Continued Fractions for Special Functions: Handbook and Software

Team:

A. Cuyt,
V.B. Petersen, J. Van Deun, B. Verdonk, H. Waadeland,
F. Backeljauw, S. Becuwe, M. Colman, T. Docx,
W.B. Jones

Universiteit Antwerpen

Sør-Trøndelag University College

Norwegian University of Science and Technology

University of Colorado at Boulder

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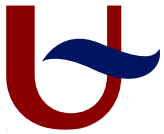
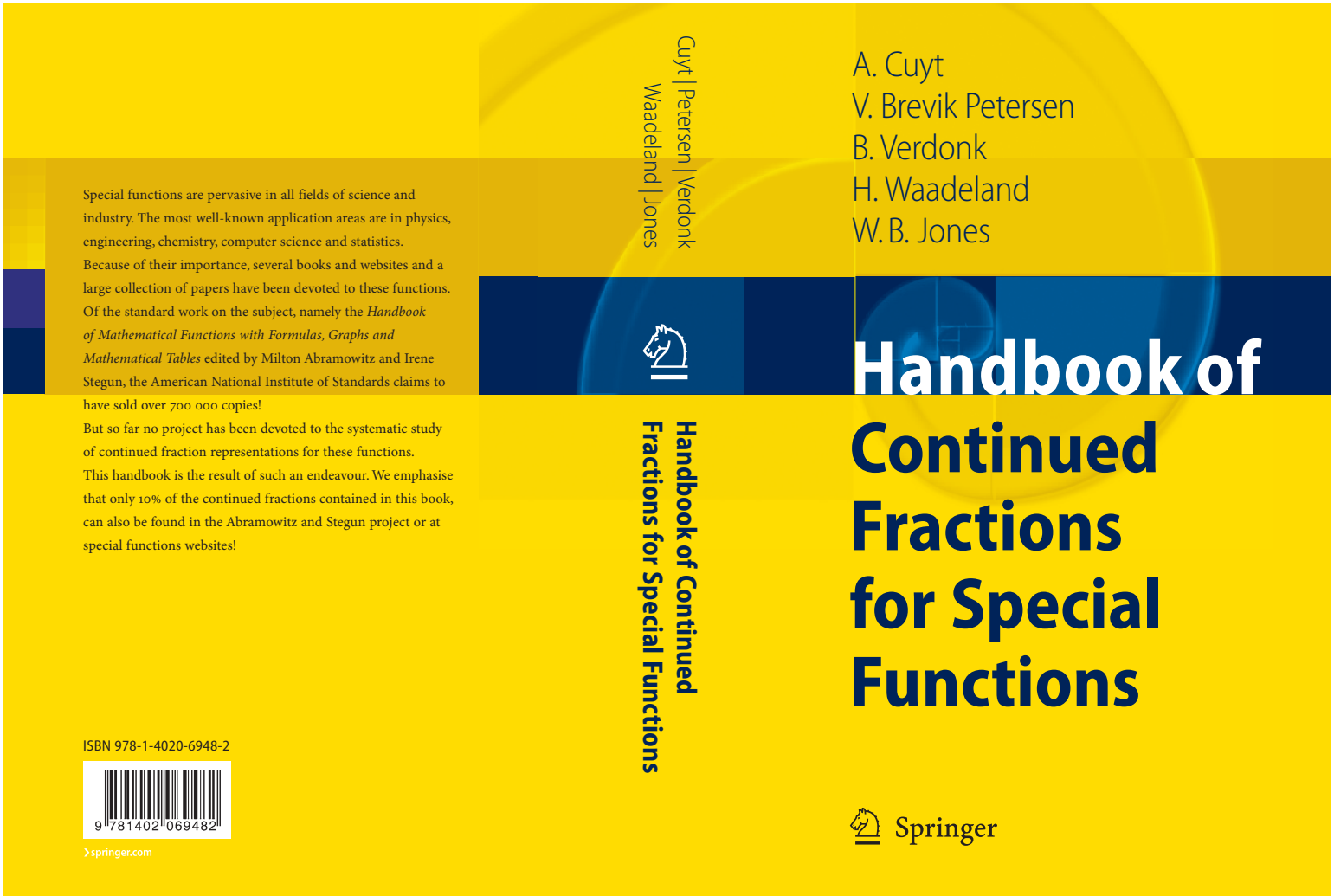


Project

- Book: *formulas* with *tables* and *graphs*
- Maple library: all *fractions*
- C++ library: all *functions*
- Web version to explore: all of the above


Special functions are pervasive in all fields of science and industry. The most well-known application areas are in physics, engineering, chemistry, computer science and statistics.
Because of their importance, several books and websites and a large collection of papers have been devoted to these functions. Of the standard work on the subject, namely the *Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables* edited by Milton Abramowitz and Irene Stegun, the American National Institute of Standards claims to have sold over 700 000 copies!
But so far no project has been devoted to the systematic study of continued fraction representations for these functions.
This handbook is the result of such an endeavour. We emphasise that only 10% of the continued fractions contained in this book, can also be found in the Abramowitz and Stegun project or at special functions websites!
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Cuyt | Petersen | Verdonk
Waadeland | Jones
Handbook of Continued Fractions for Special Functions
A. Cuyt
V. Brevik Petersen
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special function: $f(z)$

continued fraction: $b_0(z) + \mathrm{K}_{m=1}^{\infty} \frac{a_m(z)}{b_m(z)}$

$$f(z) = b_0(z) + \cfrac{a_1(z)}{b_1(z) + \cfrac{a_2(z)}{b_2(z) + \cfrac{a_3(z)}{b_3(z) + \cdots}}}$$

$$= b_0(z) + \frac{a_1(z)}{b_1(z)} + \frac{a_2(z)}{b_2(z)} + \cdots$$

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tail: $t_n(z) = K_{m=n+1}^{\infty} \frac{a_m(z)}{b_m(z)}$

Example:

$$\frac{\sqrt{1+4x}-1}{2} = \mathop{K}_{m=1}^{\infty} \frac{x}{1}, \qquad x \geqslant -\frac{1}{4}, \qquad t_n(x) = f(x)$$

$$\sqrt{2}-1 = \frac{1}{1}\,\frac{2}{1}\,\frac{1}{1}\,\frac{2}{1}\,\dots, \qquad t_{2n} \to \sqrt{2}-1$$

$$t_{2n+1} \to \sqrt{2}$$

$$1 = \mathop{K}_{m=1}^{\infty} \frac{m(m+2)}{1}, \qquad t_n \to \infty$$

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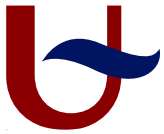
approximant:

$$f_n(z;0) = b_0(z) + \mathop{\mathrm{K}}_{m=1}^{n} \frac{a_m(z)}{b_m(z)}$$

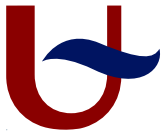
modified approximant:

$$f_n(z;w_n) = b_0(z) + \mathop{\mathrm{K}}_{m=1}^{n-1} \frac{a_m(z)}{b_m(z)} + \frac{a_n(z)}{b_n(z)+w_n}$$



Part I: Basic theory

- Basics
- Continued fraction representation of functions
- Convergence criteria
- Padé approximants
- Moment theory and orthogonal polynomials



Part II: Numerics

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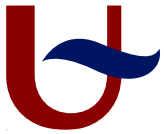
- Continued fraction construction
- Truncation error bounds
- Continued fraction evaluation



Part III: Special functions

- Incomplete gamma and related
- Error function and related
- Exponential integrals
- Hypergeometric ${}_2\mathrm{F}_1(a, n; c; z)$, $n \in \mathbb{Z}$
- Confluent hypergeometric ${}_1\mathrm{F}_1(n; b; z)$, $n \in \mathbb{Z}$
- Legendre functions
- Bessel functions of integer and fractional order
- Coulomb wave functions

Formulas

$$\operatorname{Ln}(1+z) = \frac{z}{1} + \mathop{\mathrm{K}}_{m=2}^{\infty}\left(\frac{a_m z}{1}\right), \qquad |\operatorname{Arg}(1+z)| < \pi, \qquad (11.2.2)$$

$$a_{2k} = \frac{k}{2(2k-1)}, \quad a_{2k+1} = \frac{k}{2(2k+1)}$$

$$= \frac{2z}{2+z} + \mathop{\mathrm{K}}_{m=2}^{\infty}\left(\frac{-(m-1)^2 z^2}{(2m-1)(2+z)}\right), \qquad (11.2.3)$$

$$|\operatorname{Arg}\left(1 - z^2/(2+z)^2\right)| < \pi$$

$$\operatorname{Ln}\left(\frac{1+z}{1-z}\right) = \frac{2z}{1} + \mathop{\mathrm{K}}_{m=1}^{\infty}\left(\frac{a_m z^2}{1}\right), \qquad |\operatorname{Arg}(1-z^2)| < \pi, \qquad (11.2.4)$$

$$a_m = \frac{-m^2}{(2m-1)(2m+1)}.$$

Tables

TABLE 11.2.3: Relative error of 20^{th} partial sum and 20^{th} approximants.

x	$\mathrm{Ln}(1+x)$	(11.2.1)	(11.2.2)	(11.2.3)	(6.8.8)
-0.9	$-2.302585e{+}00$	$1.5e{-}02$	$2.8e{-}06$	$5.8e{-}12$	$1.5e{-}06$
-0.4	$-5.108256e{-}01$	$2.5e{-}10$	$1.8e{-}18$	$2.2e{-}36$	$2.4e{-}19$
0.1	$9.531018e{-}02$	$4.4e{-}23$	$5.3e{-}33$	$1.9e{-}65$	$1.3e{-}34$
0.5	$4.054651e{-}01$	$1.8e{-}08$	$1.9e{-}20$	$2.3e{-}40$	$2.0e{-}21$
1.1	$7.419373e{-}01$	$2.4e{-}01$	$2.8e{-}15$	$5.3e{-}30$	$5.5e{-}16$
5	$1.791759e{+}00$	$1.0e{+}13$	$4.2e{-}08$	$1.3e{-}15$	$2.0e{-}08$
10	$2.397895e{+}00$	$1.8e{+}19$	$5.3e{-}06$	$2.1e{-}11$	$3.3e{-}06$
100	$4.615121e{+}00$	$1.0e{+}40$	$1.8e{-}02$	$3.6e{-}04$	$2.5e{-}02$

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Modification:

since in (11.2.2), $\lim_{m\to\infty} a_m z = z/4$ and

$$\lim_{m\to\infty} \frac{a_{m+1} - \frac{1}{4}}{a_m - \frac{1}{4}} = -1,$$

we find

$$w(z) = \frac{-1 + \sqrt{1+z}}{2}$$

and

$$\begin{cases} w_{2k}^{(1)}(z) = w(z) + \frac{kz}{2(2k+1)} - \frac{z}{4} \\ w_{2k+1}^{(1)}(z) = w(z) + \frac{(k+1)z}{2(2k+1)} - \frac{z}{4} \end{cases}$$

TABLE 11.2.4: Relative error of 20^{th} (modified) approximants.

x	$\mathrm{Ln}(1+x)$	(11.2.2)	(11.2.2)	(11.2.2)
-0.9	$-2.302585e{+}00$	$2.8e{-}06$	$1.9e{-}07$	$2.9e{-}10$
-0.4	$-5.108256e{-}01$	$1.8e{-}18$	$9.3e{-}20$	$4.3e{-}22$
0.1	$9.531018e{-}02$	$5.3e{-}33$	$2.7e{-}34$	$3.2e{-}37$
0.5	$4.054651e{-}01$	$1.9e{-}20$	$9.5e{-}22$	$5.5e{-}24$
1.1	$7.419373e{-}01$	$2.8e{-}15$	$1.5e{-}16$	$1.8e{-}18$
5	$1.791759e{+}00$	$4.2e{-}08$	$2.6e{-}09$	$1.1e{-}10$
10	$2.397895e{+}00$	$5.3e{-}06$	$3.7e{-}07$	$2.8e{-}08$
100	$4.615121e{+}00$	$1.8e{-}02$	$2.9e{-}03$	$9.2e{-}04$

Graphs

FIGURE 11.2.1: Complex region where $f_8(z;0)$ of (11.2.2) guarantees k significant digits for $\mathrm{Ln}(1+z)$ (from light to dark $k = 6, 7, 8, 9$).

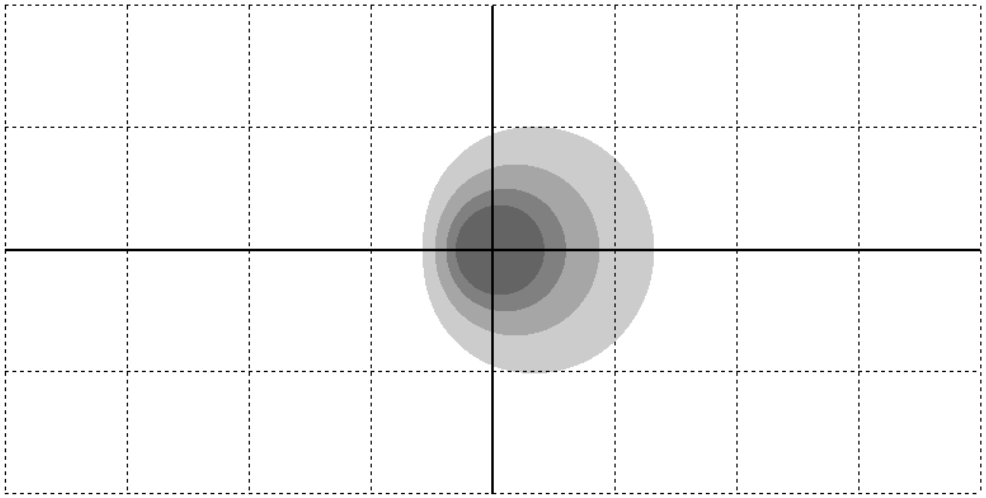
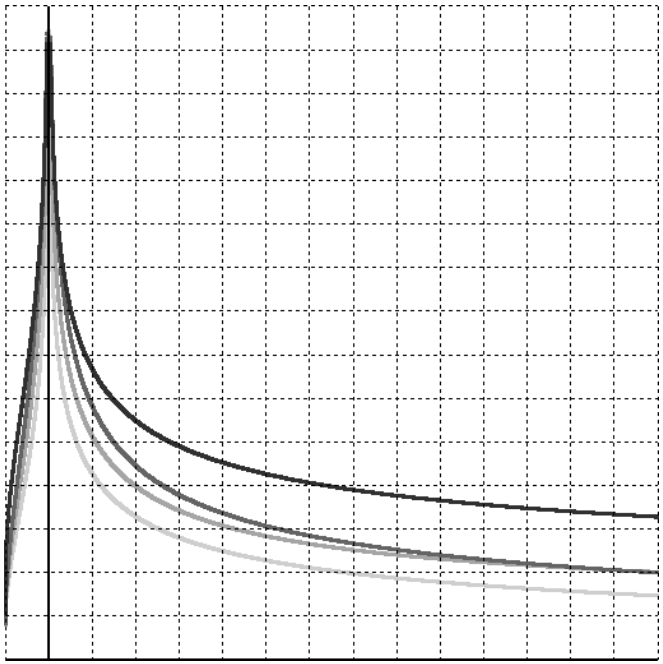


FIGURE 11.2.2: Number of significant digits guaranteed by the n^{th} classical approximant of (11.2.2) (from light to dark $n = 5, 6, 7$) and the 5^{th} modified approximant evaluated with $w_5^{(1)}(z)$ (darkest).



Demo

$$ {}_2\mathrm{F}_1(1/2, 1; 3/2; z) = \frac{1}{2\sqrt{z}} \operatorname{Ln}\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right) $$

- series representation
- continued fraction representations:
  - C-fraction (15.3.7)
  - M-fraction (15.3.12)
  - Nörlund fraction (15.3.17)

$$ {}_2F_1(a,b;c;z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!}, \qquad a,b \in \mathbb{C}, \quad c \in \mathbb{C} \setminus \mathbb{Z}_0^- \ . \tag{15.1.4} $$

$$ z\ {}_2F_1\left(1/2, 1; 3/2; z\right) = \mathop{\mathrm{K}}_{m=1}^{\infty} \left(\frac{c_m z}{1} \right), \qquad z \in \mathbb{C} \setminus [1, +\infty) \tag{15.3.7a} $$

with

$$ c_1 = 1, \quad c_m = \frac{-(m-1)^2}{4(m-1)^2 - 1}, \quad m \geq 2. \tag{15.3.7b} $$

$$ {}_2F_1(1/2, 1; 3/2; z) = \frac{1/2}{1/2 + z/2} - \frac{z}{3/2 + 3z/2} - \frac{4z}{5/2 + 5z/2} - \ldots \tag{15.3.12} $$

$$ {}_2F_1(1/2, 1; 3/2; z) = \frac{1}{1-z} + \frac{z(1-z)}{3/2 \ - 5/2 z} + \mathop{\mathrm{K}}_{m=2}^{\infty} \left(\frac{m(m-1/2)z(1-z)}{(m+1/2) - (2m+1/2)z} \right), \quad \Re z < 1/2. \tag{15.3.17} $$

TABLE 15.3.1: Relative error of the 5^{th} partial sum and 5^{th} (modified) approximants. More details can be found in the *Examples* 15.3.1, 15.3.2 and 15.3.3.

x	${}_2F_1(1/2, 1; 3/2; x)$	(15.3.7)	(15.3.12)	(15.3.17)
0.1	1.035488e+00	1.9e−08	1.4e−05	6.1e−06
0.2	1.076022e+00	7.9e−07	4.4e−04	3.4e−04
0.3	1.123054e+00	8.0e−06	3.1e−03	4.9e−03
0.4	1.178736e+00	4.7e−05	1.2e−02	4.3e−02

x	${}_2F_1(1/2, 1; 3/2; x)$	(15.3.7)	(15.3.7)	(15.3.7)
0.1	1.035488e+00	1.9e−08	2.0e−10	1.6e−12
0.2	1.076022e+00	7.9e−07	8.7e−09	1.5e−10
0.3	1.123054e+00	8.0e−06	9.4e−08	2.7e−09
0.4	1.178736e+00	4.7e−05	5.9e−07	2.5e−08

...

TABLE 15.3.2: Relative error of the 20^{th} partial sum and 20^{th} (modified) approximants. More details can be found in the *Examples* 15.3.1, 15.3.2 and 15.3.3.

x	${}_2F_1(1/2, 1; 3/2; x)$	(15.3.7)	(15.3.12)	(15.3.17)
0.1	1.035488e+00	4.0e−32	1.5e−20	1.5e−20
0.2	1.076022e+00	1.3e−25	1.5e−14	1.7e−13
0.3	1.123054e+00	1.4e−21	4.8e−11	7.6e−09
0.4	1.178736e+00	1.8e−18	1.4e−08	5.0e−05

x	${}_2F_1(1/2, 1; 3/2; x)$	(15.3.7)	(15.3.7)	(15.3.7)
0.1	1.035488e+00	4.0e−32	2.6e−35	6.5e−38
0.2	1.076022e+00	1.3e−25	8.8e−29	4.8e−31
0.3	1.123054e+00	1.4e−21	1.1e−24	9.6e−27
0.4	1.178736e+00	1.8e−18	1.4e−21	1.9e−23

...

Demo

Special functions :
continued fraction and series representations

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T. Docx

Function categories

Elementary functions
Gamma function and related functions
Error function and related integrals
Exponential integrals and related functions
Hypergeometric functions
2F1(a,b;c;z)
2F1(a,n;c;z)
2F1(a,1;c;z)
2F1(a,b;c;z) /
2F1(a,b+1;c+1;z)
2F1(a,b;c;z) /
2F1(a+1,b+1;c+1;z)
2F1(1/2,1;3/2;z)
2F1(2a,1;a+1;1/2)
Confluent hypergeometric functions
Bessel functions

Hypergeometric functions » 2F1(1/2,1;3/2;z) » tabulate

Representation

☑	(HY.1.4)	${}_2F_1(a,b;c;z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!}, \qquad a,b \in \mathbb{C}, \quad c \in \mathbb{C} \setminus \mathbb{Z}_0^-$
☑	(HY.3.7)	$z\,{}_2F_1\left(1/2, 1; 3/2; z\right) = \mathop{\mathbf{K}}_{m=1}^{\infty} \left(\frac{c_m z}{1}\right), \qquad z \in \mathbb{C} \setminus [1, +\infty)$ $c_1 = 1, \quad c_m = \frac{-(m-1)^2}{4(m-1)^2 - 1}, \quad m \geq 2$
☑	(HY.3.7)	with modification (limit-periodic)
☑	(HY.3.7)	with improved modification (limit-periodic)
☑	(HY.3.12)	${}_2F_1(1/2, 1; 3/2; z) = \frac{1/2}{1/2 + z/2} - \frac{z}{3/2 + 3z/2} - \frac{4z}{5/2 + 5z/2} - \ldots, \qquad \lvert z \rvert < 1$
☑	(HY.3.12)	with modification (limit-periodic)
☑	(HY.3.12)	with improved modification (limit-periodic)
☑	(HY.3.17)	${}_2F_1(1/2, 1; 3/2; z) = \frac{1}{1 - z} + \frac{z(1-z)}{3/2 - 5/2 z} + \mathop{\mathbf{K}}_{m=2}^{\infty} \left(\frac{m(m-1/2)z(1-z)}{(m+1/2) - (2m+1/2)z}\right), \quad \Re z < 1/2$
☑	(HY.3.17)	with modification (limit-periodic)
☑	(HY.3.17)	with improved modification (limit-periodic)

Continue

Input

parameters	none
base	10
approximant	10 (1 ≤ approximant ≤ 999)
z	.1,.15,.2,.25,.3,.35,.4,.45
output in tables	relative error absolute error

Output

10th approximant (relative error)

z	${}_2F_1(1/2,1;3/2;z)$	(HY.1.4)	(HY.3.7)	(HY.3.7)	(HY.3.7)	(HY.3.12)	(HY.3.12)	(HY.3.12)	(HY.3.17)	(HY.3.17)	(HY.3.17)
0.1	1.035488e+00	4.6e−13	2.4e−16	6.4e−19	3.0e−21	1.5e−10	4.5e−13	8.5e−15	7.5e−11	3.9e−12	5.4e−14
0.15	1.055046e+00	4.1e−11	1.8e−14	5.0e−17	3.6e−19	8.3e−09	2.8e−11	8.4e−13	7.6e−09	4.0e−10	8.6e−12
0.2	1.076022e+00	1.0e−09	4.3e−13	1.2e−15	1.2e−17	1.4e−07	5.4e−10	2.3e−11	2.4e−07	1.3e−08	4.0e−10
0.25	1.098612e+00	1.2e−08	5.4e−12	1.6e−14	2.1e−16	1.3e−06	5.5e−09	3.1e−10	4.2e−06	2.5e−07	9.8e−09
0.3	1.123054e+00	9.5e−08	4.6e−11	1.4e−13	2.3e−15	7.9e−06	3.8e−08	2.7e−09	5.1e−05	3.3e−06	1.7e−07
0.35	1.149640e+00	5.4e−07	3.0e−10	9.1e−13	1.9e−14	3.6e−05	1.9e−07	1.7e−08	4.9e−04	3.6e−05	2.4e−06
0.4	1.178736e+00	2.5e−06	1.6e−09	5.1e−12	1.3e−13	1.3e−04	8.3e−07	9.1e−08	4.0e−03	3.8e−04	3.4e−05
0.45	1.210806e+00	9.4e−06	7.5e−09	2.5e−11	7.4e−13	4.1e−04	3.0e−06	4.0e−07	3.1e−02	4.6e−03	6.7e−04

Input

parameters	none
base	10
approximant	5,10,15,20,25 (1 ≤ approximant ≤ 999)
z	.25
output in tables	relative error / absolute error

Output

z = 1/4 (relative error)

n	${}_2F_1(1/2,1;3/2;z)$	(HY.1.4)	(HY.3.7)	(HY.3.7)	(HY.3.7)	(HY.3.12)	(HY.3.12)	(HY.3.12)	(HY.3.17)	(HY.3.17)	(HY.3.17)
5	1.098612e+00	2.2e−05	2.8e−06	3.1e−08	7.3e−10	1.3e−03	2.1e−05	2.0e−06	1.4e−03	1.7e−04	1.2e−05
10	1.098612e+00	1.2e−08	5.4e−12	1.6e−14	2.1e−16	1.3e−06	5.5e−09	3.1e−10	4.2e−06	2.5e−07	9.8e−09
15	1.098612e+00	8.4e−12	1.0e−17	1.3e−20	1.2e−22	1.3e−09	2.5e−12	9.7e−14	1.4e−08	5.5e−10	1.5e−11
20	1.098612e+00	6.3e−15	2.0e−23	1.4e−26	1.0e−28	1.3e−12	1.4e−15	4.2e−17	5.1e−11	1.5e−12	3.0e−14
25	1.098612e+00	5.0e−18	3.8e−29	1.8e−32	1.0e−34	1.2e−15	8.7e−19	2.1e−20	1.9e−13	4.3e−15	7.2e−17

Maple library

symbolic:

$$f(z) = c_0 + c_1 z + c_2 z^2 + \dots$$

$$f(z) = b_0(z) + \cfrac{a_1(z)}{b_1(z) + \cfrac{a_2(z)}{b_2(z) + \cfrac{a_3(z)}{b_3(z) + \cdots}}}$$

Demo

Special functions :
continued fraction and series representations

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Function categories

- Elementary functions
- Gamma function and related functions
- Error function and related integrals
  - erf
  - dawson
  - erfc
  - Fresnel C
  - Fresnel S
- Exponential integrals and related functions
- Hypergeometric functions
- Confluent hypergeometric functions
- Bessel functions

Error function and related integrals » erfc » approximate

Representation

○	(ER.2.9)	$\operatorname{erfc}(z) = e^{-z^2} \sum_{k=0}^{\infty} \frac{(-z)^k}{\Gamma(\frac{k}{2}+1)}, \qquad z \in \mathbb{C}$
○	(ER.2.11)	$\sqrt{\pi} z e^{z^2} \operatorname{erfc}(z) \approx {}_2F_0\left(1, 1/2; -z^{-2}\right), \qquad z \to \infty, \quad \lvert\arg z\rvert < 3\pi/4$
○	(ER.2.20)	$\operatorname{erfc}(z) = \frac{z}{\sqrt{\pi}} e^{-z^2} \left(\frac{a_1}{z^2+} \frac{a_2}{1+} \frac{a_3}{z^2+} \frac{a_4}{1+} \ldots \right), \qquad \Re z > 0,$ $a_1 = 1, \quad a_m = \frac{m-1}{2}, \quad m \geq 2$
●	(ER.2.23)	$\operatorname{erfc}(z) = \frac{e^{-z^2}}{\sqrt{\pi}} \left(\frac{2z}{1+2z^2+} \mathop{\mathbf{K}}_{m=2}^{\infty} \left(\frac{-(2m-3)(2m-2)}{4m-3+2z^2} \right) \right), \qquad \Re z > 0$

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Special functions :
continued fraction and series representations

Handbook
A. Cuyt
W. B. Jones
V. Petersen
B. Verdonk
H. Waadeland

Software
F. Backeljauw
S. Becuwe
M. Colman
A. Cuyt
T. Docx

Function categories

Elementary functions
Gamma function and related functions
Error function and related integrals
- erf
- dawson
- erfc
- Fresnel C
- Fresnel S

Exponential integrals and related functions
Hypergeometric functions
Confluent hypergeometric functions
Bessel functions

Error function and related integrals » erfc » approximate

Representation

(ER.2.23) $$\operatorname{erfc}(z) = \frac{e^{-z^2}}{\sqrt{\pi}} \left(\frac{2z}{1+2z^2} + \mathop{\mathbf{K}}_{m=2}^{\infty} \left(\frac{-(2m-3)(2m-2)}{4m-3+2z^2} \right) \right), \qquad \Re z > 0$$

Input

rhs	J-fraction
parameters	none
base	10
digits	46 (5 ≤ digits ≤ 999)
approximant	13 (1 ≤ approximant ≤ 999)
z	6.5
tail estimate	none / standard / improved / user defined (simregular form)

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Error function and related integrals » erfc » approximate

Representation

(ER.2.23) $\operatorname{erfc}(z) = \frac{e^{-z^2}}{\sqrt{\pi}} \left(\frac{2z}{1+2z^2} + \underset{m=2}{\overset{\infty}{\mathbf{K}}} \left(\frac{-(2m-3)(2m-2)}{4m-3+2z^2} \right) \right), \quad \Re z > 0$

Input

rhs	J-fraction
parameters	none
base	10
digits	46 (5 ≤ digits ≤ 999)
approximant	13 (1 ≤ approximant ≤ 999)
z	6.5
tail estimate	none standard improved user defined (simregular form)

Output

value	3.842148327120647469875804525852808522764701220e-20
absolute error	1.792e-46
relative error	4.663e-27
tail estimate	0

Special functions :
continued fraction and series representations

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Error function and related integrals » erfc » approximate

Representation

(ER.2.23) $$\operatorname{erfc}(z) = \frac{e^{-z^2}}{\sqrt{\pi}} \left(\frac{2z}{1+2z^2} + \underset{m=2}{\overset{\infty}{\mathbf{K}}} \left(\frac{-(2m-3)(2m-2)}{4m-3+2z^2} \right) \right), \qquad \Re z > 0$$

Input

rhs	J-fraction
parameters	none
base	10
digits	46 (5 ≤ digits ≤ 999)
approximant	13 (1 ≤ approximant ≤ 999)
z	6.5
tail estimate	none / standard / improved / user defined (simregular form)

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Error function and related integrals » erfc » approximate

Representation

(ER.2.23) $\operatorname{erfc}(z) = \frac{e^{-z^2}}{\sqrt{\pi}} \left(\frac{2z}{1+2z^2} + \underset{m=2}{\overset{\infty}{\mathbf{K}}} \left(\frac{-(2m-3)(2m-2)}{4m-3+2z^2} \right) \right), \quad \Re z > 0$

Input

rhs	J-fraction
parameters	none
base	10
digits	46 (5 ≤ digits ≤ 999)
approximant	13 (1 ≤ approximant ≤ 999)
z	6.5
tail estimate	none / standard / improved / user defined (simregular form)

Output

value	3.842148327120647469875805009143862002332125688e-20
absolute error	4.654e-45
relative error	1.211e-25
tail estimate	-6.675000000000000000000000000000000000000000000e+01

C++ library

numeric:

$$\left|\frac{f(x) - \tilde{f}(x)}{f(x)}\right| \leqslant \alpha\,\beta^{-t+1} \Rightarrow$$

$$\tilde{f}(x) - \alpha\beta^{-t+1}\beta^{e+1} \leqslant f(x) \leqslant \tilde{f}(x) + \alpha\beta^{-t+1}\beta^{e+1}$$

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Type S.I
Type S.II
Type F.I
Type F.II
Type F.III
Web

truncation error:

$$f(x) \approx F(x)$$

$$\frac{|f(x) - F(x)|}{|f(x)|} \leqslant ?$$

round-off error:

$$\begin{array}{ccc} F(x) & \longrightarrow & \mathbf{F}(x) \\ +, -, \times, \div & & \oplus, \ominus, \otimes, \oslash \end{array}$$

$$\frac{|F(x) - \mathbf{F}(x)|}{|f(x)|} \leqslant ?$$

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Type F.III
Web

$F(x) = f_n(z; w_n), b_m(z) = 1$:

$$L_n \leqslant w_n \leqslant R_n$$

$$\frac{|f(x) - F(x)|}{|f(x)|} \leqslant \frac{R_n - L_n}{1 + L_n} \prod_{m=1}^{n-1} M_m$$

$$M_m = \max\left\{ \frac{L_m}{1 + L_m}, \frac{R_m}{1 + R_m} \right\}, \quad m = 1, \ldots, n-1$$

$$F(x) = f_n(z; w_n), b_m(z) = 1:$$

$$a_m = \alpha_m(1 + \delta_m), \quad \max(|\delta_1|, \ldots, |\delta_n|) \leqslant \frac{1}{2}\Delta\beta^{-s+1}$$

$$\frac{|F(x) - \mathbf{F}(x)|}{|f(x)|} \leqslant \frac{1}{2}(4 + \Delta)\frac{M^n - 1}{M - 1}\beta^{-s+1}$$

$$M = \max\{M_1, \ldots, M_n\}, \qquad s \geqslant t$$

Type S.I

monotonely decreasing $a_m > 0$

$$f(x) = \sum_{m=0}^{\infty} a_m x^m$$

Example:

$$\exp(x) = \sum_{m=0}^{\infty} \frac{x^m}{m!}, \qquad x \in \mathbb{R}$$

Type S.II

alternating a_m with $(-1)^m a_m > 0$ and monotonely decreasing

Example:

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+1}}{(2m+1)m!}, \qquad x \in \mathbb{R}$$

Type F.I

monotone behaviour of $a_m(x)$

$$f(x) = \mathop{\mathrm{K}}_{m=1}^{\infty} \frac{a_m(x)}{1}$$

$$= \cfrac{a_1(x)}{1 + \cfrac{a_2(x)}{1 + \cfrac{a_3(x)}{1 + \cdots}}}$$

Book
Maple
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Type S.I
Type S.II
▹ Type F.I
Type F.II
Type F.III
Web

Example:

$$\operatorname{erfc}(x) = \mathop{\mathrm{K}}_{m=1}^{\infty} \frac{a_m(x)}{1}, \qquad x > 0$$

$$= \frac{\frac{2x\exp(-x^2)}{\sqrt{\pi}(2x^2+1)}}{1} + \mathop{\mathrm{K}}_{m=1}^{\infty} \frac{\frac{-2m(2m-1)}{(2x^2+1+4m)(2x^2+1+4(m-1))}}{1}$$

$$a_m(x) \searrow -1/4$$

Type F.II

bound $a_m^{(k)}(x)$ for $m \geqslant n_k$, $1 \leqslant k \leqslant l$

$$f(x) = \prod_{k=1}^{l} f_k(x)$$

$$f_k(x) = \mathop{\mathrm{K}}_{m=1}^{\infty} \frac{a_m^{(k)}(x)}{1}$$

Book
Maple
C++
Type S.I
Type S.II
Type F.I
▹ Type F.II
Type F.III
Web

Example:

$$f_k(x) = \frac{{}_2F_1(a, k; c; x)}{{}_2F_1(a, k+1; c+1; x)} = 1 + \mathop{\mathrm{K}}_{m=1}^{\infty} \frac{a_m^{(k)} x}{1},$$

$$x < 1$$

$$a_{2m+1}^{(k)} = \frac{-(a+m)(c-k+m)}{(c+2m)(c+2m+1)}, \qquad m \geqslant 0$$

$$a_{2m}^{(k)} = \frac{-(k+m)(c-a+m)}{(c+2m-1)(c+2m)}, \qquad m \geqslant 1$$

Type F.III

each $f_k(x)$ either type F.I or type F.II

$$f(x) = f_0(x) \prod_{k=1}^{l} f_k(x)$$

$$f_k(x) = \mathop{\mathrm{K}}_{m=1}^{\infty} \frac{a_m^{(k)}(x)}{1}$$

$f_0(x)$ obtained separately

Book
Maple
C++
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Type S.II
Type F.I
Type F.II
▹ Type F.III
Web

Example:

$$J_{5/2}(x) = J_{1/2}(x)\frac{J_{3/2}(x)}{J_{1/2}(x)}\frac{J_{5/2}(x)}{J_{3/2}(x)}$$

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\sin(x)$$

$$\frac{J_{\nu+1}(x)}{J_{\nu}(x)} = \frac{x/(2\nu+2)}{1} + \mathop{\mathrm{K}}_{m=2}^{\infty}\frac{(ix)^2/(4(\nu+m-1)(\nu+m))}{1},$$

$$x \in \mathbb{R}, \nu \geqslant 0$$

$$a_m(x) \nearrow 0$$

Demo

Special functions :
continued fraction and series representations

Handbook: A. Cuyt, W. B. Jones, V. Petersen, B. Verdonk, H. Waadeland
Software: F. Backeljauw, S. Becuwe, M. Colman, A. Cuyt, T. Docx

Function categories

- Elementary functions
- Gamma function and related functions
- Error function and related integrals
  - erf
  - dawson
  - erfc
  - Fresnel C
  - Fresnel S
- Exponential integrals and related functions
- Hypergeometric functions
- Confluent hypergeometric functions
- Bessel functions

Error function and related integrals » erfc » evaluate

Representation

$$\operatorname{erfc}(0)=1,\quad \operatorname{erfc}(\infty)=0,\quad \operatorname{erfc}(-\infty)=2,$$
$$\operatorname{erfc}(-x)=2-\operatorname{erfc}(x),\quad \operatorname{erfc}(x)=1-\operatorname{erf}(x)$$

$$\operatorname{erfc}(x)=1-\operatorname{erf}(x),\qquad 0\leqslant x\leqslant 1$$

$$\operatorname{erfc}(x)=\frac{e^{-x^2}}{\sqrt{\pi}}\left(\frac{2x/(2x^2+1)}{1}+\mathop{\mathbf{K}}_{m=2}^{\infty}\frac{\frac{-(2m-3)(2m-2)}{(2x^2+4m-7)(2x^2+4m-3)}}{1}\right),\qquad x>1$$

Input

parameters	none
base	10^k k= 1 (base 2^k: 1 ≤ k ≤ 24; base 10^k: 1 ≤ k ≤ 7)
digits	40 (5 ≤ digits ≤ 999)
verbose	yes ◉ no ○
x	6.5

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Error function and related integrals » erfc » evaluate

Representation

$$\operatorname{erfc}(0)=1,\quad \operatorname{erfc}(\infty)=0,\quad \operatorname{erfc}(-\infty)=2,$$
$$\operatorname{erfc}(-x)=2-\operatorname{erfc}(x),\quad \operatorname{erfc}(x)=1-\operatorname{erf}(x)$$

$$\operatorname{erfc}(x)=1-\operatorname{erf}(x),\qquad 0\leqslant x\leqslant 1$$

$$\operatorname{erfc}(x)=\frac{e^{-x^2}}{\sqrt{\pi}}\left(\frac{2x/(2x^2+1)}{1}+\underset{m=2}{\overset{\infty}{\mathrm{K}}}\frac{\frac{-(2m-3)(2m-2)}{(2x^2+4m-7)(2x^2+4m-3)}}{1}\right),\qquad x>1$$

Input

parameters	none
base	10^k k= 1 (base 2^k: 1 ≤ k ≤ 24; base 10^k: 1 ≤ k ≤ 7)
digits	40 (5 ≤ digits ≤ 999)
verbose	yes no
x	6.5

Output

value	3.842148327120647469875804543768776621449e-20
absolute error bound	3.520e-61
relative error bound	9.162e-42
tail estimate	-3.691114343068676e-02
working precision	46
approximant	13

Function categories

Elementary functions
Gamma function and related functions
Error function and related integrals
- erf
- dawson
- erfc
- Fresnel C
- Fresnel S

Exponential integrals and related functions
Hypergeometric functions
Confluent hypergeometric functions
Bessel functions

Error function and related integrals » erfc » approximate

Representation

(ER.2.23) $$\operatorname{erfc}(z) = \frac{e^{-z^2}}{\sqrt{\pi}}\left(\frac{2z}{1+2z^2} + \mathop{\mathbf{K}}_{m=2}^{\infty}\left(\frac{-(2m-3)(2m-2)}{4m-3+2z^2}\right)\right), \qquad \Re z > 0$$

Input

rhs	J-fraction
parameters	none
base	10
digits	46 (5 ≤ digits ≤ 999)
approximant	13 (1 ≤ approximant ≤ 999)
z	6.5
tail estimate	none / standard / improved / user defined (simregular form) -3.691114343068676e-02

Output

value	3.842148327120647469875804543768776621449281104e-20
absolute error	3.597e-62
relative error	9.362e-43
tail estimate	-3.691114343068676e-02

Special functions :
continued fraction and series representations

Handbook
A. Cuyt
W. B. Jones
V. Petersen
B. Verdonk
H. Waadeland

Software
F. Backeljauw
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M. Colman
A. Cuyt
T. Docx

Function categories

Elementary functions
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Exponential integrals and related functions
Hypergeometric functions
Confluent hypergeometric functions
Bessel functions
J(nu, z)
J(n, z)
J(nu+1, z) / J(nu, z)
j(n, z)
j(n+1, z) / j(n, z)
I(nu, z)
I(n, z)
I(nu+1, z) / I(nu, z)
i(n, z)
i(n+1, z) / i(n, z)

Bessel functions » I(n, z) » evaluate

Representation

$$I_0(x) = \sum_{k=0}^{\infty} \frac{1}{k!k!}\left(\frac{x}{2}\right)^{2k}, \qquad x \in \mathbb{R}$$

$$\frac{I_{n+1}(x)}{I_n(x)} = \frac{x/(2n+2)}{1} + \mathop{\mathbf{K}}_{m=2}^{\infty} \frac{a_m(n)x^2}{1}, \qquad x \in \mathbb{R}, \quad n \in \mathbb{N}_0,$$

$$a_m(n) = \frac{1}{4(n+m-1)(n+m)}$$

$$I_n(x) = I_0(x)\prod_{k=1}^{n} \frac{I_k(x)}{I_{k-1}(x)}, \qquad x \in \mathbb{R}, \quad n \in \mathbb{N}$$

Input

parameters	n
base	10^k k= 1 (base 2^k: 1 ≤ k ≤ 24; base 10^k: 1 ≤ k ≤ 7)
digits	50 (5 ≤ digits ≤ 999)
verbose	yes ○ no ●
x	4.5
n	4

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Function categories

Elementary functions
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Confluent hypergeometric functions
Bessel functions
J(nu, z)
J(n, z)
J(nu+1, z) / J(nu, z)
j(n, z)
j(n+1, z) / j(n, z)
I(nu, z)
I(n, z)
I(nu+1, z) / I(nu, z)
i(n, z)
i(n+1, z) / i(n, z)

Bessel functions » I(n, z) » evaluate

Representation

$$I_0(x) = \sum_{k=0}^{\infty} \frac{1}{k!k!} \left(\frac{x}{2}\right)^{2k}, \qquad x \in \mathbb{R}$$

$$\frac{I_{n+1}(x)}{I_n(x)} = \frac{x/(2n+2)}{1} + \mathop{\mathbf{K}}_{m=2}^{\infty} \frac{a_m(n)x^2}{1}, \qquad x \in \mathbb{R}, \quad n \in \mathbb{N}_0,$$

$$a_m(n) = \frac{1}{4(n+m-1)(n+m)}$$

$$I_n(x) = I_0(x) \prod_{k=1}^{n} \frac{I_k(x)}{I_{k-1}(x)}, \qquad x \in \mathbb{R}, \quad n \in \mathbb{N}$$

Input

parameters	n
base	10^k k= 1 (base 2^k: 1 ≤ k ≤ 24; base 10^k: 1 ≤ k ≤ 7)
digits	50 (5 ≤ digits ≤ 999)
verbose	yes no
x	4.5
n	4

Output

value	2.7347222766930378559470450618354311463035025612657
absolute error bound	1.106e-51, 1.017e-53, 2.128e-53, 6.851e-53
relative error bound	1.256e-51, 1.471e-53, 3.818e-53, 1.486e-52
tail estimate	7.733062944939854e-03, 7.163861708630548e-03, 7.163861708630548e-03, 7.163861708630548e-03
working precision	57, 59, 58, 57, 56
approximant	25, 25, 24, 23, 33