, k) \in T \) with \( p > k \). Therefore, for any odd prime \( p > k = 5 \) and for any two triples \((p, \ell_1, 5), (p, \ell_2, 5) \in T \), two graphs \( \text{Cay}(D_p; \ell_1, 5) \) and \( \text{Cay}(D_p; \ell_2, 5) \) are isomorphic because both graphs are isomorphic to the graph \( G(2p, 5) \). In particular, by Lemma 5.2, the graphs \( \text{Cay}(D_n; \ell, 5) \) are not one-regular for the pairs \((n, \ell) = (11, 9), (11, 5), (11, 4) \) and \((11, 3) \). And, they are one-regular for \( n = 31, 41 \) and \( 61 \). Note that \( \text{Cay}(D_5; 1, 5) = \text{Cay}(D_5, \{b, ab, a^2b, a^3b, a^4b\}) \) is the complete bipartite graph \( K_{5,5} \), which is not one-regular. So, the remaining cases in the list in Lemma 5.1 are \((n, \ell) = (55, 16), (55, 26), (55, 31) \) and \((55, 36) \).

**Lemma 5.3.** For any pair \((n, \ell) \) in the list in Lemma 5.1, the Cayley graph \( \text{Cay}(D_n; \ell, 5) \) is one-regular except the pairs \((5, 1), (11, 9), (11, 5), (11, 4) \) and \((11, 3) \). Furthermore, any two such graphs which have the same number of vertices are isomorphic.

**Proof.** We need to prove it for only \( n = 55 \) because \( n = 55 \) is a unique non-prime in the list in Lemma 5.1. Note that

1. \( \text{Cay}(D_{55}; 16, 5) = \text{Cay}(D_{55}, \{b, ab, a^{17}b, a^{-2}b, a^{24}b\}) \),
2. \( \text{Cay}(D_{55}; 36, 5) = \text{Cay}(D_{55}, \{b, ab, a^{-18}b, a^{13}b, a^{-26}b\}) \),
3. \( \text{Cay}(D_{55}; 26, 5) = \text{Cay}(D_{55}, \{b, ab, a^{27}b, a^{-12}b, a^{19}b\}) \), and
4. \( \text{Cay}(D_{55}; 31, 5) = \text{Cay}(D_{55}, \{b, ab, a^{-23}b, a^{3}b, a^{-16}b\}) \).

Define three group automorphisms \( \alpha_1: D_{55} \rightarrow D_{55} \) by \( b \mapsto b, a \mapsto a^{13} \), \( \alpha_2: D_{55} \rightarrow D_{55} \) by \( b \mapsto b, a \mapsto a^{19} \) and \( \alpha_3: D_{55} \rightarrow D_{55} \) by \( b \mapsto b, a \mapsto a^{-16} \). Then, we have

\[
\{b, ab, a^{17}b, a^{-2}b, a^{24}b\}^{\alpha_1} = \{b, ab, a^{-18}b, a^{13}b, a^{-26}b\},
\]

\[
\{b, ab, a^{17}b, a^{-2}b, a^{24}b\}^{\alpha_2} = \{b, ab, a^{27}b, a^{-12}b, a^{19}b\},
\]

\[
\{b, ab, a^{17}b, a^{-2}b, a^{24}b\}^{\alpha_3} = \{b, ab, a^{-23}b, a^{3}b, a^{-16}b\}.
\]

Hence, the four graphs

\( \text{Cay}(D_{55}; 16, 5), \text{Cay}(D_{55}; 26, 5), \text{Cay}(D_{55}; 31, 5) \) and \( \text{Cay}(D_{55}; 36, 5) \)

are all isomorphic.

Now, it is enough to show that \( \text{Cay}(D_{55}; 16, 5) = \text{Cay}(D_{55}, \{b, ab, a^{17}b, a^{-2}b, a^{24}b\}) \) is one-regular. Since its arc-transitivity is already known, it suffices to show \( \text{Aut}(\text{Cay}(D_{55}; 16, 5)) = \{1\} \), where \( 1 \) is the identity. Let \( \gamma \in \text{Aut}(\text{Cay}(D_{55}; 16, 5)) \). By a direct chasing the graph up to \( N_0(1) \cup N_1(1) \cup N_2(1) \cup N_3(1) \), one can notice that among vertices in \( N_2(b) \cap N_3(1) \), the end vertex \( a^{-17}b \) of the 3-path \((1, b, b \cdot a^{17}b = a^{-17}, a^{-17}b) \) has only one adjacent vertex in \( N_2(1) \) which is just \( a^{-17} \), but all the other vertices in \( N_2(b) \cap N_3(1) \) have more than one adjacent vertex in \( N_2(1) \). Hence, \( \gamma \) should fix the 3-path \((1, b, b \cdot a^{17}b = a^{-17}, a^{-17}b) \) pointwise. Now by applying the same process to \( L_{a^{17}} \circ \gamma \circ L_{a^{-17}} \), one can see that \( \gamma \) fixes the vertices \( a^{-34} \) and \( a^{-34}b \). Continuing the same process, one can conclude that \( \gamma \) fixes \( a^{-t} \) and \( a^{-t}b \) for every multiple \( t \) of \( \gcd(17, 55) = 1 \). So, \( \gamma \) fixes all vertices of \( \text{Cay}(D_{55}; 16, 5) \) and \( \text{Cay}(D_{55}; 16, 5) \) is one-regular. \( \square \)

Now by combining Propositions 2.1, 2.2 and 4.1, Theorem 3.2 and Lemmas 5.1 and 5.3, one can obtain Theorem B.
Remark. D. Marušič and R. Nedela [17] gave a relation between tetravalent graphs admitting half-transitive actions and one-regular graphs with cyclic vertex stabilizer via concept of orientable regular maps. According to this connection (in particular, by [17, Theorem 3.2]), one can construct infinitely many finite tetravalent half-transitive graphs from the graphs in Theorem B.

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References