Abstract—In this paper a control problem for a linear stochastic system driven by a fractional Brownian motion with a cost functional that is quadratic in the state and the control is considered. An optimal control is given explicitly using fractional calculus and the control is shown to depend on the prediction of the fractional Brownian motion as well as the usual linear feedback control for the linear-quadratic control problem.

Key Words: linear quadratic Gaussian control, fractional Brownian motion, linear regulator

I. INTRODUCTION

The control of a linear stochastic system with a Brownian motion (white Gaussian noise) and a quadratic cost functional of the state and the control is probably the most well known solvable stochastic control problem for continuous time systems in Euclidean spaces (e.g. [2]). Brownian motion is typically used in a model to approximate the perturbations of a physical system because it is a Gauss-Markov process and a continuous martingale. However from empirical measurements of many physical phenomena, Brownian motion is often shown not to be an effective process to use in a model. A family of processes that seems to have wide physical applicability is fractional Brownian motion. Fractional Brownian motion is a family of Gaussian processes that was constructed by Kolmogorov [8] in his study of turbulence [13,14]. Brownian motion is a member of this family of processes. The first empirical evidence for a general fractional Brownian motion was made by Hurst [5] in his modeling of rainfall along the Nile River. Mandelbrot [9] used fractional Brownian motions to describe financial data and noted that Hurst’s statistical analysis was for fractional Brownian motion (FBM). Mandelbrot and van Ness [10] developed some of the theory for FBM. Subsequent to Hurst’s work, empirical justifications for modeling with fractional Brownian motions have been demonstrated for economic data [9], flicker noise in electronic devices [15], turbulence [18], and internet traffic [16,17].

With a wide variety of potential applications for fractional Brownian motions, it is natural to consider the control of a linear stochastic system driven by an arbitrary FBM with a quadratic cost functional. This control problem is investigated here. Some initial work on this problem has been done previously. Hu and Zhou [4] consider a scalar system with the state multiplying the noise and the family of admissible controls to be Markov controls though the uncontrolled system is not Markov. Kleptsyna, Le Breton and Viot [6,7] consider a family of admissible controls that are required to ensure that the solution of the stochastic equation for the system is well defined but the system is restricted to be scalar and the Hurst parameter $H$ that indexes the FBM is restricted to $(1/2, 1)$. Jumarie [19] considers a scalar system with Markov controls and the Hurst parameter $H \in (0, 1/2)$. In this paper for a multidimensional linear stochastic system with an FBM with $H \in (1/2, 1)$ and a quadratic cost, an optimal control is given explicitly. The approach that is given here uses fractional calculus so the optimal control is described more explicitly than in [6,7]. When the FBM is not a Brownian motion, the optimal control is the sum of two terms with one term being the linear state feedback for the case of Brownian motion and the other term being a prediction of the future FBM.
A result for the prediction of some processes related to a FBM [1] is reviewed here. Section II of this paper contains some information about the family of FBMs and some other preliminaries for the determination of the optimal control. The main result on the explicit description of the optimal control is given in Section III.

II. PRELIMINARIES

Fractional Brownian motion (FBM) is a family of Gaussian processes that is indexed by the Hurst parameter $H \in (0, 1)$. Initially a definition for an FBM is given.

Definition 1: Let $H \in (0, 1)$ be fixed. A real-valued standard fractional Brownian motion, $(B(t), t \geq 0)$, with Hurst parameter $H$ is a Gaussian process with continuous sample paths such that

$$E[B(t)] = 0$$

$$E[B(s)B(t)] = \frac{1}{2} \left( t^{2H} + s^{2H} - |t-s|^{2H} \right)$$

for all $s, t \in \mathbb{R}_+$.

An $\mathbb{R}^n$-valued standard fractional Brownian motion, $(B(t), t \geq 0)$, with Hurst parameter $H$ is an $n$-vector of independent real-valued standard fractional Brownian motions with the same Hurst parameter $H$. If $B$ is an FBM with $H = 1/2$ then $B$ is a Brownian motion. Some important properties of a fractional Brownian motion are reviewed now to give some perspective on these processes. If $(B(t), t \geq 0)$ is a real-valued standard fractional Brownian motion then for each $\alpha > 0 \ (B(\alpha t), t \geq 0)$ and $(\alpha^H B(t), t \geq 0)$ have the same probability law. This property is called self-similarity. For $H \in (1/2, 1)$

$$\sum_{n=1}^{\infty} r(n) = \infty$$

where

$$r(n) = E[B(1)(B(n+1) - B(n))]$$

This property is called long range dependence. Finally a sample path property is noted, that is,

$$\sum_{i} |B(t^{(n)}_{i+1}) - B(t^{(n)}_{i})|^p \rightarrow \begin{cases} 0 & pH > 1 \\ c(p) & pH = 1 \\ +\infty & pH < 1 \end{cases}$$

where $c(p) = E[B(1)]^p$ and $(t^{(n)}_{i}, i = 0, 1, \cdots, n)$ is a sequence of nested partitions of $[0,1]$ that become arbitrarily fine and the convergence is almost surely. Thus for $p = 2$ and $H \in (1/2, 1)$ a FBM has zero quadratic variation and for $p = 2$ and $H \in (0,1/2)$ a FBM has infinite quadratic variation so an FBM is not a semimartingale for $H$ not equal to $1/2$. A fractional calculus of Riemann and Liouville [12] plays an important role in the analysis of a FBM. Let $\alpha \in (0, 1)$ be fixed. The left-sided and the right-sided fractional (Riemann-Liouville) integrals for $\varphi \in L^1([0,T])$ are defined for almost all $t \in [0,T]$ by

$$(I^t_{0+} \varphi)(t) = \frac{1}{\Gamma(\alpha)} \int_{0}^{t} \frac{(t-s)^{\alpha-1} \varphi(s)ds}{s^{\alpha}}$$

and

$$(I^t_{T-} \varphi)(t) = \frac{1}{\Gamma(\alpha)} \int_{t}^{T} \frac{(s-t)^{\alpha-1} \varphi(s)ds}{s^{\alpha}}$$

respectively, where $\Gamma(\cdot)$ is the gamma function. The inverse operators of these fractional integrals are called fractional derivatives and can be given by their respective Weyl representations

$$(D^t_{0+} \varphi)(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_{0}^{t} \frac{\varphi(s)ds}{(t-s)^{\alpha}}$$

$$= \frac{1}{\Gamma(1-\alpha)} \left( \frac{\varphi(t)}{t^{\alpha}} + \alpha \int_{0}^{t} \frac{\varphi(s) - \varphi(t)}{(s-t)^{\alpha+1}}ds \right)$$

and

$$(D^t_{T-} \varphi)(t) = \frac{-1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_{t}^{T} \frac{\varphi(s)ds}{(s-t)^{\alpha}}$$

$$= \frac{1}{\Gamma(1-\alpha)} \left( \frac{\varphi(t)}{(T-t)^{\alpha}} + \alpha \int_{t}^{T} \frac{\varphi(s) - \varphi(t)}{(s-t)^{\alpha+1}}ds \right)$$

where $\varphi \in L^p_{0+} (L^1([0,T]))$ and $\varphi \in L^p_{T-} (L^1([0,T]))$ respectively. Often it is convenient to write $D^\alpha$ as $I^{-\alpha}$.
One indication of the relevance of fractional calculus to FBM is that the covariance of a FBM can be expressed in terms of fractional integrals or fractional derivatives. Let \( (B(t), t \geq 0) \) be a real-valued standard fractional Brownian motion. The covariance of \( B \) satisfies the following equality

\[
E[B(s)B(t)] = \rho(H) \int_0^T u_a^2 - H(r) (I_H^{-\frac{1}{2}} u_{H-\frac{1}{2}} 1_{[0,a]})(r) (I_H^{-\frac{1}{2}} u_{H-\frac{1}{2}} 1_{[0,a]})(r) dr
\]

where \( \rho(H) = \frac{2H\Gamma(H+\frac{1}{2})\Gamma(\frac{3}{2} - H)}{\Gamma(2-2H)} \).

and \( u_a(s) = s^a \) for \( a > 0, s > 0 \). The integral on the RHS shows that the covariance for a FBM can be factored in the Lebesgue space \( L^2([0,T]) \) from which it follows that an FBM can be expressed in terms of a Brownian motion and conversely. This relation between processes is important for the solution of the prediction for a fractional Brownian motion. Associated with each Gaussian process is a Hilbert space. For a FBM this Hilbert space can be given explicitly from the above equality for the covariance. Let \( H \in (0,1) \) be fixed and let \( L_H^2([0,T]) \) be the Hilbert space where \( f, g \in L_H^2 \) if

\[
<f,g>_H = \rho(H) \int_0^T u_a^2 - H(r) (I_H^{-\frac{1}{2}} f(r))(I_H^{-\frac{1}{2}} g(r)) dr
\]

The linear operator \( I^\alpha \) is understood to be the fractional derivative \( D^{-\alpha} \) for \( \alpha \in (-1,0) \).

If \( B \) is a FBM with the Hurst parameter \( H \) and \( f \in L_H^2 \) then \( \int_0^T f dB \) is a zero mean Gaussian random variable with second moment \( <f,f>_H \). An element \( f \in L_H^2 \) is a function if \( H \in (0,1/2) \) but it may be a distribution if \( H \in (1/2,1) \).

Consider the control system given by the following controlled linear stochastic differential equation with a fractional Brownian motion:

\[
\begin{align*}
\frac{dX(t)}{dt} &= (AX(t) + U(t))dt + dB(t) \quad \text{(II.1)} \\
X(0) &= X_0 \quad \text{(II.2)}
\end{align*}
\]

where \( X_0 \) is a constant vector, \( X_0, X(t) \in \mathbb{R}^n, U(t) \in \mathbb{R}^n, A \in L(\mathbb{R}^n, \mathbb{R}^n), \) and \( (B(t), t \geq 0) \) is an \( \mathbb{R}^n \)-valued standard fractional Brownian motion with \( H \in (1/2,1) \). The existence and the uniqueness of the solution of (II.1) (e.g.[20]) follows from the variation of parameters formula for linear ordinary differential equations. The quadratic cost functional \( J \) is

\[
J(U) = E[\int_0^T <QX(s), X(s)> + <U(s),U(s)> ds] + E <MX(T), X(T)>
\]

where \( Q > 0, \) and \( M \geq 0 \) are symmetric linear transformations, \( T > 0 \) is fixed, and \( <,,> \) is the standard inner product on \( \mathbb{R}^n \).

The fractional Brownian motion \( (B(t), t \in [0,T]) \) is defined on the complete probability space \((\Omega, \mathcal{F}, P)\) and \((\mathcal{F}(t), t \in [0,T])\) is the natural filtration for \( B \) on \((\Omega, \mathcal{F}, P)\). The family of admissible controls \( \mathcal{U} \) is defined as

\[
\mathcal{U} = \{ U : U \text{ is an } \mathbb{R}^n \text{-valued process adapted to } (\mathcal{F}(t), t \in [0,T]) \text{ such that } U \in L^2([0,T]) \text{ a.s.} \}
\]

The prediction of a fractional Brownian motion or of some processes related to a FBM plays an important role in the determination of an optimal control for (II.1,II.3). The minimum variance prediction of a random variable \( X \) given a \( \sigma \)-algebra \( G \) generated by a family of random variables is well known to be the conditional mean \( \mathbb{E}[X|G] \). However often it is not clear how to give an explicit, useful expression for \( \mathbb{E}[X|G] \) even in the case of Gaussian random variables where \( \mathbb{E}[X|G] \) is a linear functional of the conditioning random variables. The following result [1] which generalizes [3,11] provides an explicit expression for the conditional mean that is subsequently used in the explicit description of an optimal control for (II.1-II.3).

**Proposition.** Let \( (B(t), t \geq 0) \) be a real-valued standard fractional Brownian motion with Hurst parameter \( H \in (0,1) \) and let \( c \in L_H^2 \). Then

\[
\begin{align*}
\mathbb{E} \left[ \int_s^t c dB \mid B(r), r \in [0,s] \right] &= \int_s^t u_{-(H-1/2)} \left( I_{s-(H-1/2)} \left( I_{t-(H-1/2)} u_{H-1/2} c \right) \right) dB \\
\end{align*}
\]

where \( u_a(s) = s^a \) for \( a > 0, s > 0 \).
III. MAIN RESULT

The following theorem provides a solution to the control problem (II.1,II.3) by exhibiting an explicit optimal control. In [6] an optimal control for a scalar system is given which is consistent with this result though it is less explicit.

**Theorem.** For the control problem (II.1,II.3) and the family of admissible controls \( \mathcal{U} \), there is an optimal control \( U^* \) that can be expressed as

\[
U^*(t) = -(K(t)X(t) + V(t)) \quad (\text{III.1})
\]

where \( (K(t), t \in [0, T]) \) is the unique symmetric positive definite solution of the Riccati equation

\[
\frac{dK}{dt} = -KA - A'K + KK - Q \quad (\text{III.2})
\]

\[
K(T) = M \quad (\text{III.3})
\]

\[
V(t) = \int_t^T \Phi_K(s-t)K(s)\left[ \frac{1}{\Gamma(3/2-H)} \int_0^t r^{1/2-H} s^{H-1/2} \frac{1}{\Gamma(H-1/2)} \left( \frac{d}{dr} \int_r^t (s-p)^H-3/2 (p-r)^{H-1/2} dp \right) dB(r) \right] ds
\]  

(\text{III.4})

and \( \Phi_K \) is the fundamental matrix for the matrix equation

\[
\frac{dY}{dt} = (A - K(t))Y \quad (\text{III.5})
\]

\[
Y(0) = I \quad (\text{III.6})
\]

Comparing the optimal control (III.1) with the well known result for Brownian motion \( (H = 1/2) \) (e.g. [2]), it is clear that the optimal control (III.1) is the sum of the optimal control for Brownian motion and the (minimum variance) prediction of the optimal system response to the future noise. Only a sketch or an outline of the proof is given here. Let \( t_0 \in [0, T] \) and assume that the optimal control has been determined in the interval \([t_0, T]\). Let \( \delta > 0 \) and determine the optimal control for the interval \([t_0 - \delta, T]\) in \([0, T]\). Initially consider that there is no FBM term. The optimal control is obtained from the result for the deterministic optimal control, that is, 

\[-KX.\] By the linearity of the system consider now the FBM \( dB(t) \). The most obvious strategy would be to cancel out this FBM with a suitable control. However such a strategy violates the measurability of the family of admissible controls \( \mathcal{U} \). Thus the strategy is to make the best estimate of this FBM acting on the system in the \( L^2(\mathbb{P}) \) norm. This strategy is the conditional expectation which minimizes the error for the \( L^2(\mathbb{P}) \) norm, that is, the prediction of the future FBM acting on the optimal system. The prediction of \( dB(t) \) has to be mapped from the state space \( X \) to the control space \( U \). This map is obtained from the solution of the Hamiltonian equations or equivalently from the Lagrangian Grassmannian, so it is the linear map \( K \) that satisfies the Riccati equation (III.1). This control strategy can be verified to be optimal by considering a small variation from it.

From the Proposition it follows that

\[
\mathbb{E}\left[ \int_t^s dB \mid B(r), r \in [0, t] \right] = \int_t^s u_{-}(H-1/2) \left( I_{r-}^{-(H-1/2)} \left( I_{r-}^{-(H-1/2)} u_{-H-1/2} \right) \right) dB
\]

Recall that \( I_{t-}^{H-1/2} \) is a fractional derivative because \( 1/2 - H < 0 \). To obtain the formal conditional expectation \( \mathbb{E}[dB(s)\mid \mathcal{F}(t)] \), apply \( \frac{\partial}{\partial s} \) to both sides of the above equality.

From fractional calculus it follows that

\[
\frac{\partial}{\partial s} \left( I_{t-}^{H-1/2} \right)(s) = \frac{\partial}{\partial s} \frac{1}{\Gamma(H-1/2)} \int_0^s (q-r)^{H-3/2} q^{H-1/2} dq
\]

\[
= \frac{1}{\Gamma(H-1/2)} s^{H-1/2} (s-r)^{H-3/2}
\]

Now substitute this result in the expression for \( \mathbb{E}[dB(s)\mid \mathcal{F}(t)] \), map to the control space by \( K(s) \), apply the fundamental matrix \( \Phi_k \) and integrate on the interval \([t, T]\) to obtain

\[
V(t) = \int_t^T \Phi_K(s-t)K(s)\left[ \frac{1}{\Gamma(3/2-H)} \int_0^t r^{1/2-H} s^{H-1/2} \frac{1}{\Gamma(H-1/2)} \left( \frac{d}{dr} \int_r^t (s-p)^H-3/2 (p-r)^{H-1/2} dp \right) dB(r) \right] ds
\]

(\text{III.8})

which is (III.4) and completes the outline of the proof.

With this approach a fractional Brownian motion with \( H \in (0, 1/2) \) can also be considered. The major difference for this
case from $H \in (1/2, 1)$ is that the computations from the prediction result are somewhat more complicated. However the basic method to obtain the optimal control is the same. In future work, some applications of these results to finance will be considered.

REFERENCES


