MINIMAL DOMINATING FUNCTIONS OF CORONA PRODUCT
GRAPH OF A PATH WITH A STAR

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ABSTRACT

Domination in graphs is an emerging area of research in recent years and has been studied extensively. An introduction and an extensive overview on domination in graphs and related topics is surveyed and detailed in the two books by Haynes et al. [1, 2].

Recently dominating functions is receiving much attention. They give rise to an important classes of graphs and deep structural problems. In this paper we discuss some results on minimal dominating functions of corona product graph of a path with a star.

Key Words: Corona Product, Path, Star, Dominating Function.

Subject Classification: 68R10

1. INTRODUCTION

Domination Theory is an important branch of Graph Theory that has wide range of applications to many fields like Engineering, Communication Networks, Social Sciences, Linguistics, Physical Sciences and many others. Various domination parameters of graphs are studied by Allan, R.B. and Laskar, R.[3]. The concept of minimal dominating functions was introduced by Cockayne, E.J. and Hedetniemi, S.T. [4] and Cockayne et al. [5]. Jeelani Begum, S. [6] has studied some dominating functions of Quadratic Residue Cayley graphs.

Frucht and Harary [7] introduced a new product on two graphs $G_1$ and $G_2$ called corona product denoted by $G_1 \odot G_2$. The object is to construct a new and simple operation on two graphs $G_1$ and $G_2$ called their corona, with the property that the group of
the new graph is in general isomorphic with the wreath product of the groups of G_1 and of G_2.

The authors have studied some dominating functions of corona product graph of a cycle with a complete graph [8] and published papers on minimal dominating functions, some variations of Y – dominating functions and Y – total dominating functions [9,10,11].

In this paper we present some basic properties of corona product graph of a path with a star and some results on minimal dominating functions.

2. CORONA PRODUCT OF P_n AND K_{1,m}

The corona product of a path P_n with star K_{1,m} is a graph obtained by taking one copy of a n – vertex path P_n and n copies of K_{1,m} and then joining the i^{th} vertex of P_n to every vertex of i^{th} copy of K_{1,m} and it is denoted by P_n ∘ K_{1,m}.

We present some properties of the corona product graph P_n ∘ K_{1,m} without proofs and the proofs can be found in Siva Parvathi, M. [8].

Theorem 2.1: The graph G = P_n ∘ K_{1,m} is a connected graph.
Theorem 2.2: The degree of a vertex v_i in G = P_n ∘ K_{1,m} is given by
\[ d(v_i) = \begin{cases} 
  m + 3, & \text{if } v_i \in P_n \text{ and } 2 \leq i \leq (n - 1), \\
  m + 2, & \text{if } v_i \in P_n \text{ and } i = 1 \text{ or } n, \\
  m + 1, & \text{if } v_i \in K_{1,m} \text{ and } v_i \text{ is in first partition}, \\
  2, & \text{if } v_i \in K_{1,m} \text{ and } v_i \text{ is in second partition}. 
\end{cases} \]

Theorem 2.3: The number of vertices and edges in G = P_n ∘ K_{1,m} is given by
1. \( |V(G)| = n (m + 2) \),
2. \( |E(G)| = 2n (m + 1) - 1 \).

Theorem 2.4: The graph G = P_n ∘ K_{1,m} cannot be eulerian.
Theorem 2.5: The graph G = P_n ∘ K_{1,m} is non hamiltonian.
Theorem 2.6: The graph G = P_n ∘ K_{1,m} is not bipartite.

3. DOMINATING SETS AND DOMINATING FUNCTIONS

In this section we discuss dominating sets (DS), dominating functions (DFs) of the graph G = P_n ∘ K_{1,m}. We present some results related to minimal dominating functions of this graph.

Theorem 3.1: The domination number of G = P_n ∘ K_{1,m} is n.
Proof: Let D denote a dominating set of G.
Case 1: Suppose D contains the vertices of path P_n in G.

By the definition of the graph G which is the corona product of P_n and K_{1,m}, the i^{th} vertex v_i in P_n is adjacent to all vertices of i^{th} copy of K_{1,m}. That is the vertices in P_n dominate the vertices in all copies of K_{1,m} respectively. Therefore the vertices in D dominate all vertices of G. Thus D becomes a DS of G.

Case 2: Suppose D contains a vertex of degree m + 1 in each copy of K_{1,m} in G.

As there are n copies of K_{1,m}, it follows that |D| = n. Obviously these vertices in G dominate the vertices in all copies of K_{1,m} respectively and also a single vertex of P_n to which it is connected. Therefore the vertices of D dominate all vertices of G. Further this set is also minimal. Thus \( \gamma(G) = n \).
Theorem 3.2: Let D be a MDS of \( G = P_n \odot K_{1,m} \). Then a function \( f : V \rightarrow [0,1] \) defined by
\[
f(v) = \begin{cases} 
1, & \text{if } v \in D, \\
0, & \text{otherwise.}
\end{cases}
\]
becomes a MDF of \( G = P_n \odot K_{1,m} \).

**Proof:** Consider the graph \( G = P_n \odot K_{1,m} \) with vertex set \( V \).

We have seen in Theorem 3.1 that a DS of \( P_n \) contains all the vertices of \( K_{1,m} \) and this set is minimal. Also the set of vertices whose degree is \( m+1 \) in each copy of \( K_{1,m} \) form a minimal DS of \( G \).

Let \( D \) be a MDS of \( G \). For definiteness, let \( D \) contain the vertices of \( P_n \) in \( G \). In \( P_n \), there are two end vertices whose degree is \( m+2 \) and there are \( n-2 \) intermediate vertices whose degree is \( m+3 \) respectively in \( G \). In \( K_{1,m} \), there is one vertex of degree \( m+1 \) and there are \( m \) vertices whose degree is 2 respectively in \( G \).

**Case 1:** Let \( v \in P_n \) be such that \( d(v) = m+3 \) in \( G \).
Then \( \sum_{u \in N[v]} f(u) = 1 + 1 + f(0, \ldots, 0) = 3 \).

**Case 2:** Let \( v \in P_n \) be such that \( d(v) = m+2 \) in \( G \).
Then \( \sum_{u \in N[v]} f(u) = 1 + f(0, \ldots, 0) = 2 \).

**Case 3:** Let \( v \in K_{1,m} \) be such that \( d(v) = m+1 \) in \( G \).
Then \( \sum_{u \in N[v]} f(u) = 1 + f(0, \ldots, 0) = 1 \).

**Case 4:** Let \( v \in K_{1,m} \) be such that \( d(v) = 2 \) in \( G \).
Then \( \sum_{u \in N[v]} f(u) = 1 + 0 = 1 \).

Therefore for all possibilities, we get \( \sum_{u \in N[v]} f(u) \geq 1, \forall \ v \in V \).

This implies that \( f \) is a DF.

Now we check for the minimality of \( f \).

Define \( g : V \rightarrow [0,1] \) by
\[
g(v) = \begin{cases} 
1, & \text{for } v = v_k \in D, \\
r, & \text{for } v \in D - \{v_k\}, \\
0, & \text{otherwise.}
\end{cases}
\]

where \( 0 < r < 1 \).

Since strict inequality holds at the vertex \( v_k \in D \), it follows that \( g < f \).

**Case (i):** Let \( v \in P_n \) be such that \( d(v) = m+3 \) in \( G \).

**Sub case 1:** Let \( v_k \in N[v] \).
Then \( \sum_{u \in N[v]} g(u) = r + 1 + f(0, \ldots, 0) = r + 2 > 1 \).

**Sub case 2:** Let \( v_k \notin N[v] \).
Then \[ \sum_{u \in N[v]} g(u) = 1 + 1 + \left[ \frac{0 + \ldots + 0}{(m+1) - \infty} \right] = 3. \]

**Case (ii):** Let \( v \in P_n \) be such that \( d(v) = m + 2 \) in \( G \).

**Sub case 1:** Let \( v_k \in N[v] \).

Then \[ \sum_{u \in N[v]} g(u) = r + 1 + \left[ \frac{0 + \ldots + 0}{(m+1) - \infty} \right] = r + 1 > 1. \]

**Sub case 2:** Let \( v_k \notin N[v] \).

Then \[ \sum_{u \in N[v]} g(u) = 1 + 1 + \left[ \frac{0 + \ldots + 0}{(m+1) - \infty} \right] = 2. \]

**Case (iii):** Let \( v \in K_{1,m} \) be such that \( d(v) = m + 1 \) in \( G \).

**Sub case 1:** Let \( v_k \in N[v] \).

Then \[ \sum_{u \in N[v]} g(u) = r + \left[ \frac{0 + \ldots + 0}{(m+1) - \infty} \right] = r < 1. \]

**Sub case 2:** Let \( v_k \notin N[v] \).

Then \[ \sum_{u \in N[v]} g(u) = 1 + \left[ \frac{0 + \ldots + 0}{(m+1) - \infty} \right] = 1. \]

**Case (iv):** Let \( v \in K_{1,m} \) be such that \( d(v) = 2 \) in \( G \).

**Sub case 1:** Let \( v_k \in N[v] \).

Then \[ \sum_{u \in N[v]} g(u) = r + 0 + 0 = r < 1. \]

**Sub case 2:** Let \( v_k \notin N[v] \).

Then \[ \sum_{u \in N[v]} g(u) = 1 + 0 + 0 = 1. \]

This implies that \( \sum_{u \in N[v]} g(u) < 1 \), for some \( v \in V \).

So \( g \) is not a DF.

Since \( g \) is taken arbitrarily, it follows that there exists no \( g < f \) such that \( g \) is a DF.

Thus \( f \) is a MDF.

**Theorem 3.3:** A function \( f : V \rightarrow [0,1] \) defined by \( f(v) = \frac{1}{q} \) \( \forall v \in V \) is a DF of \( G = P_n \odot K_{1,m} \), if \( q \leq 3 \). It is a MDF if \( q = 3 \).

**Proof:** Let \( f \) be a function defined as in the hypothesis.

**Case 1:** Suppose \( q < 3 \).

**Case 1:** Let \( v \in P_n \) be such that \( d(v) = m + 3 \) in \( G \).

So \[ \sum_{u \in N[v]} f(u) = \frac{1}{q} + \frac{1}{q} + \ldots + \frac{1}{q} = \frac{m+4}{q} > 1. \] since \( q < 3 \) and \( m \geq 2 \).

**Case 2:** Let \( v \in P_n \) be such that \( d(v) = m + 2 \) in \( G \).

So \[ \sum_{u \in N[v]} f(u) = \frac{1}{q} + \frac{1}{q} + \ldots + \frac{1}{q} = \frac{m+3}{q} > 1. \] since \( q < 3 \) and \( m \geq 2 \).

**Case 3:** Let \( v \in K_{1,m} \) be such that \( d(v) = m + 1 \) in \( G \).
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So \( \sum_{u \in N[v]} f(u) = \frac{1}{q} + \frac{1}{q} + \ldots + \frac{1}{q} = \frac{m+2}{q} > 1 \), since \( q < 3 \) and \( m \geq 2 \).

**Case 4:** Let \( v \in K_{1,m} \) be such that \( d(v) = 2 \) in \( G \).

So \( \sum_{u \in N[v]} f(u) = \frac{1}{q} + \frac{1}{q} + \frac{1}{q} = \frac{3}{q} > 1 \), since \( q < 3 \).

Therefore, for all possibilities, we get \( \sum_{u \in N[v]} f(u) > 1, \forall \ v \in V \).

This implies that \( f \) is a DF.

Now we check for the minimality of \( f \).

Define \( g : V \to [0,1] \) by

\[
g(v) = \begin{cases} 
  r, & \text{if } v = v_k \in V, \\
  1, & \text{otherwise.}
\end{cases}
\]

where \( 0 < r < \frac{1}{q} \).

Since strict inequality holds at a vertex \( v_k \) of \( V \), it follows that \( g < f \).

**Case (i):** Let \( v \in P_n \) be such that \( d(v) = m + 3 \) in \( G \).

**Sub case 1:** Let \( v_k \in N[v] \).

Then \( \sum_{u \in N[v]} g(u) = r + \frac{1}{q} + \frac{1}{q} + \ldots + \frac{1}{q} = \frac{m+4}{q} \), since \( q < 3 \) and \( m \geq 2 \).

**Sub case 2:** Let \( v_k \notin N[v] \).

The \( \sum_{u \in N[v]} g(u) = \frac{1}{q} + \frac{1}{q} + \ldots + \frac{1}{q} = \frac{m+4}{q} \), since \( q < 3 \) and \( m \geq 2 \).

**Case (ii):** Let \( v \in P_n \) be such that \( d(v) = m + 2 \) in \( G \).

**Sub case 1:** Let \( v_k \in N[v] \).

Then \( \sum_{u \in N[v]} g(u) = r + \frac{1}{q} + \frac{1}{q} + \ldots + \frac{1}{q} = \frac{m+3}{q} \), since \( q < 3 \) and \( m \geq 2 \).

**Sub case 2:** Let \( v_k \notin N[v] \).

Then \( \sum_{u \in N[v]} g(u) = \frac{1}{q} + \frac{1}{q} + \ldots + \frac{1}{q} = \frac{m+3}{q} \), since \( q < 3 \) and \( m \geq 2 \).

**Case (iii):** Let \( v \in K_{1,m} \) be such that \( d(v) = m + 1 \) in \( G \).

**Sub case 1:** Let \( v_k \in N[v] \).

Then \( \sum_{u \in N[v]} g(u) = r + \frac{1}{q} + \frac{1}{q} + \ldots + \frac{1}{q} = \frac{m+2}{q} \), since \( q < 3 \) and \( m \geq 2 \).

**Sub case 2:** Let \( v_k \notin N[v] \).
So $\sum_{u\in N[v]} g(u) = \frac{1}{q} + \frac{1}{q} + \ldots + \frac{1}{q} = \frac{m+2}{q}$, since $q < 3$ and $m \geq 2$.

**Case (iv):** Let $v \in K_{1, m}$ be such that $d(v) = 2$ in $G$.

**Sub case 1:** Let $v_j \in N[v]$.

Then $\sum_{u\in N[v]} g(u) = r + \frac{1}{q} + \frac{1}{q} < \frac{3}{q}$, since $q < 3$, it follows that $\frac{3}{q} > 1$.

**Sub case 2:** Let $v_j \notin N[v]$.

Then $\sum_{u\in N[v]} g(u) = \frac{1}{q} + \frac{1}{q} + \frac{1}{q} = \frac{3}{q} > 1$, since $q < 3$.

Hence, it follows that $\sum_{u\in N[v]} g(u) \geq 1$, $\forall v \in V$.

Thus $g$ is a DF.

This implies that $f$ is not a MDF.

**Case II:** Suppose $q = 3$.

Substituting $q = 3$ in case I, for $v \in P_n$ such that $d(v) = m + 3$, we have

$$\sum_{u\in N[v]} f(u) = \frac{1}{3} + \frac{1}{3} + \ldots + \frac{1}{3} = \frac{m+4}{3} = \frac{1}{3} + \frac{m+1}{3} > 1,$$

and for $v \in P_n$ such that $d(v) = m + 2$, we have

$$\sum_{u\in N[v]} f(u) = \frac{1}{3} + \frac{1}{3} + \ldots + \frac{1}{3} = \frac{m+3}{3} = \frac{1}{3} + \frac{m}{3} > 1.$$  

Also for $v \in K_{1, m}$ such that $d(v) = m + 1$, we have

$$\sum_{u\in N[v]} f(u) = \frac{1}{3} + \frac{1}{3} + \ldots + \frac{1}{3} = \frac{m+2}{3} = \frac{m+1}{3} > 1, \text{ since } m \geq 2.$$  

And for $v \in K_{1, m}$ such that $d(v) = 2$, we have

$$\sum_{u\in N[v]} f(u) = \frac{1}{3} + \frac{1}{3} + \frac{1}{3} = \frac{3}{3} = 1.$$  

Therefore for all possibilities, we get $\sum_{u\in N[v]} f(u) \geq 1$, $\forall v \in V$.

This implies that $f$ is a DF. Now we check for the minimality of $f$.

Define $g : V \rightarrow [0,1]$ by

$$g(v) = \begin{cases} 
  r, & \text{for } v = v_k \in V, \\
  \frac{1}{q}, & \text{otherwise}.
\end{cases}$$

where $0 < r < \frac{1}{q}$.

Since strict inequality holds at a vertex $v_{k}$ of $V$, it follows that $g < f$. Then we can show as in case (i) that for $v \in P_n$ such that $d(v) = m + 3$,
\[ \sum_{u \in N[v]} g(u) = r + \frac{1}{q} + \frac{1}{q} + \ldots + \frac{1}{q} > 1, \text{ if } v_k \in N[v], \]

and \[ \sum_{u \in N[v]} g(u) = \frac{1}{q} + \frac{1}{q} + \ldots + \frac{1}{q} = \frac{m+4}{3} = 1 + \frac{m+1}{3} > 1, \text{ if } v_k \notin N[v]. \]

Again as in case (ii), for \( v \in P_n \) such that \( d(v) = m + 2 \),

We have \[ \sum_{u \in N[v]} g(u) = r + \frac{1}{q} + \frac{1}{q} + \ldots + \frac{1}{q} > 1, \text{ if } v_k \in N[v]. \]

And \[ \sum_{u \in N[v]} g(u) = \frac{1}{q} + \frac{1}{q} + \ldots + \frac{1}{q} = \frac{m+3}{3} = 1 + \frac{m+2}{3} > 1, \text{ if } v_k \notin N[v]. \]

Again we can see as in case (iii) that for \( v \in K_{1,m} \) such that \( d(v) = m + 1 \),

\[ \sum_{u \in N[v]} g(u) = r + \frac{1}{q} + \frac{1}{q} + \ldots + \frac{1}{q} > 1, \text{ if } v_k \in N[v]. \]

And \[ \sum_{u \in N[v]} g(u) = \frac{1}{q} + \frac{1}{q} + \ldots + \frac{1}{q} = \frac{m+2}{3} > 1, \text{ if } v_k \notin N[v]. \]

Similarly we can show as in case (iv) that for \( v \in K_{1,m} \) such that \( d(v) = 2 \),

\[ \sum_{u \in N[v]} g(u) = r + \frac{1}{q} + \frac{1}{q} + \ldots + \frac{1}{q} > 1, \text{ if } v_k \in N[v], \]

and \[ \sum_{u \in N[v]} g(u) = \frac{1}{q} + \frac{1}{q} + \ldots + \frac{1}{q} = \frac{3}{3} = 1, \text{ if } v_k \notin N[v]. \]

This implies that \( \sum_{u \in N[v]} g(u) < 1 \), for some \( v \in V \).

So \( g \) is not a DF.

Since \( g \) is defined arbitrarily, it follows that there exists no \( g < f \) such that \( g \) is a DF.

Thus \( f \) is a MDF.

**Theorem 3.4:** A function \( f : V \to [0, 1] \) defined by \( f(v) = \frac{p}{q}, \forall v \in V \) where \( p = \min (m, n) \) and \( q = \max (m, n) \) is a DF of \( G = P_n \circ K_{1,m} \) if \( \frac{p}{q} \geq \frac{1}{3} \).

Otherwise it is not a DF. Also it becomes MDF if \( \frac{p}{q} = \frac{1}{3} \).

**Proof:** Let \( f : V \to [0, 1] \) be defined by \( f(v) = \frac{p}{q}, \forall v \in V \) where \( p = \min (m, n) \) and \( q = \max (m, n) \). Clearly \( \frac{p}{q} > 0 \).

**Case 1:** Let \( v \in P_n \) be such that \( d(v) = m + 3 \) in \( G \).

So \[ \sum_{u \in N[v]} f(u) = \frac{p}{q} + \frac{p}{q} + \ldots + \frac{p}{q} = \frac{(m+4)p}{q}. \]

**Case 2:** Let \( v \in P_n \) be such that \( d(v) = m + 2 \) in \( G \).
So \( \sum_{u \in N[v]} f(u) = \frac{P + P + \ldots + P}{q} = (m + 3)\frac{P}{q} \).

**Case 3:** Let \( v \in K_{1,m} \) be such that \( d(v) = m + 1 \) in \( G \).

So \( \sum_{u \in N[v]} f(u) = \frac{P + P + \ldots + P}{q} = (m + 2)\frac{P}{q} \).

**Case 4:** Let \( v \in K_{1,m} \) be such that \( d(v) = 2 \) in \( G \).

So \( \sum_{u \in N[v]} f(u) = \frac{P + P + P}{q} = 3\frac{P}{q} \).

From the above four cases, we observe that \( f \) is a DF if \( \frac{P}{q} \geq \frac{1}{3} \).

Otherwise \( f \) is not a DF.

**Case 5:** Suppose \( \frac{P}{q} > \frac{1}{3} \).

Clearly \( f \) is a DF.

Now we check for the minimality of \( f \).

Define \( g : V \rightarrow [0,1] \) by

\[
g(v) = \begin{cases} 
q, & \text{for } v = v_k \in V, \\
\frac{P}{q}, & \text{otherwise}. 
\end{cases}
\]

where \( 0 < r < \frac{P}{q} \).

Since strict inequality holds at a vertex \( v_k \) of \( V \), it follows that \( g < f \).

**Case (i):** Let \( v \in P_n \) be such that \( d(v) = m + 3 \) in \( G \).

**Sub case 1:** Let \( v_1 \in N[v] \).

Then \( \sum_{u \in N[v]} g(u) = r + \frac{P + P + \ldots + P}{q} < \frac{P}{q} + (m + 3)\frac{P}{q} = (m + 4)\frac{P}{q} > 1 \), since \( \frac{P}{q} > \frac{1}{3} \) and \( m \geq 2 \).

**Sub case 2:** Let \( v_2 \in N[v] \).

Then \( \sum_{u \in N[v]} g(u) = \frac{P + P + \ldots + P}{q} = (m + 4)\frac{P}{q} > 1 \), since \( \frac{P}{q} > \frac{1}{3} \) and \( m \geq 2 \).

**Case (ii):** Let \( v \in P_n \) be such that \( d(v) = m + 2 \) in \( G \).

**Sub case 1:** Let \( v_1 \in N[v] \).

Then \( \sum_{u \in N[v]} g(u) = r + \frac{P + P + \ldots + P}{q} < \frac{P}{q} + (m + 2)\frac{P}{q} = (m + 3)\frac{P}{q} > 1 \), since \( \frac{P}{q} > \frac{1}{3} \) and \( m \geq 2 \).

**Sub case 2:** Let \( v_2 \in N[v] \).

Then \( \sum_{u \in N[v]} g(u) = \frac{P + P + \ldots + P}{q} = (m + 3)\frac{P}{q} > 1 \), since \( \frac{P}{q} > \frac{1}{3} \) and \( m \geq 2 \).

**Case (iii):** Let \( v \in K_{1,m} \) be such that \( d(v) = m + 1 \) in \( G \).
Sub case 1: Let \( v_k \in \mathcal{N}[v] \).

Then
\[
\sum_{u \in \mathcal{N}[v]} g(u) = \frac{r + \frac{p}{q} + \frac{p}{q} + \ldots}{(m+1)\text{-times}} + \frac{p}{q} = (m+2)\frac{p}{q} > 1, \quad \text{since } \frac{p}{q} > \frac{1}{3} \text{ and } m \geq 2.
\]

Sub case 2: Let \( v_k \notin \mathcal{N}[v] \).

Then
\[
\sum_{u \in \mathcal{N}[v]} g(u) = \frac{p}{q} + \frac{p}{q} + \frac{p}{q} = (m+2)\frac{p}{q} > 1, \quad \text{since } \frac{p}{q} > \frac{1}{3} \text{ and } m \geq 2.
\]

Case (iv): Let \( v \in K_{1,m} \) be such that \( d(v) = 2 \) in \( G \).

Sub case 1: Let \( v_k \in \mathcal{N}[v] \).

Then
\[
\sum_{u \in \mathcal{N}[v]} g(u) = \frac{r + \frac{p}{q} + \frac{p}{q}}{3} > 1, \quad \text{since } \frac{p}{q} > \frac{1}{3}.
\]

Sub case 2: Let \( v_k \notin \mathcal{N}[v] \).

Then
\[
\sum_{u \in \mathcal{N}[v]} g(u) = \frac{p}{q} + \frac{p}{q} + \frac{p}{q} = 3 \frac{p}{q} > 1, \quad \text{since } \frac{p}{q} > \frac{1}{3},
\]

Hence, it follows that \( \sum_{u \in \mathcal{N}[v]} g(u) > 1, \forall \ v \in V \).

Thus \( g \) is a DF. This implies that \( f \) is not a MDF.

Case 6: Suppose \( \frac{p}{q} = \frac{1}{3} \).

As in case 1, we can show for \( v \in P_n \) such that \( d(v) = m + 3 \),
\[
\sum_{u \in \mathcal{N}[v]} f(u) = \frac{p}{q} + \frac{p}{q} + \ldots = (m+4)\left(\frac{p}{q}\right) = (m+4)\left(\frac{1}{3}\right) = 1 + \frac{m+1}{3} > 1.
\]

Again as in case 2, for \( v \in P_n \) such that \( d(v) = m + 2 \), we have
\[
\sum_{u \in \mathcal{N}[v]} f(u) = \frac{p}{q} + \frac{p}{q} + \ldots = (m+3)\left(\frac{p}{q}\right) = (m+3)\left(\frac{1}{3}\right) = 1 + \frac{m}{3} > 1.
\]

We can show as in case 3, for \( v \in K_{1,m} \) such that \( d(v) = m + 1 \),
\[
\sum_{u \in \mathcal{N}[v]} f(u) = \frac{p}{q} + \frac{p}{q} + \ldots = (m+2)\left(\frac{p}{q}\right) = (m+2)\left(\frac{1}{3}\right) = 1 + \frac{m-1}{3} > 1.
\]

Again as in case 4, we can show for \( v \in K_{1,m} \) such that \( d(v) = 2 \),
\[
\sum_{u \in \mathcal{N}[v]} f(u) = \frac{p}{q} + \frac{p}{q} + \frac{p}{q} = 3 \frac{p}{q} = 3 \frac{1}{3} = 1.
\]

Therefore for all possibilities, we get \( \sum_{u \in \mathcal{N}[v]} f(u) \geq 1, \forall \ v \in V \).

This implies that \( f \) is a DF.

Now we check for the minimality of \( f \).

Define \( g : V \to [0,1] \) by
\[
g(v) = \begin{cases} 
  r, & \text{for } v = v_k \in V, \\
  \frac{p}{q}, & \text{otherwise}.
\end{cases}
\]
where \(0 < r < \frac{p}{q}\).
Since strict inequality holds at a vertex \(v_k\) of \(V\), it follows that \(g < f\).

Then we can show as in case (i), (ii) and (iii) of case 5 that
\[
\sum_{u \in N[v]} g(u) > 1, \text{ if } v \in P_n \text{ and } v_k \in N[v] \text{ or } v_k \notin N[v].
\]

Now we can show as in case (iv) of case 5 that
\[
\sum_{u \in N[v]} g(u) = r + \frac{p}{q} + \frac{p}{q} < 1, \text{ if } v \in K_1,3 \text{ and } v_k \in N[v].
\]

And \[
\sum_{u \in N[v]} g(u) = \frac{p}{q} + \frac{p}{q} + \frac{p}{q} = 3\left(\frac{p}{q}\right) = 3\left(\frac{1}{3}\right) = 1, \text{ if } v \in K_1,3 \text{ and } v_k \notin N[v].
\]

This implies that \(\sum_{u \in N[v]} g(u) < 1\), for some \(v \in V\). So \(g\) is not a DF.

Since \(g\) is defined arbitrarily, it follows that there exists no \(g < f\) such that \(g\) is a DF. Thus \(f\) is a MDF.

**ILLUSTRATION**

![Diagram](image)

The function \(f\) takes the value 1 for vertices of \(P_5\) and value 0 for vertices in each copy of \(K_{1,3}\)
REFERENCES