Outage-Optimal Power Adaptation and Allocation for Truncated HARQ

Mohammed Jabi, Leszek Szczechinski, Mustapha Benjillali\(^{(1)}\), and Fabrice Labeau\(^{(2)}\)

INRS-EMT, Montreal, Canada
(\(^{(1)}\) INPT, Rabat, Morocco

\(^{(2)}\) McGill University, Montreal, Canada

\{jabi, leszek\}@emt.inrs.ca, benjillali@ieee.org, fabrice.labeau@mcgill.ca

Abstract—In this work, we analyze hybrid ARQ (HARQ) protocols used over independent block fading channels. We assume that the transmitter is unaware of the channel state information (CSI) at the moment of transmission but knows the statistics of the channel. We consider two scenarios with respect to the feedback received by the transmitter: (i) “conventional”, one-bit feedback carrying the message about the decoding success/failure (ACK/NACK), and (ii) the multi-bit feedback scheme when, on top of ACK/NACK, the receiver feeds back information about the state of the decoder, which is equivalent to the CSI experienced by the receiver in the past (unsuccesful) transmissions. The feedback can then be used to allocate power across the HARQ transmission attempts (in the case of one-bit feedback) or adapt the power for each re-transmission as a function of multi-bit feedback. The objective in both cases is the minimization of the outage probability under peak and long term average power constraints. We solve the optimization problems using the dynamic programming framework. The obtained results quantify the advantage of the multi-bit feedback over the conventional approach and indicate that power optimization can provide significant gains over the conventional constant-power HARQ transmission even in the presence of peak-power constraints.

I. INTRODUCTION

To secure reliable data transmission and improve robustness against noisy channels, two fundamental techniques are commonly used: forward error correction (FEC) and automatic repeat request (ARQ) [1]. In FEC schemes, an error correcting code is used for combating transmission errors. In ARQ schemes, an error detecting code is used and a retransmission is requested with a negative acknowledgment (NACK) sent to the transmitter via a feedback channel. A HARQ scheme combines ARQ and FEC, and provides a better performance compared to each scheme alone [2]. In typical HARQ protocols, retransmission-request is repeated until the codeword is received without errors—in which case a positive acknowledgment (ACK) is sent on the feedback channel—or a maximum number of transmissions is attained; this particular case is called truncated HARQ [3], [4]. HARQ schemes are classified basically into two categories: the Chase combining HARQ (CC-HARQ) [5], [6], where all retransmitted packets are identical, and the incremental redundancy HARQ (IR-HARQ) [7] where each retransmission carries a different piece of the "mother code" that generates the complete coded version of the message.

In this paper, we design the power strategies which minimize the outage probability subject to both long-term average and peak power constraints for IR-HARQ and CC-HARQ protocols in block fading channels. We analyze both cases when one bit ACK/NACK or multi-bits feedback is available at the transmitter. The multi-bit feedback scenario covers the case when the transmitter may obtain the channel state information (CSI) from the receiver through the feedback channel, but due to long communication/processing delays, the CSI is fully outdated (i.e., independent of the CSI in the subsequent transmissions).

To increase HARQ's performance, many power policies have been proposed in the litterature. In [8], a power adaptation was proposed to increase the throughput in the case of a discretized CSI. An asymptotically optimal power control algorithm that attains the diversity limit in long-term static channels has been presented in [9]. The case of long-term static channels was also studied in [10], where the authors determined the optimal power assignment strategy to minimize the total average transmission power subject to outage probability constraints. The optimization of energy efficiency with a packet error rate (PER) constraint was solved as a geometric programming problem in [11] for the case of space-time coded HARQ and in [12] for CC-HARQ over independent Rayleigh block fading channels. In [13], the authors derived an optimal power allocation scheme which minimizes the packet drop probability under a total average transmit power constraint for IR-HARQ with two transmissions. A suboptimal feedback and power adaptation rule was proposed for multiple-input multiple-output (MIMO) IR-HARQ block fading channels in [14], achieving the optimal outage diversity.

The objective of this paper is to assess the value of the multi-bit feedback for power designing schemes in HARQ and the main contributions of this work are:

1) We show how to use the well-known dynamic programming (DP) methods [15] to find the optimal power adaptation policies for truncated HARQ in order to minimize the outage probability under constraints on long-term average and peak allowed power. The method can be applied for both CC-HARQ and IR-HARQ and for any channel with a continuous cumulative distribution function. Unlike [14], when the proposed power strategies
are sub-optimal in terms of outage performance, our power policies are optimal in terms of outage.

2) We show how to optimize the power-allocation policy for IR-HARQ and CC-HARQ on Nakagami-\(m\) fading channel. The optimal solutions are given in parametrized closed form for an arbitrary number of transmissions. We note that only two transmissions were allowed in [13], while in [12] and [13] only Rayleigh block fading channels were considered.

3) We provide numerical results for practically interesting wireless channel models, comparing the outage probability with various power adaptation/ allocation methods.

The rest of the paper is organized as follows. In Sec. II, we introduce the adopted system model and, in Sec. III, we define the optimization problem. We show the optimization method for power adaptation policies in Sec. IV and the power allocation is treated in Sec. V. We provide numerical examples that illustrate the advantages obtained using the optimal power policies in Sec. VI. Conclusions are drawn in Sec. VII.

II. SYSTEM MODEL

We consider a block-fading model where the channel between the transmitter and the receiver is varying (fading) randomly from one transmission to another but stays invariant during each of the transmissions, thus the signal received on the \(k\)-th ARQ transmission round is given by

\[
y_k = \sqrt{\gamma_k} \cdot P_k(\text{CSI}_{k-1}) \cdot x_k + z_k, \quad k = 1, \ldots, K
\]

where \(z_k\) is a zero-mean, unit-variance Gaussian noise, \(x_k\) is the unit-variance transmitted signal, \(P_k(\text{CSI}_{k-1}) \geq 0\) is the transmit power and is a function of the previous realization of the channel \(\text{CSI}_{k-1} = [\gamma_1, \gamma_2, \ldots, \gamma_{k-1}]\), where \(\sqrt{\gamma_k}\) defines the CSI, i.e., the instantaneous channel gain which is assumed perfectly known at the receiver but unknown to the transmitter. Thus, the transmitter cannot adjust the communication rate in the \(k\)-th transmission.

To recover from decoding errors, the transmitter can transmit coded versions of a data packet at most \(K\) times. On top of the conventional signaling between the transmitter and the receiver (ACK/NACK messages), we also allow the receiver to send the CSI collected during unsuccessful transmission attempts back to the transmitter (entirely defined through channel realizations \(\gamma_k\)). The transmitter should be able to adapt the transmit power during the \(k\)-th transmission attempt using the knowledge of \(\gamma_1, \ldots, \gamma_{k-1}\). Thus, we will talk about power adaptation when the CSI is used to adjust the power in each transmission. On the other hand, the power allocation covers the case when only the “conventional” one-bit feedback (ACK/NACK) is available. In this case, the transmitter responds to the reception of a NACK message by retransmitting the packet with a different power.

We assume that \(\gamma_k\) changes on an independent and identically distributed (i.i.d.) basis from slot to slot with \(\gamma_k \sim \mathbb{E} \gamma_k \cdot [\gamma]\), where \(\mathbb{E}[\cdot]\) denotes the mathematical expectation calculated with respect to \(\gamma\). The independence of \(\gamma_k\) can be justified by the practical scenario where the successive transmissions are not sent in adjacent time instants and, being sufficiently well separated, the realizations of the channel become—to all practical extent—independent [17].

Most of the derivations will be done in abstraction of the particular fading distribution, but in numerical examples, we consider the popular Nakagami-\(m\) fading profile. Hence, channel gains \(\gamma_k\) follow a gamma distribution with a probability density function (PDF) \(p_\gamma(x)\) given by

\[
p_\gamma(x) = \frac{m^m}{\Gamma(m)} x^{m-1} e^{-mx/\gamma}, \quad x > 0,
\]

and the cumulative density function (CDF) \(F_\gamma(x)\) is given by

\[
F_\gamma(x) = 1 - \frac{\Gamma(m, mx/\gamma)}{\Gamma(m)}, \quad 0 < x,
\]

where \(\Gamma(x)\) and \(\Gamma(s, x)\) denote respectively the gamma function and the upper incomplete gamma function.

We assume the decoding is successful if the average accumulation mutual information at the receiver is larger than the overall transmission rate for IR-HARQ. In the case of CC-HARQ, the decoding is successful if the accumulated SNR is larger than an SNR threshold. Thus, the decoding fails after \(k\) transmissions with probability

\[
f_k = \begin{cases} 
\Pr \left\{ \sum_{l=1}^{k} \log_2 \left(1 + \gamma_l \cdot P_l(\text{CSI}_{l-1})\right) < R \right\}, & \text{for IR-HARQ} \\
\Pr \left\{ \sum_{l=1}^{k} \log_2 \left(1 + \gamma_l \cdot P_l(\text{CSI}_{l-1})\right) < R \right\} & \text{for CC-HARQ}
\end{cases}
\]

\[
= \Pr \left\{ I_k < i_\text{th} \right\}
\]

\[
= F_{I_k}(i_\text{th}) = \int_0^{i_\text{th}} p_{I_k}(x) dx,
\]

where

\[
I_k = \begin{cases} 
\sum_{l=1}^{k} C_l, & \text{for IR-HARQ} \\
\sum_{l=1}^{k} \sigma_l, & \text{for CC-HARQ}
\end{cases}
\]

and \(\sigma_l = \gamma_l \cdot P_l(\text{CSI}_{l-1})\), \(C_l = \log_2 \left(1 + \gamma_l \cdot P_l(\text{CSI}_{l-1})\right)\), \(p_{I_k}(x)\) is the PDF of \(I_k\) and

\[
i_\text{th} = \begin{cases} R, & \text{for IR-HARQ} \\
\frac{1}{2} R - 1, & \text{for CC-HARQ}
\end{cases}
\]

With these notations, the scenarios we consider are defined as follows:

- Constant power (CP) HARQ, where \(P_k(\text{CSI}_{k-1}) \equiv \bar{P}\), i.e., the power is the same throughout retransmissions.
- Variable Power (VP) HARQ, where the CSI feedback is ignored (or, simply not available) and the power varies solely as a function of the transmission’s index, i.e., \(P_k(\text{CSI}_{k-1}) \equiv \bar{P} \cdot \mathbb{I}(I_{k-1} \leq i_\text{th})\) where \(\mathbb{I}(x) = 1\) if \(x\) is true, and 0 otherwise, since the \(k\)-th transmission is necessary only if the previous \((k-1)\) transmissions were unsuccessful. Finding \(\bar{P}\) is a problem of power allocation.
- Adaptive Power (AP) HARQ, where the power is modified in each transmission attempt using the CSI provided over the feedback channel. From (3), the decoding error
event in the $k^\text{th}$ transmission depends uniquely on $I_{k-1}$ and $\gamma_k$ (which is unknown, and cannot be predicted from the previous CSI $\gamma_1$, ..., $\gamma_{k-1}$ due to the independence assumption). Consequently, $I_{k-1}$ (which is a scalar representation of the vector $\text{CSI}_{k-1}$) is the only parameter eventually required to adapt the power $P_k(\text{CSI}_{k-1})$ via a scalar function

$$P_k(\text{CSI}_{k-1}) \equiv \bar{P}_k(I_{k-1}) \cdot \mathbb{I}(I_{k-1} \leq i_{th}), \quad k = 1, \ldots, K$$

(7)

where $I_0 \triangleq 0$ by definition. Finding $\bar{P}_k(I_{k-1})$ is a problem of power adaptation.

For simplicity, we assume that the transmitter has a perfect knowledge of $I_{k-1}$, that is, we ignore all eventual transmission and discretization errors. We hence evaluate the maximum gain that can be achieved using information about the channel $I_{k-1}$.

III. Optimization Problem

According to the reward-renewal theorem [16], the long-term average consumed power is the ratio between the average transmit power between two consecutive renewals (sending a new data packet) $\mathbb{E}_{\text{CSI}_K}[P]$ and the expected number of transmissions $\mathbb{E}_{\text{CSI}_K}[T]$ needed to deliver the packet with up to $K$ transmission attempts [8], [17]

$$\bar{P} \triangleq \frac{\mathbb{E}_{\text{CSI}_K}[P]}{\mathbb{E}_{\text{CSI}_K}[T]} = \sum_{k=1}^{K} \frac{\sum_{k=1}^{K-1} P_k(\text{CSI}_{k-1})}{\sum_{k=0}^{K-1} f_k},$$

(8)

where $f_k$ is the probability of a decoding failure after $k$ transmission attempts given by (3), and $\mathbb{E}_{\text{CSI}_K}[P_k(\text{CSI}_{k-1})]$ is the expected transmit power during the $k^\text{th}$ transmission attempt, obtained by considering all the events yielding the $k^\text{th}$ transmission, i.e., the event $I_{k-1} < i_{th}$.

In this work, we aim at minimizing the outage probability $P_{out} = f_K$ with respect to the power policy $\{P_k(\text{CSI}_{k-1})\}_{k=1}^{K}$ for a given long-term average power $\bar{P}_{\text{max}}$, peak allowed power $P_{\text{max}}$ and a transmission rate $R$. Taking (8) into consideration, the optimization problem can be formulated as follows

$$\min_{P_1, P_2(\text{CSI}_1), \ldots, P_K(\text{CSI}_{K-1})} f_K,$$

s.t.

$$\begin{cases} \bar{P} \leq \bar{P}_{\text{max}} \\ 0 \leq P_k(\text{CSI}_{k-1}) \leq P_{\text{max}}, \quad 1 \leq k \leq K \end{cases}$$

(9)

We define the Lagrangian function $L : \mathbb{R}_+^{(K-1)+1} \times \mathbb{R} \rightarrow \mathbb{R}$ associated with the problem (9) as

$$L(P_1, P_2(\text{CSI}_1), \ldots, P_K(\text{CSI}_{K-1}), \lambda) = f_K + \lambda \cdot \left( \sum_{k=1}^{K} \mathbb{E}_{\text{CSI}_k}[P_k(\text{CSI}_{k-1})] - \bar{P}_{\text{max}} \sum_{k=0}^{K-1} f_k \right),$$

(10)

where $\text{CSI}_k$ is discretized over $N$ points. We left implicit all power constraints $0 \leq P_k(\text{CSI}_{k-1}) \leq P_{\text{max}}$. Without any loss of generality, we consider $\bar{P}_{\text{max}} = 1$ in what follows.

IV. Outage-Optimal Power Adaptation

For power adaptation (7), the expected transmit power during the $k^\text{th}$ transmission is given by

$$\mathbb{E}_{\text{CSI}_k}[P_k(\text{CSI}_{k-1})] = \mathbb{E}_{I_{k-1}}[\tilde{P}_k(I_{k-1})] = \int_{0}^{i_{th}} \tilde{P}_k(x)p_{I_{k-1}}(x)dx.$$  

(11)

Thus the Lagrangian function $L$ defined in (10) can be written as:

$$L \left( \tilde{P}_1, \tilde{P}_2(I_1), \ldots, \tilde{P}_K(I_{K-1}), \lambda \right) = f_K + \lambda \cdot \left( \sum_{k=1}^{K} \mathbb{E}_{I_{k-1}}[\tilde{P}_k(I_{k-1})] - \sum_{k=0}^{K-1} f_k \right).$$

(12)

To solve the primal problem in (9), it is difficult to use the Karush–Kuhn–Tucker (KKT) conditions on the Lagrangian function (12) since it requires to solve analytically a system of an infinite number of equations where, in addition, closed form expressions of $f_k$ (for $2 \leq k \leq K$) are unknown. To overcome these difficulties, we can solve the dual problem. In the general case, the dual problem provides a solution which is a lower bound to the solution of (9); the difference between the lower bound and the true optimum is called the “duality gap”.

However, according to a result in [18, Theorem 1], optimization problems with expectations over non-convex functions of random variables in both objective and constraint functions have a zero duality gap, given that the PDF of the random variable of interest has no points of strictly positive probability (i.e., its CDF is continuous). For this reason, we express the outage probability and the average transmit power (11) as a function of the channel gains $\gamma_k$, $k = 1, \ldots, K$, with a continuous CDF, indeed verifying the above requirement

$$f_K = \Pr \{I_K < i_{th}\} = \mathbb{E}_{\gamma_1, \gamma_2, \ldots, \gamma_K}[\mathbb{I}(I_K < i_{th})],$$

(13)

and

$$\mathbb{E}_{\text{CSI}_K}[P] = \tilde{P}_1 + \sum_{k=2}^{K} \mathbb{E}_{\gamma_1, \gamma_2, \ldots, \gamma_{k-1}}[\tilde{P}_k(I_{k-1})] = \tilde{P}_1 + \sum_{k=2}^{K} \mathbb{E}_{\gamma_1, \gamma_2, \ldots, \gamma_{k-1}}[\tilde{P}_k(I_{k-1})].$$

(14)

Thus, the Lagrangian function (12) can be written as

$$L \left( \tilde{P}_1, \tilde{P}_2(I_1), \ldots, \tilde{P}_K(I_{K-1}), \lambda \right) = \mathbb{E}_{\gamma_1, \gamma_2, \ldots, \gamma_K}[\mathbb{I}(I_K < i_{th})] + \lambda \cdot \left( \tilde{P}_1 + \sum_{k=2}^{K} \tilde{P}_k(I_{k-1}) - \sum_{k=0}^{K-1} \mathbb{I}(I_K < i_{th}) \right).$$

(15)

Since $P_1$ is independent of any random variable, and in order to use correctly the result in [18, Theorem 1], i.e., both objective and constraint functions must be expectations over
non-convex functions of random variables, we introduce a sub-
optimization problem for each value of \( \tilde{P}_1 > 0 \)
\[
\hat{f}_K(P_1) \triangleq \min_{f_2(1), \ldots, f_K(I_{K-1})} f_K,
\]
(16)
s.t.
\[
\sum_{k=2}^{K} \mathbb{E}_{I_{k-1}}[\tilde{P}_k(I_{k-1})] - \sum_{k=2}^{K-1} f_k \leq 1 + f_1 - \tilde{P}_1.
\]
(17)

The optimal solution of (9) is then given by
\[
\min_{P_1} \hat{f}_K(P_1).
\]
(18)

Defining the Lagrange dual function \( d : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R} \) as
\[
d(\tilde{P}_1, \lambda) \triangleq \min_{P_2(I_1), \ldots, \tilde{P}_K(I_{K-1}), \lambda} L(\tilde{P}_1, \tilde{P}_2(I_1), \ldots, \tilde{P}_K(I_{K-1}), \lambda),
\]
the dual optimization problem is given by
\[
D(\tilde{P}_1) = \min_{\lambda \geq 0} d(\tilde{P}_1, \lambda).
\]
(20)

Note that, since problem (16) and its dual (20) have a zero
duality gap, we can guarantee that \( D(\tilde{P}_1) = \hat{f}_K(P_1) \) for \( \tilde{P}_1 > 0 \).

Finally, (19) can be rewritten in a recursive form character-
izing dynamic programming (DP) optimization
\[
d(\tilde{P}_1, \lambda) = J_1(I_0)
\]
\[
J_1(I_0) = \left\{ -\lambda \cdot \mathbb{E}_{\gamma_1}[\mathbb{I}(I_1 < i_{th})] + \lambda \cdot \tilde{P}_1(I_0) + \mathbb{E}_{\gamma_1}[J_2(I_1)] \right\}
\]
(21)
\[
J_2(I_1) = \min_{\tilde{P}_2(I_1)} \left\{ -\lambda \cdot \mathbb{E}_{\gamma_2}[\mathbb{I}(I_2 < i_{th})] + \lambda \cdot \tilde{P}_2(I_1) + \mathbb{E}_{\gamma_2}[J_3(I_2)] \right\}
\]
(22)
\[
\vdots
\]
\[
J_{k}(I_{k-1}) = \min_{\tilde{P}_k(I_{k-1})} \left\{ -\lambda \cdot \mathbb{E}_{\gamma_k}[\mathbb{I}(I_k < i_{th})] + \lambda \cdot \tilde{P}_k(I_{k-1}) + \mathbb{E}_{\gamma_k}[J_{k+1}(I_k)] \right\}
\]
(23)
\[
\vdots
\]
\[
J_{K}(I_{K-1}) = \min_{\tilde{P}_K(I_{K-1})} \left\{ \gamma_k \cdot \tilde{P}_K(I_{K-1}) + \mathbb{E}_{\gamma_k}[\mathbb{I}(I_K < i_{th})] \right\},
\]
(24)

where \( I_k \) is a function of \( I_{k-1}, \tilde{P}_k(I_{k-1}) \), and \( \gamma_k \) in the form
\[
I_k = \begin{cases} 
I_{k-1} + \log \left( 1 + \gamma_k \tilde{P}_k(I_{k-1}) \right), & \text{for IR-HARQ} \\
I_{k-1} + \gamma_k \tilde{P}_k(I_{k-1}), & \text{for CC-HARQ}
\end{cases}
\]
(25)

For a given \( I_k \)—noting that \( I_k \in [0, i_{th}] \) and should be
discretized over \( N \) points—we can optimize the value of
the function \( \tilde{P}_{k+1}(I_k) \) provided that the function \( J_k(I_{k+1}) \) is known. Thus, the global optimization of the possibly non-
convex problem in (16) over the set of \( N^{K-1} \) values is reduced
to a series of \( (K-1) \cdot N \) one-dimensional optimizations thanks
to the DP formulation equations in (22)-(24).

A. Radio silence and calculation of the outage

In this context of IR-HARQ transmissions, we have
\[
\mathbb{E}_{\gamma_k}[\mathbb{I}(I_k < i_{th})] = F_{\gamma_k} \left( \frac{2^{R_{\tilde{I}_k}-1} - 1}{P_k} \right).
\]
(26)
The condition to guarantee a minimum in the last DP step is
that the derivative of the function under minimization in (24)
equals zero, i.e.,
\[
u(\tilde{P}_K) = \lambda - \frac{2^{R_{\tilde{I}_k}-1} - 1}{P_k^2} \cdot \gamma_k \left( \frac{2^{R_{\tilde{I}_k}-1} - 1}{P_k} \right)
\]
\[
= \lambda - \frac{1}{2^{R_{\tilde{I}_k}-1} - 1} \cdot \gamma_k \left( \frac{2^{R_{\tilde{I}_k}-1} - 1}{P_k} \right) = 0,
\]
(27)

where it is easy to show that \( q(x) \triangleq x^2 p_{\gamma_k}(x) \) satisfies
\( q(x) > 0, q(0) = 0, \) and \( q(\infty) = 0, \) and hence, \( q(x) \) has a maximum \( q_{\text{max}} = \max_x q(x). \) Since the derivative \( u(0) = \lambda \) and \( u(\tilde{P}_K) \) must be locally non-increasing around \( \tilde{P}_K = 0 \)
i.e., \( u'(0) \leq 0, \) the solution of (27) does not exist if
\( \lambda \cdot (2^{R_{\tilde{I}_k}-1} - 1) > q_{\text{max}}; \) meaning that the minimum is
obtained by setting \( \tilde{P}_K = 0 \) and yielding \( J_k(I_{K-1}) = 1. \)
When \( \lambda \cdot (2^{R_{\tilde{I}_k}-1} - 1) < q_{\text{max}}, \) \( u(\tilde{P}_K) \) has at least two zeros\(^1\) and the optimal solution corresponds to the point where the
second derivative is positive.

In Fig. 1, we show the adaptation policy \( \tilde{P}_k(x) \) in the case of IR-HARQ with \( K = 4. \) As we see, the optimal
solution requires a “radio silence”, that is, knowing in the \( k^{th} \)
transmission that the accumulated mutual information at the receiver is below a threshold \( i_{0,k-1}, \) the transmitter decides to
remain silent (zero transmit power) until the maximum number of
transmissions is attained. This “silence time” guarantees that the power is “saved” when the transmitter does not have a “reasonable hope” of successfully terminating the transmission.

---

1In the case of a Rayleigh fading channel, there are exactly two zeros: one corresponds to a local maximum, and the other to the local minimum.
To calculate the outage probability, we use (4) where the CDF of $I_k$ is $F_{k_0}(x)$ and taking into consideration that $\hat{P}_k(x) = 0$ for $x \in [0, i_{0,k-1}]$. We obtain

$$F_{I_k}(x) = \Pr \left\{ I_{k-1} + \log(1 + \gamma_k \cdot \hat{P}_k(I_{k-1})) < x \right\}$$

$$= \left\{ \begin{array}{ll}
F_{I_{k-1}}(x), & \text{if } x < i_{0,k-1} \\
F_{I_{k-1}}(i_{0,k-1}) + \int_{i_{0,k-1}}^x F_{y} \left( \frac{2x - y}{\hat{P}_k(y)} \right) p_{I_{k-1}}(y) \, dy, & \text{if } x > i_{0,k-1}
\end{array} \right.$$

(28)

which depends on the PDF $p_{I_{k-1}}(y)$ of $I_{k-1}$.

The differentiation of (28) yields a recursive relationship for the PDF

$$p_{I_k}(x) = \left\{ \begin{array}{ll}
p_{I_{k-1}}(x), & \text{if } x < i_{0,k-1} \\
\int_{i_{0,k}}^x \frac{\log(2)2x - y}{\hat{P}_k(y)} p_{I_{k-1}}(y) \, dy \times p_{y} \left( \frac{2x - y}{\hat{P}_k(y)} \right) \, dy, & \text{if } x > i_{0,k-1}
\end{array} \right.$$

(29)

where

$$p_{I_{k-1}}(x) = \frac{\log(2)2x}{P_{I_{k-1}}} \cdot p_{y} \left( \frac{2x - 1}{P_{I_{k-1}}} \right).$$

(30)

Considering $F_{I_k}(x) = \Pr \{ I_{k-1} + \gamma_k \cdot \hat{P}_k(I_{k-1}) < x \}$, the same analysis as (28) and (29) can be done in the case of CC-HARQ but we omit the derivation for lack of space.

V. OUTAGE OPTIMAL POWER ALLOCATION

In this section, we consider the problem of optimal power allocation (i.e., $P_k(CS\mathcal{L}_{k-1}) \equiv \hat{P}_k \cdot I(I_{k-1} \leq i_{k})$). The expected power consumed in the $k$th transmission attempt is given by

$$E_{CS\mathcal{L}_k}[P] = E_{I_{k-1}}[\hat{P}_k \cdot I(I_{k-1} \leq i_{k})] = \hat{P}_k \cdot f_{k-1}$$

(31)

and the long-term average power (8) by

$$\mathcal{P} = \frac{\sum_{k=1}^{K} \hat{P}_k \cdot f_{k-1}}{\sum_{k=0}^{K-1} f_k}.$$  

(32)

Thus, the Lagrangian function $L$ defined in (10) can be expressed as

$$L(\hat{P}_1, \hat{P}_2, \ldots, \hat{P}_K, \lambda) = f_K + \lambda \cdot \left( \sum_{k=1}^{K} \hat{P}_k f_{k-1} - \sum_{k=0}^{K-1} f_k \right).$$

(33)

We note that [12] and [13] define the long term average power as $\mathcal{P} = \sum_{k=1}^{K} \hat{P}_k \cdot f_{k-1}$, which is an approximation valid in high SNR regime, where $\sum_{k=0}^{K-1} f_k \approx 1$. Thus, we expect that we obtain the same results as in [12] and [13] in the high SNR regime in the case of Rayleigh ($m = 1$) fading channel model.

To cast (33) into the DP formulation, we have to find the “states” $S_k$ such that: (i) $f_k$ may be calculated from $S_k$, and (ii) state $S_{k+1}$ may be obtained from $S_k$ and $\hat{P}_{k+1}$. Because the closed form expressions of $f_k$ are unknown, we use an accurate approximation to express $f_k$ in terms of $\{P_l\}_{l=1}^{K}$, as shown in the Appendix, we can obtain the following relationship:

$$f_k \approx \frac{h_k}{P_{kmax}} \cdot f_{k-1}, \quad \text{for } 1 \leq k \leq K,$$

(34)

where the parameter $h_k$ is independent from $\hat{P}_k$ and $m_k$ is the parameter of the Nakagami-$m$ channel at the $k$th transmission. Considering $f_0 = 1$, the optimization problem in (33) can be reformulated recursively as follows

$$L(\hat{P}_1, \hat{P}_2, \ldots, \hat{P}_K, \lambda) = J_1(f_0)$$

$$J_1(f_0) = \lambda \cdot (\hat{P}_1 - 1) \cdot f_0 + J_2(f_1)$$

$$J_2(f_1) = \lambda \cdot (\hat{P}_2 - 1) \cdot f_1 + J_3(f_2)$$

$$\vdots$$

$$J_{K-1}(f_{K-1}) = \lambda \cdot (\hat{P}_K - 1) \cdot f_{K-1} + f_K$$

(35)

From the KKT necessary conditions, and starting from (37), we find a unique (therefore, the optimal) solution as

$$\hat{P}_k =$$

(38)

$$\min \left\{ \begin{array}{l}
\left( \frac{m_k \cdot g_{k+1} \cdot h_k}{\lambda} \right)^{\frac{m_k + 1}{m_k}}, P_{max}, \quad \text{for } 1 \leq k \leq K - 1
\end{array} \right.$$

(39)

where

$$g_k = \left\{ \begin{array}{ll}
\lambda \cdot (\hat{P}_k - 1) + \frac{g_{k+1} \cdot h_k}{\hat{P}_{kmax}}, & \text{for } 1 \leq k \leq K - 1
\lambda \cdot (\hat{P}_K - 1) + \frac{h_k}{\hat{P}_K}, & \text{for } k = K
\end{array} \right.$$  

(40)

VI. NUMERICAL EXAMPLES

Fig. 2 and Fig. 3 present the optimal outage probability $P_{out}$ in the case of IR-HARQ and CC-HARQ respectively. We show both cases when $P_{max} = 5$ and $P_{max} = \infty$. For comparison purposes, we also plot the equal power outage probability. We can see clearly that the optimal results are better compared with CP. For instance, in the case of IR-HARQ when $K = 4$, $P_{max} = \infty$ and for a value of the outage probability equal to $10^{-6}$, $P_{out}$ of AP presents a gain close to $5$ dB relative to CP, while $P_{out}$ of VP presents a gain close to $2.5$ dB relative to CP.

When $P_{max} = 5$ the gain of the optimal outage compared with CP starts to decrease after a specific value $\gamma_0$ of the average SNR $\gamma$. For example, in the case of allocation when $K = 4$: $\gamma_0 \approx 0$ dB for IR-HARQ and $\gamma_0 \approx 2$ dB for CC-HARQ. This is justified by the fact that the constraint $P_{max} = 5$ becomes active for $\gamma \geq \gamma_0$, as can be seen in Fig. 4 for IR-HARQ.

We note that the gain of the power adaptation over allocation strategies, is not only a function of the maximum number
of transmissions $K$, but it also depends on the adopted HARQ scheme (IR-HARQ or CC-HARQ) as it is clear from comparison between Fig. 2 and Fig. 3.

Fig. 2. Optimal outage probability when using the optimal power adaptation (AP) and allocation (VP) compared with outage probability of constant-power transmission (CP) (i.e., $P_k = P = 1$, $\forall k$) in the case of IR-HARQ, $K = 2, 4$ and Nakagami-m fading with $m = 2; R = 1.5$, unconstrained peak (i.e., $P_{\text{max}} = \infty$) and constrained peak (i.e., $P_{\text{max}} = 5$) cases are shown.

Fig. 3. Optimal outage probability when using the optimal power adaptation (AP) and allocation (VP) compared with outage probability of constant-power transmission (CP) (i.e., $P_k = P = 1$, $\forall k$) in the case of CC-HARQ, $K = 2, 4$ and all variables correspond to Nakagami-m fading with $m = 2; R = 1.5$, unconstrained peak (i.e., $P_{\text{max}} = \infty$) and constrained peak (i.e., $P_{\text{max}} = 5$) cases are shown.

VII. CONCLUSION

In this paper, we analyzed the impact of multi-bit feedback for improvement of the HARQ protocol in terms of the outage probability. We analyzed both HARQ with Chase combining and Incremental Redundancy combining in Nakagami-m block fading channels. We conclude that the appropriate (optimized) variation of the power throughout the transmissions leads to notable gains over the constant power HARQ. Adding multi-bit feedback improves further the performance and the gains grow with the allowed number of transmissions.

As future work, it would be interesting to evaluate the outage optimal power adaptation when outdated CSI is dis-

Fig. 4. Optimal allocation policies $\hat{P}_k$ as a function of $\gamma$ in the case of IR-HARQ when $K = 4$, $m = 2$, $R = 1.5$. Both cases of unconstrained (i.e., $P_{\text{max}} = \infty$) and constrained peak power (i.e., $P_{\text{max}} = 5$) are shown for comparison.

APPENDIX

In this appendix, we aim to determine the expression of $h_k$ in (34) for the case of IR-HARQ and CC-HARQ. For that, we will derive a simple and accurate approximation of $f_k$ defined in (3). Clearly, calculating the outage probability in (3) for power allocation scheme requires the derivation of the CDF of the sum of $k$ independent random variables: $C_k = \log_2 \left(1 + \gamma_k \hat{P}_k\right)$ in the case of IR-HARQ and $\sigma_k = \gamma_k \hat{P}_k$ in the case of CC-HARQ.

A. CC-HARQ

We use a simple and accurate method to evaluate the outage probability at the output of maximum ratio combining (MRC) receivers in arbitrarily fading channels introduced in [21]. The approximation is based on the so-called saddle-point approximation (SPA) [19], [20], which is a simple and accurate method to approximate the CDF of a random variable. For the special case of Nakagami-$m$ fading channels, the outage probability can be approximated by [21]

$$ f_k \approx \left(\frac{\exp(1) \cdot \gamma_k}{\tilde{m}_K}\right) \cdot \frac{1}{\sqrt{2\pi m_K}} \cdot \prod_{k=1}^{K} \left(\frac{m_k}{\gamma_k \hat{P}_k}\right)^{m_k}, \quad (41) $$

where $\tilde{m}_K = \sum_{k=1}^{K} m_k$. In this case, we can easily show that $h_k$, required in (34), is given by

$$ h_k = \begin{cases} \left(\frac{\exp(1) \cdot \gamma_k m_k}{\gamma_k \tilde{m}_K}\right)^{m_k} \cdot \left(1 - \frac{m_k}{\tilde{m}_K}\right)^{\hat{f}_{\tilde{m}_K-1+0.5}}, & \text{for} \ 2 \leq k \leq K \\ \frac{1}{\sqrt{2\pi m_k}} \left(\frac{\exp(1) \cdot \gamma_k}{\gamma_1}\right)^{m_1}, & \text{for} \ k = 1 \end{cases} \quad (42) $$

Let

$$ \hat{h}_k = \begin{cases} \left(\frac{\exp(1) \cdot \gamma_k m_k}{\gamma_k \tilde{m}_K}\right)^{m_k} \cdot \left(1 - \frac{m_k}{\tilde{m}_K}\right)^{\hat{f}_{\tilde{m}_K-1+0.5}}, & \text{for} \ 2 \leq k \leq K \\ \frac{1}{\sqrt{2\pi m_k}} \left(\frac{\exp(1) \cdot \gamma_k}{\gamma_1}\right)^{m_1}, & \text{for} \ k = 1 \end{cases} \quad (43) $$
Similar analysis as [21] can be done to approximate outage probability in the case of IR-HARQ. We omit analysis’s details here for lack of space. For IR-HARQ in the case of Nakagami-\(m\) fading channels, the outage probability can be approximated by [23]

\[
f_K \approx \tilde{f}_K = \frac{R \cdot e^{\tilde{m}_K}}{\tilde{m}_K \sqrt{2\pi \sum_{k=1}^{K} \tilde{\kappa}''_k \left( \frac{\tilde{m}_k}{R} \right) \prod_{l=1}^{K} \left( \frac{\log(2)Rm_k}{m_k\tilde{T}_k} \right)^{m_k}}} \cdot \left( 1 + \frac{\log(2)Rm_k}{m_k} \right) \left( \frac{1}{P_k} \right)^{m_k},
\]

(44)

where

\[
\tilde{\kappa}''_k(\tilde{s}) = \frac{(m_k + 1)}{\tilde{s}^2} - \frac{1}{(\tilde{s} - m_k \log(2))}.
\]

(45)

Thus, for \(2 \leq k \leq K\) the expression of \(h_k\), required in (34), is given by

\[
h_k = \frac{\tilde{m}_{k-1}}{m_k} \left( \frac{\log(2)R \cdot m_{k} \exp(1)}{\tilde{T}_k} \right)^{m_k} \cdot \sqrt{\sum_{l=1}^{k} \tilde{\kappa}''_l \left( \frac{\tilde{m}_{k-1}}{R} \right) \prod_{l=1}^{k-1} \left( \frac{\log(2)Rm_l}{m_l} \right) \left( \frac{1}{m_{k-1}} \right)^{m_l} \left( 1 - \frac{\log(2)Rm_l}{m_l} \right) /
\]

\[
\prod_{l=1}^{k-1} \left( \frac{1}{m_{l-1}} \right)^{m_l} \left( 1 - \frac{\log(2)Rm_l}{m_l} \right)
\]

(46)

and

\[
h_1 = \frac{R \cdot e^{m_1}}{m_1 \sqrt{2\pi \tilde{\kappa}''_1 \left( \frac{\tilde{m}_1}{R} \right)}} \left( 1 + \log(2)R \right) \left( \frac{\log(2)R}{\tilde{s} e^{m_1}} \right)^{m_1}.
\]

(47)

We do not show the lengthy derivations that lead to (44), we refer to [23]. Instead, we show the accuracy of the approximation in Fig. 5.

**Fig. 5.** Exact outage probability \(f_K\) compared with the SPA approximation \(\tilde{f}_K\) (44) in the case of IR-HARQ for Nakagami-\(m\) when \(K = 1, 2, 3, 4, R = 1\) and \(m = 2\).