BULLETIN OF THE POLISH ACADEMY OF SCIENCES
TECHNICAL SCIENCES, Vol. 63, No. 2, 2015
DOI: 10.1515/bpasts-2015-0055

Finite-time stability and stabilization of singular state-delay systems using improved estimation of a lower bound on a Lyapunov-like functional

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Abstract. In this paper, the problems of finite-time stability and stabilization for a class of singular time-delay systems are studied. Using the Lyapunov-like functional (LLF) with (exponential or power) weighting function and a new estimation method for the lower bound on LLF, some sufficient stability conditions are introduced. It is shown that the weighting function significantly reduces the conservatism of the stability criteria in comparison to estimation of the lower bound on LLF without this function. To solve the finite-time stabilization problem, a stabilizing linear state controller is designed by exploiting the cone complementarity linearization algorithm. Two numerical examples are given to illustrate the effectiveness of the proposed method.

Key words: singular systems, time-delay, LMIs, finite-time stability, stabilizing controller, cone complementarity linearization algorithm.

1. Introduction

The class of singular systems has been extensively studied in the past years due to the fact that the singular model is a natural representation of practical systems and can better describe a large class of systems (power systems, electrical systems, social economic systems and chemical systems) than regular ones [1–4]. In general, the singular representation consists of differential and algebraic equations, and hence it is a generalized representation of the state-space system. These systems are also known as descriptor systems, implicit systems, generalized state-space systems, differential-algebraic systems or semi-state systems. Analysis and design of the singular systems cannot be easily treated in the same way as that of regular systems. Namely, it is well known that singular systems are much more complicated than regular ones, because they cannot be regular and impulse free. Therefore, the regularity and absence of impulses must be considered simultaneously with other system properties (stability, robustness, controllability. . .).

In many practical systems (chemical engineering systems, large-scale electric network control, aircraft attitude control, flexible arm control of robots, etc.) a time-delay often appears [5–7]. When a time-delay is small, then it can be ignored, but if not, it may cause instability in the system. For the sake of infinite dimension of the singular time-delay systems, their dynamic behaviour is more difficult to analyse in comparison to singular non-delay systems. For this reason, over the past decades, there has been increasing interest for the stability analysis of singular time-delay systems and many results have been reported [8–13].

Often, Lyapunov asymptotic stability for practical applications is not suitable enough, because there are some cases where large values of the state are not acceptable. For example, in a chemical process, the state variables (such as temperature, humidity, pressure, and so on) are expected to be controlled within certain bounds for fixed time interval. In these cases, the stability concept on finite-time interval (finite-time stability concept) is possible to use. In the existing literature, there are two concepts of finite-time stability (FTS) with a very different meaning. The first concept of FTS requires that the state of system does not exceed a specified bound in a given finite-time interval [14–27], while in the second concept the term FTS is used to describe system whose state approaches equilibrium point in a finite time [27, 28]. In our paper, the first concept of FTS is considered. Further, Amato et al. [14] have extended the definition of FTS to the definition of finite-time boundedness (FTB) to take into account the presence of external disturbances.

A little work has been done for the finite-time stability and stabilization of singular time-delay systems. Some results on FTS can be found in [17–20] for singular systems and [21–25] for singular time-delay systems. Paper [21] presents an overview of existing results in the field of FTS and gives new sufficient conditions. Stability conditions are derived using an approach based on Lyapunov-like functionals (LLF). Paper [22] introduces the concepts of practical stability for nonlinear descriptor systems with time-delays in terms of two measurements. Based on Lyapunov functions and the comparison principle, a criterion, by which the problem of a descriptor system with time delay is reduced to that of a standard state-space system without time delays, is derived. In [23]

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new stability conditions have been derived using classical and LMI approach based on the Lyapunov-like functions and their properties on the subspace of consistent initial conditions. The problem of FTB and stabilization of singular time-delay system with time-varying exogenous disturbances is defined in [24]. By combining Lyapunov-like approach and matrix inequality technique, a sufficient condition of FTB and stabilization is given by a set of LMIs and nonlinear constraints. In reference [24], in the process of majoritarian of LLF an integral term with delayed states is omitted. The problem FTS of the singular discrete-time systems is studied in [25, 26].

In this paper, we extend the existing methods of FTS [22–24] by introducing the exponential ($e^{\gamma(t-\theta)}$) and power ($\mu^{t-\theta}$) weighting functions in the LLF. Also, by using integral inequalities with delayed states, the estimation of the lower bound of LLF is improved. As a result of this, less conservative stability results were obtained with regard to some existing ones in the literature [23, 24]. Based on the cone complementarity linearization algorithm, an efficient approach is proposed to design state feedback controller such that the resultant closed-loop singular time-delay systems is finite-time stable. Finally, two numerical examples are provided to show the advantage of developed results.

Throughout this article we use the following notation. Superscript "T" stands for matrix transposition. $\Re^n$ denotes the n-dimensional Euclidean space and $\Re^{n\times m}$ is the set of all real matrices of dimension $n\times m$. $X>0$ means that X is real symmetric and positive definite, and $X>Y$ means that the matrix $X-Y$ is positive definite. $diag\{\cdots\}$ and $trace(\cdots)$ denote block-diagonal matrix and trace of matrix. In symmetric block matrices or long matrix expressions, we use an asterisk $(*)$ to represent a term that is induced by symmetry. Matrices are assumed to be compatible for algebraic operations if their dimensions are not explicitly stated.

2. Problem formulation

Consider the following singular linear continuous time-delay system:

$$\widehat{E}\dot{\widehat{x}}(t)=\widehat{A}\widehat{x}(t)+\widehat{A}_d\widehat{x}(t-\tau)+\widehat{B}u(t) \tag{1}$$

with a known compatible vector valued function of the initial conditions

$$\widehat{x}(t)=\widehat{\phi}(t),\quad t\in[-\tau,0], \tag{2}$$

where $\widehat{x}(t)\in\Re^n$ is the state vector, $u(t)\in\Re^m$ is the control input, $\widehat{A}\in\Re^{n\times n}$, $\widehat{A}_d\in\Re^{n\times n}$ and $\widehat{B}\in\Re^{n\times m}$ are known constant matrices and τ is constant time delay. The matrix $\widehat{E}\in\Re^{n\times n}$ may be singular, and it is assumed that $\mathrm{rank}(\widehat{E})=r\le n$.

Since $\mathrm{rank}(\widehat{E})=r\le n$, there exist invertible matrices $M$and N [8] such that

$$M\widehat{E}N=\begin{bmatrix} I_r & 0\\ 0 & 0\end{bmatrix}\triangleq E. \tag{3}$$

Then, based on the nonsingular transformation

$$x=N^{-1}\widehat{x} \tag{4}$$

the system (1) can be transformed into the following form:

$$\begin{gathered}E\dot{x}(t)=Ax(t)+A_dx(t-\tau)+Bu(t),\\ x(t)=\phi(t),\quad \phi(t)=N^{-1}\widehat{\phi}(t),\quad t\in[-\tau,0],\end{gathered} \tag{5}$$

where $x(t)=[x_1^T(t)\ x_2^T(t)]^T$ is new (temporary) state vector ($x_1\in\Re^r$, $x_2\in\Re^{n-r}$) and

$$\begin{gathered}A=M\widehat{A}N=\begin{bmatrix} A_{11} & A_{12}\\ A_{21} & A_{22}\end{bmatrix},\\ A_d=M\widehat{A}_dN=\begin{bmatrix} A_{d11} & A_{d12}\\ A_{d21} & A_{d22}\end{bmatrix},\\ B=M\widehat{B}=\begin{bmatrix} B_1\\ B_2\end{bmatrix},\\ A_{11},A_{d11}\in\Re^{r\times r},\quad A_{22},A_{d22}\in\Re^{(n-r)\times(n-r)},\\ A_{12},A_{d12}\in\Re^{r\times(n-r)},\quad A_{21},A_{d21}\in\Re^{(n-r)\times r},\\ B_1\in\Re^{r\times m},\quad B_2\in\Re^{(n-r)\times m}\end{gathered} \tag{6}$$

The aim of this paper is to develop finite-time stability and stabilization of the system (1). In order to do this, the following definitions will be used in the proof of the main results.

Definition 1. [8] The matrix pair (E,A) is said to be regular if $\det\,(sE-A)$ is not identically zero.

Definition 2. [8] The matrix pair (E,A) is said to be impulse free if $\deg(\det\,(sE-A))=\mathrm{rank}(E)$. Otherwise the matrix pair is said to be impulsive.

The singular time delay system (5) with $u(t)=0$ may have an impulsive solution. However, the regularity and absence of impulses of the pair (E,A) ensure the existence and uniqueness of an impulse free solution of the unforced system (5) on $[0,\infty)$. The existence of the solutions is defined in the following lemma.

Lemma 1. [8] Suppose that the matrix pair (E,A) is regular and impulse free, then the solution of (5) with $u(t)=0$ exists and is impulse free and unique on $[0,\infty)$.

Based on the previous definitions, we introduce the next definitions for singular time-delay system (5).

Definition 3. The singular time-delay system (5) with $u(t)=0$ is said to be regular and impulse free, if the matrix pair (E,A) is regular and impulse free.

Definition 4. Singular time-delay system (5) with $u(t)\equiv 0$ is said to be finite-time stable with respect to (α,β,T), $0<\alpha<\beta$, if

$$\begin{gathered}\sup_{t\in[-\tau,0]}\phi^T(t)\phi(t)\le\alpha\Rightarrow x^T(t)E^TEx(t)<\beta,\\ \forall t\in[0,T],\qquad 0<T<+\infty.\end{gathered} \tag{7}$$

Remark 1. Using the state transformation (4), the expression (7) can be written in the following equivalent form:

$$\begin{gathered}\sup_{t\in[-\tau,0]}\widehat{\phi}^T(t)N^{-T}N^{-1}\widehat{\phi}(t)\le\alpha\\ \Rightarrow\widehat{x}^T(t)\widehat{E}^TM^TM\widehat{E}\widehat{x}(t)\le\beta,\\ \forall t\in[0,T]\end{gathered} \tag{8}$$

which is suitable for the definition of FTS of the system (1). Thus, the connection between the FTS of the systems (1) and (5) can be simply established.

We consider the following stabilizing controller:

$$u(t) = Kx(t), \qquad K \in \Re^{m\times n}, \tag{9}$$

where K is design parameter that has to be determined such that the closed-loop system is FTS. Substituting (9) in (5), the resultant closed-loop system is described by:

$$\begin{aligned} \dot{x}(t) = A_K x(t) + A_d x(t-\tau), \qquad A_K = A + BK, \\ x(t) = \phi(t), \qquad t \in [-\tau, 0]. \end{aligned} \tag{10}$$

3. Main results

3.1. Finite-time stability analysis In this section we give new finite-time stability criteria by introducing the exponential ($e^{\gamma(t-\theta)}$) and power ($\mu^{t-\theta}$) weighting functions in the LLF and two delay integral inequality that is used in the process of majorization of LLF.

Theorem 1. The singular time-delay system (5) with $u(t) = 0$ is regular, impulse free and FTS with respect to (α, β, T) if there exist a nonnegative scalar γ, positive scalars λ_1, λ_2, λ_3, λ_4, non-singular matrix

$$P = \begin{bmatrix} P_{11} & P_{12} \\ 0 & P_{22} \end{bmatrix}, \quad P_{11} = P_{11}^T > 0 \tag{11}$$

positive definite symmetric matrices

$$\begin{aligned} Q = \begin{bmatrix} Q_1 & 0 \\ 0 & Q_2 \end{bmatrix} > 0, \quad R = \begin{bmatrix} P_{11} & R_{12} \\ * & R_{22} \end{bmatrix} > 0, \\ Q_1 = Q_1^T > 0, \qquad Q_2 = Q_2^T > 0 \end{aligned} \tag{12}$$

such that the following conditions hold

$$\Omega = \begin{bmatrix} A^T P^T + PA + Q - \gamma PE & PA_d \\ * & -(1+\gamma\rho)Q \end{bmatrix} < 0, \tag{13}$$

$$\lambda_1 I < R, \quad \lambda_2 I > P_{11}, \quad \lambda_3 I < Q < \lambda_4 I \tag{14}$$

$$\alpha\left[\lambda_2 + \rho\lambda_4\right] e^{\gamma T} - \beta\left[\lambda_1 + \rho\lambda_3\right] < 0 \tag{15}$$

where $\rho = (e^{\gamma\tau} - 1)/\gamma \geq 0$.

Proof. First of all, we show the singular system (5) is regular and impulse free under the conditions of the theorem. Suppose a condition (13) holds. Then we have

$$A^T P^T + PA + Q - \gamma PE < 0, \quad Q > 0. \tag{16}$$

From (16) follows:

$$A^T P^T + PA - \gamma PE < 0. \tag{17}$$

Using (11) and (17), we get:

$$\begin{aligned} &A^T P^T + PA - \gamma PE \\ &= \begin{bmatrix} a^* & A_{21}^T P_{22}^T + P_{11}A_{12} + P_{12}A_{22} \\ * & A_{22}^T P_{22}^T + P_{22}A_{22} \end{bmatrix} < 0. \end{aligned} \tag{18}$$

where

$$a^* = A_{11}^T P_{11}^T + A_{21}^T P_{12}^T + P_{11}A_{11} + P_{12}A_{21} - \gamma P_{11}.$$

From (18) we deduce:

$$A_{22}^T P_{22} + P_{22}A_{22} < 0 \tag{19}$$

from which we conclude that A_{22} is non-singular. Otherwise, supposing that A_{22} is singular, there must exist a nonzero vector $\eta \in \Re^{n-r}$, which ensures that $A_{22}\eta = 0$. Then we can conclude that $\eta^T\left(A_{22}^T P_{22} + P_{22}A_{22}\right)\eta = 0$, which is in contradiction with (19); so A_{22} is non-singular.

Using proprieties of determinant, we have

$$\begin{aligned} \det(sE - A) &= \det\left(\begin{bmatrix} sI_r - A_{11} & -A_{12} \\ -A_{21} & -A_{22} \end{bmatrix}\right) \\ &= \det(-A_{22})\det\left(sI_r - (A_{11} - A_{12}A_{22}^{-1}A_{21})\right) \end{aligned} \tag{20}$$

which means that $\det(sE - A) \not\equiv 0$ and $\deg\det(zE - A) = r = \operatorname{rank}(E)$. Therefore, (E, A) is regular and impulse free and, based on Definition 3, the unforced system (5) is regular and impulse free.

Next, we will show the unforced ($u(t) = 0$) singular time-delay system (5) is finite-time stable. Let us choose the following LLF with the exponential weighting function $e^{\gamma(t-\theta)}$:

$$V\left(x(t)\right) = x^T(t)PEx(t) + \int_{t-\tau}^{t} e^{\gamma(t-\theta)} x^T(\theta)Qx(\theta)d\theta. \tag{21}$$

From (11) and (12) we have $PE = E^T P^T \geq 0$ and $PE = E^T RE$. Then, we have:

$$\begin{aligned} \dot{V}(x(t)) &= \dot{x}^T(t)PEx(t) + x^T(t)PE\dot{x}(t) \\ &+ \frac{d}{dt}\int_{t-\tau}^{t} e^{\gamma(t-\theta)}x^T(\theta)Qx(\theta)d\theta \\ &= x^T(t)(A^T P + PA)x(t) + 2x^T(t)PA_d x(t-\tau) \\ &-\gamma x^T(t)PEx(t) + \gamma x^T(t)PEx(t) \\ &+ \int_{t-\tau}^{t} \frac{d}{dt}\left(e^{\gamma(t-\theta)}x^T(\theta)Qx(\theta)\right)d\theta \\ &+ e^{\gamma(t-\theta)}x^T(\theta)Qx(\theta)\Big|_{\theta=t} \times \frac{d}{dt}t \\ &- e^{\gamma(t-\theta)}x^T(\theta)Qx(\theta)\Big|_{\theta=t-\tau} \times \frac{d}{dt}(t-\tau) \\ &= x^T(t)(A^T P + PA - \gamma PE)x(t) \\ &+2x^T(t)PA_d x(t-\tau) + \gamma V_1(x(t)) \\ &+\gamma\int_{t-\tau}^{t} e^{\gamma(t-\theta)}x^T(\theta)Qx(\theta)d\theta + x^T(t)Qx(t) \\ &-e^{\gamma\tau}x^T(t-\tau)Qx(t-\tau) = \gamma V(x(t)) \\ &+ \begin{bmatrix} x(t) \\ x(t-\tau) \end{bmatrix}^T \begin{bmatrix} A^T P^T + PA + Q - \gamma PE & PA_d \\ * & -e^{\gamma\tau}Q \end{bmatrix} \\ &\begin{bmatrix} x(t) \\ x(t-\tau) \end{bmatrix} = \gamma V(x(t)) + \xi(t)^T\Omega\xi(t), \end{aligned} \tag{22}$$

where $\xi(t) = \begin{bmatrix} x(t)^T & x(t-\tau)^T \end{bmatrix}^T$ and $e^{\gamma\tau} = 1 + \rho\gamma$. Based on (13) it is easy to see that

$$\dot{V}(x(t)) < \gamma V(x(t)) \tag{23}$$

Integrating (23) from 0 to t, with $t \in [0, T]$, we have:

$$V(x(t)) < e^{\gamma t} V(x(0)). \tag{24}$$

Furthermore:

$$\begin{aligned} V(x(0)) &= x^T(0)PEx(0) + \int_{-\tau}^{0} e^{-\gamma\theta} x^T(\theta)Qx(\theta)d\theta \\ &\le \lambda_{\max}(P_{11})x^T(0)x(0) + \lambda_{\max}(Q)\int_{-\tau}^{0} e^{-\gamma\theta} x^T(\theta)x(\theta)d\theta \\ &\le \lambda_{\max}(P_{11})\alpha + \lambda_{\max}(Q)\alpha\frac{e^{\gamma\tau}-1}{\gamma} \\ &\le \alpha\left[\lambda_{\max}(P_{11}) + \rho\lambda_{\max}(Q)\right]. \end{aligned} \tag{25}$$

On the other hand, the majorization of LLF at the time t gives the following (first) delay integral inequality:

$$\begin{aligned} V(x(t)) &= x^T(t)PEx(t) + \int_{t-\tau}^{t} e^{\gamma(t-\theta)} x^T(\theta)Qx(\theta)d\theta \\ &\ge x^T(t)E^T REx(t) + \int_{t-\tau}^{t} e^{\gamma(t-\theta)} x^T(\theta)E^T QEx(\theta)d\theta \\ &> \lambda_{\min}(R)x^T(t)E^T E^T x(t) \\ &+ \lambda_{\min}(Q)\int_{t-\tau}^{t} e^{\gamma(t-\theta)} x^T(\theta)E^T Ex(\theta)d\theta. \end{aligned} \tag{26}$$

Combining (24)–(26) we obtain:

$$\begin{aligned} &\lambda_{\min}(R)x^T(t)E^T E^T x(t) \\ &+\lambda_{\min}(Q)\int_{t-\tau}^{t} e^{\gamma(t-\theta)} x^T(\theta)E^T Ex(\theta)d\theta \\ &< e^{\gamma t}\alpha\left[\lambda_{\max}(P_{11}) + \rho\,\lambda_{\max}(Q)\right]. \end{aligned} \tag{27}$$

If the following inequality is satisfied:

$$\begin{aligned} &e^{\gamma t}\alpha\left[\lambda_{\max}(P_{11}) + \rho\,\lambda_{\max}(Q)\right] \\ &< \beta\left[\lambda_{\min}(R) + \rho\lambda_{\min}(Q)\right], \\ &\forall t \in [0, T] \end{aligned} \tag{28}$$

then:

$$\begin{aligned} &\lambda_{\min}(R)x^T(t)E^T E^T x(t) \\ &+\lambda_{\min}(Q)\int_{t-\tau}^{t} e^{\gamma(t-\theta)} x^T(\theta)E^T Ex(\theta)d\theta \\ &< \beta\left[\lambda_{\min}(R) + \rho\lambda_{\min}(Q)\right], \\ &\forall t \in [0, T]. \end{aligned} \tag{29}$$

From the last inequality and (28), the following condition holds:

$$x^T(t)E^T Ex(t) < \beta, \quad \text{for all} \quad t \in [0, T]. \tag{30}$$

Further, from (28) follows:

$$\begin{aligned} &\alpha\left[\lambda_{\max}(P_{11}) + \rho\lambda_{\max}(Q)\right] e^{\gamma T} \\ &< \beta\left[\lambda_{\min}(R) + \rho\lambda_{\min}(Q)\right]. \end{aligned} \tag{31}$$

Let

$$\begin{aligned} &0 < \lambda_1 < \lambda_{\min}(R), \quad \lambda_2 > \lambda_{\max}(P_{11}), \\ &\lambda_3 < \lambda_{\min}(Q), \quad \lambda_4 > \lambda_{\max}(Q). \end{aligned} \tag{32}$$

Then, (14) and (15) are satisfied. This completes the proof.

Remark 2. Instead of the exponential weighting function $e^{\gamma(t-\theta)}$, it is possible, too, to use other functions in LLF (21). For example, by using the power weighting function $\mu^{t-\theta}$ in LLF

$$V(x(t)) = x^T(t)PEx(t) + \int_{t-\tau}^{t} \mu^{t-\theta} x^T(\theta)Qx(\theta)d\theta \tag{33}$$

the following stability criterion can be obtained.

Theorem 2. The singular time-delay system (5) with $u(t) = 0$ is regular, impulse free and FTS with respect to (α, β, T) if there exist a nonnegative scalar μ, positive scalars λ_1, λ_2, λ_3, λ_4, non-singular matrix P and positive definite symmetric matrices Q and$R$ defined by (11) and (12), such that the following conditions hold:

$$\Sigma = \begin{bmatrix} A^T P^T + PA + Q - \ln\mu PE & PA_d \\ * & -(1+\rho\ln\mu)Q \end{bmatrix} < 0, \tag{34}$$

$$\lambda_1 I < R, \quad \lambda_2 I > P_{11}, \quad \lambda_3 I < Q < \lambda_4 I, \tag{35}$$

$$\alpha\left[\lambda_2 + \rho\lambda_4\right]\mu^T - \beta\left[\lambda_1 + \rho\lambda_3\right] < 0, \tag{36}$$

where $\rho = (\mu^\tau - 1)/\ln\mu \ge 0$.

Remark 3. The proof of this theorem can be obtained by following the same proof procedure as given in Theorem 1. However, the previous result can be directly obtained from Theorem 1 by introducing the following change of variables: $\gamma = \ln\mu$.

Remark 4. In the previous criteria, two innovations are introduced in order to reduce the conservatism of FTS in the existing literature [22–24]. The first innovation is application of LLF with the exponential or the power weighting functions, and the second one is the utilization of delay integral inequalities (26) and (28) for the estimation of the lower bound of LLF. Using the exponential weighting function in LLF (21), the inequality $\dot{V}(x(t)) < \gamma V(x(t))$ is obtained provided that $\Omega < 0$, without any approximation, which is not the case in the existing literature. A similar conclusion can be obtained by using the power weighting function. Further, unlike the inequality (26), the following non-delay and non-integral inequality is used in the existing literature, for the estimation of the lower bound of LLF:

$$V(x(t)) \ge \lambda_{\min}(R)x^T(t)E^T E^T x(t). \tag{37}$$

Obviously, the previous inequality is more restrictive in the comparison with the inequality (26). Accordingly, it should be expected that the Theorems 1–2 give less conservative results in regard to the existing results (see Example 1 as the confirmation).

In the following, we will consider the individual influence of each innovation to the conservatism of FTS. Firstly, we use the innovation with the exponential weighting function, without the second innovation (instead of inequality (26), we use (37)). In this case, the following result is obtained.

Corollary 1. The singular time-delay system (5) with $u(t) = 0$ is regular, impulse free and FTS with respect to (α, β, T) if there exist a nonnegative scalar γ, positive scalars λ_1, λ_2, λ_3, non-singular matrix P and positive definite symmetric matrices Q and R defined by (11) and (12), such that the (13) and the following condition hold:

$$\lambda_1 I < R, \quad \lambda_2 I > P_{11}, \quad \lambda_3 I > Q, \tag{38}$$

$$\alpha\left[\lambda_2 + \rho\lambda_3\right] e^{\gamma T} - \beta\lambda_1 < 0. \tag{39}$$

Remark 5. The previous criterion is more conservative compared to Theorem 1 (see Example 1). However, it is still a less restrictive than the criteria given in [22–24], because of the existence of the exponential weighting function $e^{\gamma(t-s)}$ in LLF. If the weighting function is omitted in LLK, then we get the following simple LLF with two terms:

$$V(x(t)) = x^T(t)PEx(t) + \int_{t-\tau}^{t} x^T(\theta)Qx(\theta)d\theta \tag{40}$$

which is identical to the functional that used in [22, 23]. In [24], an additional (third) term is inserted in LLF in order to achieve the delay-dependent stability. However, due to the omission of both the exponential weighting function in LLF and inequality (26), the criteria given in [24] are also restrictive (see Example 1).

Next, we present a stability criterion that uses delay integral inequalities (26) and (28) and LLF (40) without weighting function.

Corollary 2. The singular time-delay system (5) with $u(t) = 0$ is regular, impulse free and FTS with respect to (α, β, T) if there exist a nonnegative scalar γ, positive scalars λ_1, λ_2, λ_3, λ_4, non-singular matrix P and positive definite symmetric matrices Q and R defined by (11) and (12), such that (14), (15) and the following condition hold:

$$\Omega = \begin{bmatrix} A^T P^T + PA + Q - \gamma PE & PA_d \\ * & -Q \end{bmatrix} < 0. \tag{41}$$

The last case of a stability criterion, which shall be considered, is based on the LLF (40) without both exponential weighting function and delay integral inequalities (26) and (28). Instead of (26) we use (37).

Corollary 3. The singular time-delay system (5) with $u(t) = 0$ is regular, impulse free and FTS with respect to (α, β, T) if there exist a nonnegative scalar γ, positive scalars λ_1, λ_2, λ_3, non-singular matrix P and positive definite symmetric matrices Q and R defined by (11) and (12), such that the conditions (38), (39) and (41) hold.

Remark 6. In [22, Theorem 6], [23, Corollary 1] and [24, Theorem 1], the similar results are obtained. Compared with the results of Theorem 1 and Corollaries 1–2, the previous criterion gives the most restrictive results (see Example 1). As the references [22–24] do not use the exponential weighting function and the delay integral inequalities (26) and (28), their results are very restrictive in comparison to Theorem 1 and Corollary 1–2. (see Example 1).

Remark 7. Based on some numerical computations (see Example 1), it can be shown that the weighting function in LLF has a dominant influence on the restrictiveness of FTS, as opposed to the delay integral inequalities (26) and (28), whose influence is lower.

3.2. Finite-time stabilization. Using Theorem 1–2 and Corollary 1–3, we are in a position to design a state feedback controller (9) such that the resultant closed-loop system (10) is regular, impulse free and finite-time stable. For practical reasons, only the problem of the stabilization based on Theorem 1 has been addressed below.

Theorem 3. The closed-loop singular time-delay system (10) is regular, impulse free and finite-time stable with respect to (α, β, T) if there exist a nonnegative scalar γ, positive scalars λ_1, λ_2, λ_3, λ_4, non-singular matrix X

$$X = \begin{bmatrix} X_{11} & 0 \\ X_{21} & X_{22} \end{bmatrix}, \qquad X_{11} = X_{11}^T > 0 \tag{42}$$

positive definite symmetric matrices Y, R

$$R = \begin{bmatrix} R_{11} & R_{12} \\ * & R_{22}l \end{bmatrix}, \qquad R_{11} = X_{11}^{-1} \tag{43}$$

and matrix Lsuch that the following conditions hold

$$\Gamma = \begin{bmatrix} X^T A^T + AX + BL + L^T B^T + Y - \gamma EX & A_d X \\ * & -(1+\gamma\rho)Y \end{bmatrix} < 0, \tag{44}$$

$$\lambda_1 I < R, \quad \begin{bmatrix} X_{11} & I \\ * & \lambda_2 I \end{bmatrix} > 0, \tag{45}$$

$$\begin{bmatrix} Y & X^T \\ * & \lambda_3^{-1} I \end{bmatrix} > 0, \quad \begin{bmatrix} Y^{-1} & X^{-1} \\ * & \lambda_4 I \end{bmatrix} > 0, \tag{46}$$

$$\alpha\left[\lambda_2 + \rho\lambda_4\right] e^{\gamma T} - \beta\left[\lambda_1 + \rho\lambda_3\right] < 0, \tag{47}$$

where $\rho = (e^{\gamma\tau} - 1)/\gamma \geq 0$. The state feedback controller is $u(t) = Kx(t) = LX^{-1}x(t)$.

Proof. By applying the congruence transformation with non-singular matrix $diag\left\{P^{-T},\ P^{-T}\right\}$ to the matrix Ω results to

$$\begin{bmatrix} b^* & A_d P^{-T} \\ * & -(1+\gamma\rho)P^{-1}QP^{-T} \end{bmatrix} < 0, \tag{48}$$

where

$$b^* = P^{-1}A^T + AP^{-T} + P^{-1}QP^{-T} - \gamma EP^{-T}$$

Let

$$X \triangleq P^{-T} > 0, \quad Y = Y^T \triangleq P^{-1}QP^{-T} > 0 \tag{49}$$

then

$$\begin{bmatrix} X^T A^T + AX + Y - \gamma EX & A_d X \\ * & -(1+\gamma\rho)Y \end{bmatrix} < 0 \quad (50)$$

From (11), (12), $PE = E^T P^T \geq 0$ and $PE = E^T RE$ we have

$$X = P^{-T} = \begin{bmatrix} P_{11} & P_{12} \\ 0 & P_{22} \end{bmatrix}^{-T}$$

$$= \begin{bmatrix} P_{11}^{-1} & -P_{11}^{-1} P_{12} P_{22}^{-1} \\ 0 & P_{22}^{-1} \end{bmatrix}^T \quad (51)$$

$$= \begin{bmatrix} P_{11}^{-T} & 0 \\ -P_{22}^{-T} P_{12}^T P_{11}^{-T} & P_{22}^{-T} \end{bmatrix} = \begin{bmatrix} X_{11} & 0 \\ X_{21} & X_{22} \end{bmatrix},$$

$$PE = X^{-T} E = E^T P^T = E^T X^{-1}, \quad (52)$$

$$X_{11} = P_{11}^{-T} = P_{11}^{-1} = R_{11}^{-1}, \quad (53)$$

$$PE = X^{-T} E = E^T RE. \quad (54)$$

Replacing A in (50) with $A + BK$ yields:

$$\begin{bmatrix} \begin{matrix} X^T A^T + AX + X^T K^T B^T \\ +BKX + Y - \gamma EX \end{matrix} & A_d X \\ * & -(1+\gamma\rho)Y \end{bmatrix} < 0. \quad (55)$$

If we adopt $K \triangleq LX^{-1}$, then (44) is obtained.

Let

$$\begin{aligned} 0 < \lambda_1 < \lambda_{\min}(R) > 0, \\ \lambda_2 > \lambda_{\max}(X_{11}^{-1}) > 0, \\ \lambda_3 < \lambda_{\min}(X^{-T} Y X^{-1}) > 0, \\ \lambda_4 < \lambda_{\max}(X^{-T} Y X^{-1}) > 0. \end{aligned} \quad (56)$$

Then, form (14), (15), (31), (32) and (56) we get (47) and:

$$\begin{aligned} \lambda_1 I < R, \quad \lambda_2 I > X_{11}^{-1}, \\ \lambda_3 I < X^{-T} Y X^{-1}, \quad \lambda_4 I > X^{-T} Y X^{-1}. \end{aligned} \quad (57)$$

Using the Schur complement, from (57) we get (45), (46). The proof is completed.

Remark 9. It should be pointed out that the stabilization problem in Theorem 3 is non-convex feasibility problem [29] due to the existence of the nonlinear terms γ, λ_3, R_{11}, X and Y in (43)–(47). Consequently, a global minimum of the stability problem cannot be found by tools for the convex optimization. However, by using some appropriate transformations and fixing parameter γ, this non-convex feasibility problem can be turned into a sequential optimization problem subject to LMI constraints.

In the following, we define an algorithm for numerical solution of the above nonlinear stabilization problem by using Theorem 3. If we define

$$\mu = \lambda_3^{-1}, \quad U = X^{-1}, \quad V = Y^{-1} \quad (58)$$

then (43), (46) and (58) can be approximately translated into

$$\begin{aligned} &\begin{bmatrix} Y & X^T \\ X & \mu I \end{bmatrix} > 0, \quad \begin{bmatrix} V & U \\ U^T & \lambda_4 I \end{bmatrix} > 0, \\ &\begin{bmatrix} R_{11} & I \\ I & X_{11} \end{bmatrix} \geq 0, \quad \begin{bmatrix} X & I \\ I & U \end{bmatrix} \geq 0, \\ &\begin{bmatrix} Y & I \\ I & V \end{bmatrix} \geq 0, \quad \begin{bmatrix} \lambda_3 & 1 \\ 1 & \mu \end{bmatrix} \geq 0. \end{aligned} \quad (59)$$

Using the cone complementarity algorithm [30], we formulate the following minimization problem instead of the original non-convex feasibility problems defined in Theorem 3.

Problem 1.

$$\min \{ trace\,(X_{11} R_{11}) + trace\,(XU + YV) + \lambda_3 \mu \} \quad (60)$$

subject to (42), (44), (45), (47) and (59).

If the solution of Problem 1 is $2n + r + 1$, then the conditions in Theorem 3 are solvable.

An iterative algorithm that solves the above nonlinear optimization problem is developed below.

Algorithm 1.

Step 1. Adopt a value of computational precision $\delta > 0$. Choose a sufficiently small $\gamma > 0$ such that there exists a feasible solution to (42), (44), (45), (47) and (59).

Step 2. Find a feasible set $(X^0, Y^0, R^0, U^0, V^0, L^0, \lambda_1^0, \lambda_2^0, \lambda_3^0)$ satisfying (42), (44), (45), (47) and (59). If the condition

$$\begin{aligned} \big| trace\,(R_{11}^0 X_{11}^0) + trace\,(U^0 X^0 + V^0 Y^0) \\ + \lambda_3 \mu - (2n + r + 1) \big| < \delta \end{aligned} \quad (61)$$

is satisfied, the controller gain can be given by $K = L^0 (X^0)^{-1}$. Otherwise, set $k = 0$ and go to the next step.

Step 3. Solve the following LMI optimization problem for $(X, Y, R, U, V, L, \lambda_1, \lambda_2, \lambda_3)$

$$\begin{aligned} \min \big\{ trace\,(R_{11}^k X_{11} + X_{11}^k R_{11}) + \lambda_3^k \mu + \mu^k \lambda \\ + trace\,(U^k X + X^k U + V^k Y + Y^k V) \big\} \end{aligned} \quad (62)$$

subject to (42), (44), (45), (47) and (59)

and set $X^{k+1} = X$, $Y^{k+1} = Y$, $R^{k+1} = R$, $U = U^{k+1}$, $V^{k+1} = V$, $\lambda_3^{k+1} = \lambda_3$, $\mu^{k+1} = \mu$, $L^{k+1} = L$.

Step 4. If the condition

$$\begin{aligned} \big| trace\,(R_{11}^{k+1} X_{11}^{k+1}) \\ + trace\,(U^{k+1} X^{k+1} + V^{k+1} Y^{k+1}) \\ + \lambda_3^{k+1} \mu^{k+1} - (2n + r + 1) \big| < \delta \end{aligned} \quad (63)$$

is satisfied, then controller gain is chosen as: $K = L^{k+1} (X^{k+1})^{-1}$. Otherwise, set $k = k + 1$ and go to Step 3. If $k > N$, where N is the maximum number of iterations allowed, then exit.

Remark 10. The conditions in Theorem 3 are expressed in the form of nonlinear matrix inequalities as non-convex feasibility problem (see Remark 9) with unknown variables. Assumptions of the variables are obtained in the proof of Theorem 3, which is based on the proof of Theorem 1. By using the cone complementarity algorithm [30], for a fixed value of the parameter γ, this non-convex feasibility problem is transformed into a sequential optimization problem subject to linear matrix inequalities (LMIs) (see Problem 1 and Algorithm 1). If there exist a solution of the sequential optimization problem, then the assumptions of Theorem 3 are satisfied and considered system is FTS.

4. Numerical examples and simulation

Now, we present numerical examples that illustrate the performance of the proposed results.

Example 1. Consider the singular time-delay system (5) with:

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} -2 & 1 & 0 \\ 0 & -2 & 0 \\ -1 & 0 & -2 \end{bmatrix},$$
$$A_d = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^T, \tag{64}$$
$$\tau = 1.$$

In order to verify the stability properties of the open-loop system (5), the system operation is simulated under the conditions $\phi(t) = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^T$, $t \in [-\tau, 0]$. Figure 1 shows the norm of the state vector $x^T(t)E^TEx(t)$ of the open-loop system (5). It is observed that the open-loop system (5) is not asymptotically stable. However, despite this fact, we will find the conditions under which the system is FTS. In other words, we will find the upper bound of T, T_m, so system (5) is regular, impulse free and FTS with respect $\alpha = 3$ and $\beta \in \{10,\ 20,\ 50,\ 100,\ 200,\ 500,\ 1000,\ 10000\}$. Table 1 lists the comparison of T_m for different values of the parameter β by using various methods: Theorem 1, Corollary 1–3 and [23, Corollary 1]. Based on the system simulation, the theoretical upper bound of the parameter T, T_m^t, is estimated from the norm of state vector and also shown in Table 1. Also, Table 1 contains the corresponding values of the parameter γ.

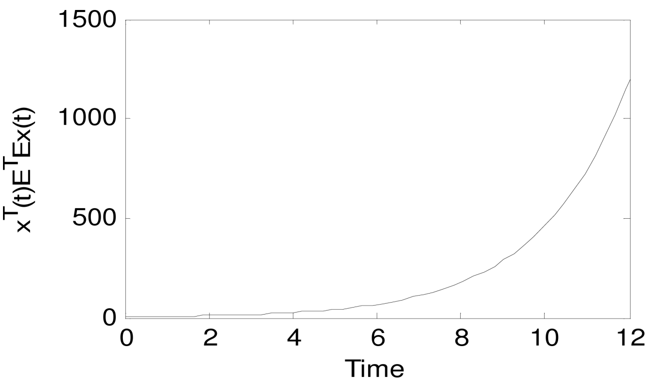

Fig. 1. The norm $x^T(t)E^TEx(t)$ of the state vector of the open-loop system (5)

Figures 2 and 3 show the dependences $\beta(T_m)$ and $\gamma(\beta)$, respectively, using the various methods (Theorem 1, Corollary 1–3 and the system simulation). Based on Fig. 2 and the data from Table 1, it can be seen that Theorem 1 is the least restrictive, as opposed to the Corollary 3 and [23, Corollary 1].

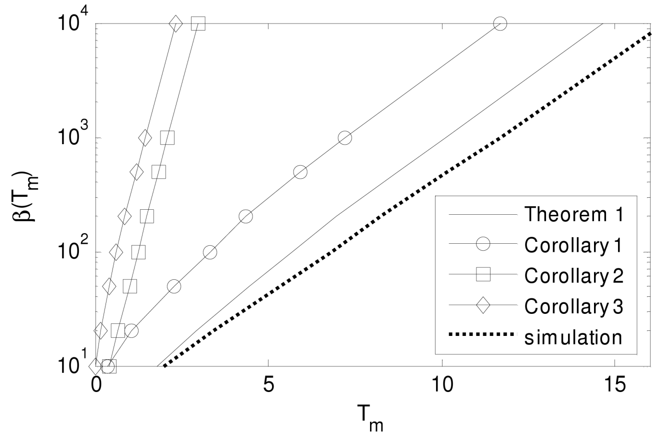

Fig. 2. The dependence $\beta(T_m)$ for different criteria

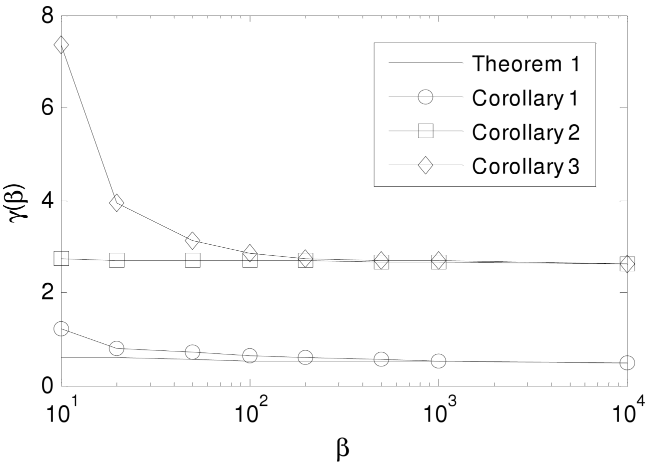

Fig. 3. The dependence $\gamma(\beta)$ for different criteria

The basic properties of the above criteria are given in Table 2. Obviously, the exponential weighting function $e^{\gamma(t-s)}$ in LLF has the greatest impact in reduction of the conservatism, while the influence of the inequalities (26) and (28) with delayed states is somewhat smaller.

Table 1 and Fig. 3 show that the parameter γ is slightly sensitive to the changes of parameter β by applying Theorem 1 and Corollary 2, but it is sensitive by using Corollary 1 and Corollary 3. Therefore, the utilization of the delay integral inequalities (26) and (28) provides the insensitivity of the parameter γ with respect to the parameter β.

Table 1
The upper bound T_m, the theoretical upper bound T_m^t and the parameter γ for different values of the parameter β

Applied method	β	10	20	50	100	200	500	1000	10000
Theorem 1	T_m	1.81	2.93	4.44	5.69	6.99	8.75	10.11	14.66
	γ	0.62	0.62	0.56	0.53	0.52	0.51	0.51	0.50
Corollary 1	T_m	0.39	1.06	2.29	3.3	4.38	5.95	7.22	11.67
	γ	1.23	0.79	0.7	0.65	0.60	0.55	0.53	0.50
Corollary 2	T_m	0.40	0.65	0.99	1.24	1.50	1.84	2.10	2.96
	γ	2.73	2.70	2.70	2.68	2.68	2.66	2.66	2.63
Corollary 3 [23, Corollary 1]	T_m	0.04	0.16	0.41	0.63	0.87	1.20	1.46	2.32
	γ	7.34	3.93	3.11	2.84	2.74	2.68	2.69	2.64
Simulation	T_m^t	1.98	3.44	5.35	6.80	8.25	10.16	11.67	16.43

Table 2
A summary of the basic properties of the obtained results (the presence of the exponential weighting functions $e^{\gamma(t-s)}$, the presence of the delay integral inequalities (26) and (28) and conservatism)

Method	The presence of the exponential weighting functions $e^{\gamma(t-s)}$	The presence of the delay integral inequalities (26) and (28)	Conservatism
Theorem 1	yes	yes	the smallest
Corollary 1	yes	no	smaller
Corollary 2	no	yes	small
Corollary 3	no	no	large
[23]	no	no	large

Based on Theorem 1 for $\gamma = 0.53$ and $(\alpha, \beta, T) = (3, 100, 5.69)$, a feasible solution is

$$P = \begin{bmatrix} 4.70 \cdot 10^2 & -1.13 \cdot 10^1 & -9.94 \cdot 10^1 \\ -1.13 \cdot 10^1 & 9.244 \cdot 10^2 & 1.97 \cdot 10^2 \\ 0 & 0 & 9.19 \cdot 10^2 \end{bmatrix},$$

$$Q = \begin{bmatrix} 1.56 \cdot 10^3 & -3.14 \cdot 10^2 & 0 \\ -3.14 \cdot 10^2 & 1.98 \cdot 10^3 & 0 \\ 0 & 0 & 2.16 \cdot 10^3 \end{bmatrix},$$

$$R = \begin{bmatrix} 4.70 \cdot 10^2 & -1.13 \cdot 10^1 & 0 \\ -1.13 \cdot 10^1 & 9.24 \cdot 10^2 & 0 \\ 0 & 0 & 1.82 \cdot 10^3 \end{bmatrix},$$

$$\lambda_1 = 4.69 \cdot 10^2, \quad \lambda_2 = 9.25 \cdot 10^2,$$
$$\lambda_3 = 1.39 \cdot 10^3, \quad \lambda_4 = 2.16 \cdot 10^3$$

Accordingly, the system (5) is regular, impulse free and FTS with respect to $(\alpha, \beta, T) = (3, 100, 5.69)$.

Example 2. Consider the singular time-delay system (5) where E, A, A_d and τ are defined by (64) and $B = [1\ \ 1\ \ 1]^T$. In order to show validity of our results for closed-loop system, we design a state feedback controller $u(t) = Kx(t)$ such that the resultant closed-loop system (10) is finite-time stable. Solving this control problem by using Theorem 3 and Algorithm 1 for $\alpha = 3$, $\beta = 3.1$, $T = 500$, $\gamma = 10^{-5}$ and $\Delta = 10^{-4}$, we find a feasible solution whit the controller gain:

$$K = \begin{bmatrix} -1.512 \cdot 10^2 & -1.529 \cdot 10^2 & -1.491 \cdot 10^2 \end{bmatrix}.$$

Therefore, the closed-loop system is finite-time stable with respect to $(3, 3.1, 500)$.

Figure 4 shows the norm $x^T(t)E^TEx(t)$ of the closed-loop system for the initial conditions $\phi(t) = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^T$, $t \in [-\tau, 0]$.

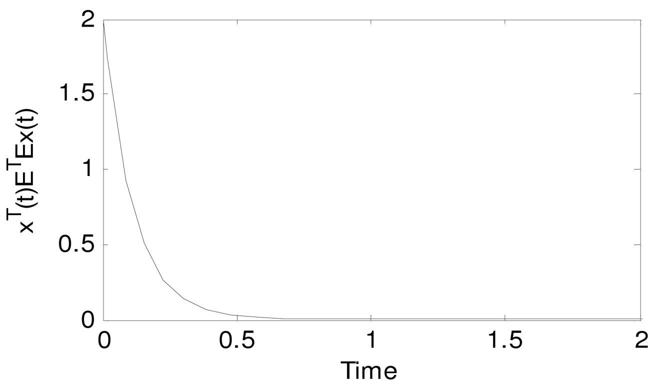

Fig. 4. The norm $x^T(t)E^TEx(t)$ of the state vector of the closed-loop system (10)

5. Conclusions

In this paper, the problems of the finite-time stability and stabilization have been investigated for a class of linear singular time-delay systems. Using the Lyapunov-like functional with exponential or power weighting function and corresponding integral inequalities with delayed states, some sufficient conditions, which guarantee that the singular time-delay system is finite-time stabile, are derived. It has been concluded that the weighting function in Lyapunov-like function has a dominant influence on the conservativeness of the obtained results, until the integral inequalities with delayed states has a lower impact. Starting from these results, a sufficient condition of the finite-time stabilization is derived in the form of a nonlinear feasible problem, which is solved by using a cone complementarity linearization algorithm.

Acknowledgements. This work was supported by the Ministry of Science and Technological Development of Republic of Serbia under the Grant ON174001.

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