The Role of Information in Multiagent Coordination

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Abstract—The goal in networked control of multiagent systems is to derive desirable collective behavior through the design of local control algorithms. The information available to the individual agents, either attained through communication or sensing, invariably defines the space of admissible control laws. Hence, informational restrictions impose constraints on achievable performance guarantees. This paper provides one such constraint with regards to the efficiency of the resulting stable solutions for a class of networked resource allocation problems with submodular objective functions. When the agents have full information regarding the mission space, the efficiency of the resulting stable solutions is guaranteed to be within 50% of optimal. However, when the agents have only localized information about the mission space, which is a common feature of many well-studied control designs, the efficiency of the resulting stable solutions can be $1/n$ of optimal, where $n$ is the number of agents. Consequently, in general such schemes cannot guarantee that systems comprised of $n$ agents can perform any better than a system comprised of a single agent for identical environmental conditions. The last part of this paper highlights an algorithm that overcomes this limitation by allowing the agents to communicate minimally with neighboring agents.

I. INTRODUCTION

Multiagent systems have the potential to impact all aspects of society with applications ranging from environmental monitoring and surveillance to the delivery of service and goods. Realizing the many benefits of multiagent systems hinges on the ability to derive desirable collective behavior through the design of local agent control algorithms. A significant challenge associated with the design of such networked control systems is analytically dealing with the resulting enmeshed environment where individual agents act independently in response to partial and overlapping information.

In response to these challenges, there has been significant research attention pertaining to the design of networked control systems for highly specific application domains including sensor allocation [1], [5], [13], robotic networks [2], [6], [16], data fusion [15], [18], among many more. In many of these cases, the dominant focus has been on the development of local agent control algorithms that guarantee that the collective behavior will reach a stable allocation while adhering to stringent information demands. The questions pertaining to the efficiency of this emergent behavior, and more generally the relationship between informational restrictions and potential efficiency guarantees, has largely been unexplored.

Consider the well-studied sensor coverage problem as an illustrative example [13]. Here, the goal is to design local control algorithms for a collection of sensors that enables the sensors, each acting independently in response to local information, to collectively reach an allocation that optimizes a given system-level objective. The need for locality emerges from the vastness of the mission space, coupled with communication and sensing limitations, that prohibit individual sensors from having global knowledge pertaining to the state of mission space. Much of the literature has inherited a natural notion of locality where individual sensors have information pertaining to their local environment as defined by neighboring sensors, e.g., Voronoi partitions. Building off the foundational work by Lloyd in [3], [9], several variants of Lloyd’s algorithm have been proposed as control algorithms for such mobile sensor networks [1], [4], [5], [13]. While many of these algorithms guarantee convergence to a stable allocation, the efficiency of this allocation remains uncharacterized.

The goal of this paper is to characterize how informational restrictions impact achievable efficiency guarantees in multiagent systems. Throughout, we focus on a general class of networked resource allocation problems where the goal is to allocate a collection of agents across a mission space to optimize a given submodular system-level objective function. Within this class of resource allocation problems, in Theorem 3.1 we derive a fundamental relationship between a measure of informational redundancy in the agents’ control laws and the achievable efficiency guarantees associated with the emergent collective behavior. A consequence of this characterization is that many of the algorithms studied in the existing literature inherit extremely poor worst-case efficiency guarantees as a result of their stringent informational restrictions. For example, in general it is impossible to guarantee that the performance associated with $n$ sensors will be better than the performance associated with just a single sensor when applying Lloyd’s algorithm to the aforementioned sensor coverage problem. It is important to emphasize that this is not a result of Lloyd’s algorithm being a poor design choice; rather, these efficiency guarantees are a direct result of the stringent informational restrictions on the agents’ control laws.

The paper proceeds by modeling the interactions of the agents as a strategic-form game where the agents’ objective functions are required to satisfy two fairly general conditions that are well-studied in the context of network resource allocations [11], [19] (see Section II). Here, game theory provides a general mathematical framework for analyzing distributed systems where individual decision-making entities respond independently in response to incomplete information. Further, the rest points associated with a given networked control system can often be interpreted as “equilibria” of an underlying strategic-form game. Within this
model, we introduce a refinement of the well-studied solution concept of Nash equilibrium (see Definition 2.1) which allows for variations in the agents’ informational capabilities. Finally, in Theorem 3.1 we show how a given measure of informational dependence directly translates to both lower and upper bounds on the efficiency of such equilibria. Here, the take away point is that a system-designer inherits extremely poor system-level efficiency guarantees when agents have limited information regarding the mission space and the behavior of other agents.

The last part of this paper focuses on the question of whether limited system-level information can be shared between the agents to rectify this efficiency loss. Focusing on a simple class of sensor coverage problems, in Section IV we provide an algorithm that improves the efficiency guarantees from 1/n to 1/2 by allowing the agents to share minimal degrees of information. Further, the presented analysis provides insight as to the root of this inefficiency, which can be interpreted as disparity in the agents’ payoffs, hence suggesting that such an algorithm has the ability to be extended to alternative domains as well.

II. MODEL

We consider the problem of distributed resource allocation where there exists a set of agents \( N = \{1, 2, \ldots, n\} \) that are to be allocated to a given mission space \( C \). The allocation of the agents is represented by the tuple \( x = (x_1, \ldots, x_n) \) where \( x_i \) denotes the location of agent \( i \), i.e., \( x_i \in C \). We assume that the set of possible locations for each agent is \( X_i = C \cup \{\emptyset\} \) where \( \emptyset \) represents the decision where the agent is inactive. The goal of this problem is to allocate the agents over the mission space to optimize \((\max, \geq)\) a given global objective of the form \( G : X \to \mathbb{R^+} \) where \( X = \prod_{i \in N} X_i \) represents the set of possible joint allocations. Here, we focus on global objectives that are both \textit{increasing} and \textit{submodular}. Increasing implies that for any allocation \( x \in X \) and agent set \( S \subseteq N \), we have \( G(x_S) \geq G(x_{S \setminus \{i\}}) \) where \( x_S = \{x_i\}_{i \in S}. \)

The objective \( G \) is submodular if for any allocation \( x \in X \), agent sets \( S \subseteq T \subseteq N \), and agent \( i \in S \),
\[
G(x_S) - G(x_{S \setminus \{i\}}) \geq G(x_T) - G(x_{T \setminus \{i\}}). \tag{1}
\]
Submodularity corresponds to a notion of decreasing marginal returns which is very common in a variety of engineering applications [7], [19]. We denote the optimal allocation as \( x^* \in \arg \max_{x \in X} G(x) \).

The goal of this paper is to understand how informational restrictions to the agents impact the efficiency of the stable solutions associated with distributed control algorithms where the individual agents make decisions independently in response to incomplete information. To that end, we model the interactions of the agents as a strategic-form game with an agent set \( N \), where each agent \( i \in N \) has an action set \( X_i \) and a utility function \( U_i : X \to \mathbb{R} \). Rather than study a specific form of utility functions, here we focus on utility functions that satisfy two general properties set forth in [19]. First, for any allocation \( x \in X \), each agent’s utility is greater than or equal to the agent’s marginal contribution to the system-level objective, i.e.,
\[
U_i(x) \geq G(x) - G(x_{-i}), \tag{2}
\]
where \( x_{-i} \) is shorthand notation for \( x_{N \setminus \{i\}} \). Second, the sum of the agents’ utilities is less than or equal to the system-level objective, i.e.,
\[
\sum_{i \in N} U_i(x) \leq G(x). \tag{3}
\]

Utility functions satisfying these two properties have been extensively studied in the game-theoretic literature [11], [17], [19]. Further, as we will see in the ensuing section, there are several classes of utility functions studied in the literature that do in fact satisfy these properties. Lastly, we refer to the tuple \( \{N, \{X_i\}_{i \in N}, \{U_i\}_{i \in N}, G\} \) as a resource allocation game.

A. An Illustrative Example

Consider the well-studied sensor coverage problem where there exists a finite set of sensors (or agents) \( N = \{1, 2, \ldots, n\} \) to be allocated over a mission space \( C \) which is characterized by a closed interval, i.e., \( C \subseteq \mathbb{R} \), and a weighting function \( v : C \to \mathbb{R} \) which identifies the relative importance of regions in the interval [13]. Each agent can be positioned at any point in the mission space, i.e., \( X_i = C \), and so the set of possible allocations is \( X = \prod_{i \in N} X_i \). The global objective \( G \) is submodular and of the form
\[
G(x) = \int_C \max_{c \in N} v(c)g(|x_i - c|) dc \tag{4}
\]
where \( g : \mathbb{R^+} \to \mathbb{R^+} \) is a non-increasing function.

One of the most commonly studied agent utility functions in the sensor coverage literature is
\[
U_i(x) = \int_{V_i(x)} v(c)g(|x_i - c|) dc \tag{5}
\]
where \( V_i(x) = \{c \in C : |x_i - c| \leq \min_{j \neq i} |x_j - c| \} \) denotes the Voronoi partition of agent \( i \) given the allocation \( x \). Observe that the utility functions in (5) satisfy both properties (2) and (3), where (3) is satisfied with equality. On the other hand, one of the most commonly studied agent utility functions in the game-theoretic literature, termed \textit{marginal contribution utility} [10], is of the form
\[
U_i(x) = \int_{V_i(x)} v(c) \left( g(|x_i - c|) - \max_{j \neq i} g(|x_j - c|) \right) dc \tag{6}
\]
as each agent’s utility is equal to their marginal contribution to the global objective. Once again, observe that the utility

Footnote 1: More formally, given an allocation \( x \), the component \( x_S \) can be viewed as an allocation \( x_i' \in X \) where \( x_i' = x_i \) for all \( i \in S \) and \( x_i' = \emptyset \) for all \( i \notin S \). However, we will merely express this allocation as \( x_S \) and omit explicitly highlighting the agents in \( N \setminus S \) for brevity.
functions in (6) satisfy both properties (2) and (3), where now (2) is satisfied with equality.\(^3\)

### B. Local Nash Equilibria

Consider the following generalization of pure Nash equilibria that captures variations in information to the agents. Given a particular allocation vector, we model the information available to each agent \(i \in N\) by an information set \(F_i(x) \subseteq C\) where \(F_i(x)\) identifies all possible decisions for which agent \(i\) can evaluate his utility given that the decision of all other agents remains \(x_{-i}\). More formally, each agent is capable of evaluating the agent’s utility for all allocations of the form \(\{(x'_i, x_{-i}) : x'_i \in F_i(x)\}\) given an allocation \(x\).

**Definition 2.1 (Local Nash Equilibrium):** An allocation \(x\) is a local Nash equilibrium with respect to the information sets \(\{F_i\}_{i \in N}\) if for each agent \(i \in N\) we have

\[
U_i(x_i, x_{-i}) \geq U_i(x'_i, x_{-i}), \quad \forall x'_i \in F_i(x).
\]  

(7)

We note the following special cases of Definition 2.1. First, when \(F_i(x) = X_i\) for all agents \(i \in N\), Definition 2.1 takes on the form of a Nash equilibrium. Alternatively, when \(F_i(x) = \{x'_i \in X_i : ||x'_i - x_i|| \leq \epsilon\}\) for small \(\epsilon > 0\), Definition 2.1 corresponds to the rest points of a distributed gradient ascent process where the individual agents respond to their local utility functions \(\{U_i\}_{i \in N}\).

### III. Efficiency Guarantees

The focus of this paper will be on understanding how the structure of the information sets \(\{F_i\}_{i \in N}\) impacts the efficiency of the resulting local Nash equilibria with respect to the system level objective. We begin by defining the information index associated with the information sets \(\{F_i\}_{i \in N}\) as

\[
f = \min_{x \in X} \min_{y \in G} \{i \in N : y \in F_i(x)\}.
\]  

(8)

The information index highlights a worst-case measure of redundancy in information across the mission space. Figure 1 provides some illustrative examples to shed light into the definition of the information index in (8). It turns out that the information index is intimately related to the efficiency of the resulting equilibria as shown in the following theorem.

**Theorem 3.1:** Consider the class of resource allocation games introduced in Section II that satisfies the following three properties:

(i) The global objective \(G\) is submodular.

(ii) For any allocation \(x \in X\) and agent \(i \in N\), the agent’s utility function satisfies \(U_i(x) \geq G(x) - G(x_{-i})\).

(iii) For any allocation \(x \in X\), the sum of agents’ utility functions satisfies \(\sum_{i \in N} U_i(x) \leq G(x)\).

If the information index associated with the information sets \(\{F_i\}_{i \in N}\) is \(f \geq 1\) and a local Nash equilibria exists, then the efficiency of any local Nash equilibrium is bounded below by

\[
G(x) \geq \left(\frac{f}{n + f}\right) G(x^*)
\]  

(9)

where \(x\) and \(x^*\) represent a local Nash equilibrium and optimal allocation respectively.

Theorem 3.1 demonstrates that the structure of the agents’ information sets plays a crucial role in determining the efficacy of the resulting local Nash equilibrium. While the above theorem only provides a lower bound on the efficiency associated with any local Nash equilibrium, in Example 3.1 we provide examples in the context of sensor coverage that prove this bounds is tight asymptotically in \(n\).

The following special cases highlight how informational restrictions impact efficiency guarantees.

**Full information:** In the case when \(F_i(x) = X_i\) for all agents \(i \in N\), we have an equivalence between local Nash equilibria and Nash equilibrium. Accordingly, the information index for this case is \(f = n\) and the bound set forth in Theorem 3.1 becomes \(G(x) \geq \left(\frac{1}{2}\right) G(x^*)\). Hence, any Nash equilibrium is within 50\% of optimal. This special case recaptures the results regarding valid utility games presented in [19].

**Localized information:** Alternatively, consider the situation where the agents’ information sets \(\{F_i\}_{i \in N}\) satisfy \(\cup_{i \in N} F_i(x) = C\) and \(F_i(x) \cap F_j(x) = \emptyset\) for any allocation \(x \in X\) and any agent \(i \neq j\). In this case, the information index is \(f = 1\) and the bound set forth in Theorem 3.1 becomes

\[
G(x) \geq \left(\frac{1}{n + 1}\right) G(x^*)\].

This lower bound suggests that under such information constraints, which are reminiscent of conditions associated with the rest points of a distributed gradient ascent process where the individual agents respond to their local utility functions \(\{U_i\}_{i \in N}\), it is not possible to guarantee that a system comprising of \(n\) agents can perform...
better than a system comprising of just a single agent.\footnote{In the context of the aforementioned sensor coverage problem, it is straightforward to verify that the underlying information index associated with Lloyd’s algorithm (and its variants) satisfies $f \leq 1$ as agents only use information regarding their Voronoi partition when formulating a decision. Consequently, it is straightforward to construct instances, similar in spirit to Example 3.1, where the rest points associated with Lloyd’s algorithms satisfy $G(x) = \frac{1}{2}G(x^*)$. This even holds true for closed and convex mission spaces $C \subseteq \mathbb{R}^2$ with uniform weighting.}

We now provide the proof of Theorem 3.1. The proof can be viewed as a generalization of the proof in [19] where the analysis is extended to include the introduced information sets.

\textbf{Proof:} With a slight abuse of notation, we extend the definition of the global objective $G$ to include all numbers of agents ranging from $\{0, 1, \ldots, 2n\}$. Let $x^*$ and $x \in X$ represent an optimal and equilibrium allocation, each consisting of $n$ agents, respectively. We begin by introducing the following function which captures a notion of marginal gains: for any set $C \subseteq C$ and allocation $x \in X$, let

$$M(x, C) = \max_{y \in C} G(y, x) - G(x)$$

(10)

capture the maximum gain with regards to the system-level objective that can be attributed to adding a new agent to the allocation $x$ in the space $C$, i.e., going from $n$ to $n+1$ agents by placing a new agent at a location $y \in C$. Accordingly, if $x$ represents an equilibrium then we have that for each agent $i \in N$

$$U_i(x) = \max_{x_i' \in F_i(x)} U_i(x'_i, x_{-i}),$$

(11)

$$\geq \max_{x_i' \in F_i(x)} G(x'_i, x_{-i}) - G(x_{-i}),$$

(12)

$$\geq \max_{x_i' \in F_i(x)} G(x'_i, x) - G(x),$$

(13)

$$= M(x, F_i(x)),$$

(14)

where (11) follows from the definition of local Nash equilibrium, (12) follows from (2), and (13) follows from the submodularity of $G$. Therefore, we have that

$$G(x) \geq \sum_{i \in N} U_i(x) \geq \sum_{i \in N} M(x, F_i(x)).$$

(15)

We begin by deriving a lower bound on the global objective associated with a local Nash equilibrium $x$. First, without loss of generality, renumber the agents such that $U_1(x) \geq U_2(x) \geq \ldots \geq U_n(x)$. The value associated with the information index provides insight into the value associated with the utility of each agent $i \in N$. To that end, let

$$M^*(x) = \max_{i \in N} M(x, F_i(x)).$$

(16)

Since $x$ is an equilibrium allocation, an information index $f$ implies that for all agents $i \in \{1, \ldots, f\}$

$$U_i(x) \geq M^*(x).$$

To see this, let $y \in C$ be the location such that $M^*(x) = G(y, x) - G(x)$, by the definition of the information index, we know that the cardinality of the set $S_y = \{i \in N : y \in F_i(x)\}$ must be at least $f$, i.e., $|S_y| \geq f$. Accordingly, we know that for any agent $i \in S_y$

$$U_i(x) \geq U_i(y, x_{-i}),$$

(17)

$$\geq G(y, x_{-i}) - G(x_{-i}),$$

(18)

$$\geq G(y, x) - G(x),$$

(19)

$$= M^*(x).$$

(20)

Therefore, we know that for any equilibrium with information index $f$ we have that

$$G(x) \geq \sum_{i \in N} U_i(x) \geq f M^*(x).$$

(21)

We now focus on a derivation of an upper bound on the global objective associated with the optimal allocation. This upper bound takes on the form

$$G(x^*) \leq G(x^*, x),$$

(22)

$$= G(x) + \sum_{i \in N} \left( G(x^*_i, x) - G(x^*_{i-1}, x) \right),$$

(23)

$$\leq G(x) + \sum_{i \in N} \left( G(x^*_i, x) - G(x) \right),$$

(24)

where the notation $x^*_{i-1} = x, \ldots, x_i$ captures the first $i$ terms of the allocation $x$ (22) follows from the fact that $G$ is increasing in the agent set, (23) follows from the submodularity of $G$, and (24) follows from the definition of $M^*(x)$. Lastly, combining (21) and (24) gives us the desired result

$$\frac{G(x)}{G(x^*)} \geq \frac{G(x)}{G(x) + n M^*(x)},$$

(25)

$$\geq \frac{f M^*(x)}{f M^*(x) + n M^*(x)},$$

(26)

$$= \frac{f}{f + n}.$$

This completes the proof. \hfill \blacksquare

The efficiency bound presented in Theorem 3.1 provides a lower bound associated with the global objective evaluated at a local Nash equilibrium. This bound highlights a fundamen-
tal relationship between the introduced information index and the resulting efficiency guarantees of local Nash equilibria. The following example demonstrates that this bound is not conservative as scenarios can be derived that have local Nash equilibria with an efficiency close to the prescribed bound.

\textbf{Example 3.1:} Consider a finite set of sensors $N = \{1, 2, \ldots, n\}$ to be allocated on a discretized linear mission space $C = \{1, \ldots, m\}$ where each region $c \in C$ is associated with a value $v_c \in \mathbb{R}^+$ that defines the relative worth of that region as illustrated in Figure 2. Here, we focus on a global objective of the form

$$G(x) = \sum_{c \in C} \max_{i \in N} v_c \cdot g(|x_i - c|),$$

(27)

where $g : \mathbb{R}^+ \to \mathbb{R}^+$ is non-increasing and satisfies $\lim_{z \to \infty} g(z) = 0$. The regions labeled with “...” represent a large number of repeated regions with value 0 such that for any distinct $c, c' \in C$ where $v_c, v_{c'} > 0$, we have that $g(c, c') \approx 0$. For brevity, we will focus on the situation where the sensors’ utility functions are of the form (6). When
all agents are on sectors with non-zero values, each agent’s utility function is of the form

\[ U_i(x) \approx \begin{cases} 0 & \text{if } x_i = x_j \text{ for some } j \neq i, \\ v_{x_i} & \text{otherwise.} \end{cases} \]  

(26)

Furthermore, note that if all sensors are at distinct locations with values strictly greater than 0, then \( G(x) = \sum_{i \in N} U_i(x) \).

Figure 2 highlights a specific example with 5 agents and an information index \( f = 2 \). The allocation given in Figure 2 represents a local Nash equilibrium and the global objective is \( 2 + 3\epsilon \). The optimal allocation is for all agents to be allocated at distinct regions with a value of 1 and the global objective is 5. Hence, as \( \epsilon \to 0 \), we attain \( G(x) = (2/5)G(x^*) \). By extending this example to \( n \) agents in the logical manner, we would obtain \( G(x) \approx (f/n)G(x^*) \) which approaches the bound set forth in Theorem 3.1. Finally, note that similar examples could easily be construed to illustrate the same phenomena for alternative utility functions.

The worst-case situation highlighted in Figure 2 exhibits a large disparity in the payoffs of the agents, i.e., 1 versus \( \epsilon \). In line with this theme, we point out the following observations associated with Theorem 3.1 that encapsulate this phenomena.

- **Observation #1**: We begin by extending the definition of \( M(x, C) \) in (10) as follows: for any \( k \in \{1, 2, \ldots, n\} \), let

\[ M_k(x, C) = \max_{y_1, \ldots, y_k \in C} G(y_1, \ldots, y_k, x) - G(x). \]  

(27)

Here, \( M_k(x, C) \) captures the potential benefit associated with adding \( k \) agents to the set \( C \) given the allocation \( x \). Note that \( M_1(x, C) = M(x, C) \) as in (10). An improved bound to (9) is

\[ \frac{G(x)}{G(x^*)} \geq \frac{G(x)}{G(x) + \max_{y_1, \ldots, y_n \in C} \sum_{y_1, \ldots, y_n \geq 0} M_{y_k}(x, F_k(x))} \]  

(28)

which follows immediately from the following upper bound of the optimal allocation

\[ G(x^*) \leq G(x^*, x) \leq G(x) + \max_{y_1, \ldots, y_n \in C} \sum_{y_1, \ldots, y_n \geq 0} M_{y_k}(x, F_k(x)). \]

Here, (28) can be viewed as an ex-post measure of the efficiency associated with a given allocation that arises from a distributed process.

- **Observation #2**: Since the agents utility functions satisfy (2), we know that \( U_i(x) \geq M_k(x, F_i(x)) \) for any \( k \geq 1 \). Hence, as we saw in Figure 2, disparity in agents’ payoffs at a local Nash equilibrium can lead to inefficiencies. Note that if there is not a large disparity in the agents’ utilities at a local Nash equilibrium, i.e., \( U_i(x) \geq M^*(x) \) for all agents \( i \in N \), then the bound in (9) becomes \( G(x) \geq (1/2)G(x^*) \).

IV. IMPROVING EFFICIENCY GUARANTEES THROUGH COMMUNICATION

The previous section demonstrates that when the agents have full information regarding the mission space, i.e., \( F_i(x) = C \) for all allocations \( x \in X \), the efficiency of a local Nash equilibrium is within 50% of optimal. However, when agents have limited knowledge regarding the mission space, i.e., \( F_i(x) \neq C \), then the efficiency associated with a local Nash equilibrium can be quite poor. The root of this inefficiency is the disparity in the utility of the agents at a local Nash equilibrium. In this section, we provide an algorithm that guarantees that the efficiency of the limiting behavior will be within 50% of optimal while adhering to local information constraints. The key component of this algorithm that enables overcoming the bounds derived in the previous section is adding a layer of communication on top of the algorithm which provides the agents with limited additional knowledge regarding the mission space. Intuitively, such information is used to destabilize local Nash equilibria with a high disparity in agents’ payoffs.

A. Preliminaries

In this section we focus purely on the class of sensor coverage problems over a discretized linear mission space as introduced in Example 3.1. The mission space \( C \) consists of a sequence of regions \( C = \{0, \ldots, m\} \). Without loss of generalities, we adopt the convention that the indices assigned to the agents will correspond to their allocation in the mission space where the leftmost agent will be agent 1 while the rightmost agent will be agent \( n \). That is, for any allocation \( x \in X \), we have \( x_j \leq x_{j+1} \) for all agents \( j \in \{1, \ldots, n-1\} \).

The following algorithm will associate with each agent a location coupled with a partition of the mission space. More formally, we represent an admissible partition associated with a given allocation \( x \in X \) as \( \mathcal{P}(x) = \{\mathcal{P}_1(x), \mathcal{P}_2(x), \ldots, \mathcal{P}_n(x)\} \) where \( x_i \in \mathcal{P}_i(x) \) for all agents \( i \in N \), e.g., Voronoi partition. We will typically denote a partition \( \mathcal{P}(x) \) as merely \( \mathcal{P} = \{\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n\} \) when the underlying allocation is clear. Given an allocation \( x \in X \) and a partition \( \mathcal{P} \), we define the utility of agent \( i \in N \) as

\[ U_i(x, \mathcal{P}) = \sum_{c \in \mathcal{P}_i} v_c \cdot g(|x_i - c|). \]  

(29)

Accordingly, for any allocation \( x \) and partition \( \mathcal{P} \), we have that

\[ G(x) \geq \sum_{i \in N} U_i(x, \mathcal{P}) \]
where we obtain equality when \( P \) represents a Voronoi partition associated with the allocation \( x \).

As the previous section suggests, a key part associated with the forthcoming algorithm will be focusing on the "smallest" agent. With that in mind, define

\[
\begin{align*}
    i_{\text{min}}(x, P) &\in \arg \min_{i \in N} U_i(x, P), \\
    U_{\text{min}}(x, P) &= \min_{i \in N} U_i(x, P).
\end{align*}
\]

We begin by introducing some notation which will be convenient for the forthcoming analysis. First, we extend the definition of the global objective to only account for sections of the mission space \( C \subseteq C \) as

\[
G(x; C) = \sum_{x \in C} \max \{ v_c \cdot g(\|x_i - c\|) \}. \tag{32}
\]

Next, we introduce a mild variant of the definition of \( M_k \) in (27) as

\[
M_k(x; C) = \max_{y_1, \ldots, y_k \in C} G(y_1, \ldots, y_k, x; C) - G(x; C) \tag{33}
\]

to explicitly capture marginal gain in a section of the mission space. Next, we define the optimal assignments

\[
B_k(x; C) = \arg \max_{y_1, \ldots, y_k \in C} G(y_1, \ldots, y_k, x; C) - G(x; C) \tag{34}
\]

where we adopt the convention that a tuple \( (y_1, \ldots, y_k) \) in \( B_k(x; C) \) satisfies \( y_1 \leq y_2 \leq \cdots \leq y_k \). Lastly, we define \( P^* \) as the Voronoi partitions associated with the allocation \( x \) on the mission space \( C \subseteq C \). The lack of uniqueness of the Voronoi partitions derives from the finiteness of the mission space.

B. Algorithm Description

We begin with an informal description of the algorithm. At each time instance, a pair of neighboring agents \( p = \{i, i+1\} \) is given the opportunity to revise their collective decision. The total partition of the pair is \( P_p = P_i \cup P_{i+1} \) and so this decision involves specifying locations within this partition and how to divide the partition. If the objective function for the pair \( p \) satisfies \( M_3(0; P_p) - M_2(0; P_p) \leq U_{\text{min}}(x, P) \), this means that there is no advantage to the minimum agent relocating to the section \( P_p \); hence, the agents optimally divide their partition among the two of them. Alternatively, if \( M_3(0; P_p) - M_2(0; P_p) > U_{\text{min}}(x, P) \), this means that there is an advantage to the minimum agent relocating to the section \( P_p \); hence, the agents optimally divide their partition for three agents (as if the minimum agent was in their partition), and effectively leave the section of their partition closest to the minimum agent open. In essence, this algorithm seeks to establish a process by which valuable regions are "passed" to the minimum agent.

We will now specify the details of our algorithm. This algorithm produces a sequence of allocation partition pairs \( \{x(k), P(k)\}_{k=1,2,\ldots} \) that converges to an allocation partition pair \( \{x^*, P^*\} \) which is at least 50% efficient. We refer to \( x(t) \) as the allocation and \( P(t) \) as the partition at time \( t \) where \( t \in \{0, 1, \ldots\} \). The allocation partition pair at time \( t \), i.e., \( \{x(t), P(t)\} \), is chosen as follows: For simplicity, let \( \{x, P\} = \{x(t-1), P(t-1)\} \).

Step 0: Let \( i_{\text{min}}(x, P) \) be the identify of the agent with minimal utility at time \( t-1 \).

Step 1: Select one pair of agents \( p = \{i, i+1\} \subseteq N \) uniformly at random. Define \( P_p = P_i \cup P_{i+1} \). Without loss of generalities, suppose \( i_{\text{min}}(x, P) \geq i \).

Step 2a: Suppose \( i_{\text{min}}(x, P) \in p \) or \( M_2(0; P_p) - M_3(0; P_p) \leq U_{\text{min}}(x, P) \). If \( (x_i, x_{i+1}) \in B_2(0; P_p) \), the location of agents \( i \) and \( i+1 \) are unchanged, i.e., \( x_i(t) = x_i \) and \( x_{i+1}(t) = x_{i+1} \). Otherwise, let \( (x_i(t), x_{i+1}(t)) \in B_2(0; P_p) \) be any joint allocation. The partition of agents \( i \) and \( i+1 \) are then set as

\[
\begin{align*}
P_i(t) &= P_i^*(x_i(t), x_{i+1}(t); P_p), \\
P_{i+1}(t) &= P_{i+1}^*(x_i(t), x_{i+1}(t); P_p),
\end{align*}
\]

where \( P_i^*(\cdot) \) denotes the Voronoi partition associated with the \( i \)-th agent.

Step 2b: Otherwise, suppose \( M_2(0; P_p) - M_3(0; P_p) > U_{\text{min}}(x, P) \) and consider any triple \( (y_1, y_2, y_3) \in B_3(0; P_p) \). Since \( i_{\text{min}}(x, P) > i+1 \), then the new location and partition of agents \( i \) and \( i+1 \) are \( x_i(t) = y_1 \) and \( x_{i+1}(t) = y_2 \), and

\[
\begin{align*}
P_i(t) &= P_i^*(y_1, y_2, y_3; P_p), \\
P_{i+1}(t) &= P_{i+1}^*(y_1, y_2, y_3; P_p) \cup P_3^*(y_1, y_2, y_3; P_p).
\end{align*}
\]

Step 3: The location and partition of any other agent \( j \not\in \{i, i+1\} \) is unchanged, i.e., \( x_j(t) = x_j \) and \( P_j(t) = P_j \).

Step 4: If \( i_{\text{min}}(x, P) \in \arg \min_{i \in N} U_i(x(t), P(t)) \), then the agent with minimal utility remains unchanged. Otherwise, pick an agent according to any underlying process.

The following theorem characterizes the asymptotic guarantees associated with this algorithm. Note that the informational demands associated with this algorithm give rise to an information index that satisfies \( f \leq 2 \).

Theorem 4.1: Consider the class of sensor coverage problems over a discretized linear mission space as introduced in Section IV-A. The algorithm described in Section IV-B converges almost surely to an allocation partition pair \( (x^*, P^*) \) that satisfies \( P = P^*(x^*) \) and \( G(x)/G(x^*) \geq 0.5 \) where \( x^* \) is the optimal allocation.

We begin by introducing the following set definitions which identify agents whose partitions could benefit the most from adding an additional agent, i.e.,

\[
\begin{align*}
S(x, P) &= \arg \max_{i \in N} M_1(x_i; P_i), \\
V(x, P) &= \max_{i \in N} M_1(x_i; P_i).
\end{align*}
\]

Further, we divide the set of admissible allocation partition pairs, which we will broadly refer to as states, into the following four classes:

- Let \( Z_1 \) represent the set of states \( (x, P) \) such that \( V(x, P) > U_{\text{min}}(x, P) \).
- Let \( Z_2 \) represent the set of states \( (x, P) \) such that \( V(x, P) \leq U_{\text{min}}(x, P) \).
- Let \( Z_3 \subseteq Z_2 \) represent the set of states \( (x, P) \) such that for any pair of neighboring agents \( p = \{i, i+1\} \subseteq N \), we have that \( M_3(0; P_p) - M_2(0; P_p) \leq U_{\text{min}}(x, P) \).
- Let \( Z_4 \subseteq Z_3 \) represent the set of states \( (x, P) \) such that for any pair of neighboring agents \( p = \{i, i+1\} \subseteq N \), we have that \( U_i(x, P) + U_{i+1}(x, P) = M_2(0; P_p \cup P_{i+1}) \).
First, note that any state \((x, P) \in Z_4\) is a rest point of our algorithm. Furthermore, note that the optimal allocation partition pair \((x^*, P^*(x^*)) \in Z_4\), hence \(Z_4\) is non-empty. The proof of this theorem will consist of three claims that focus on the state classes \(Z_1, Z_2, Z_3,\) and \(Z_4\). The central theme of these claims will be showing that our algorithm guarantees that any state \((x, P)\) will transition to a new state \((x', P')\) in \(Z_4\) in a finite number of iterations with positive probability bounded away from 0. The following Lyapunov function is central to the arguments used for each claim: for any state \((x, P)\)

\[
\phi(x, P) = [V(x, P) - U_{\min}(x, P)]_+ + \sum_{i \in N} U_i(x, P)
\]  

(35)

Roughly speaking, this Lyapunov function captures the maximum global objective that could be attained by moving the agent with the minimum utility to another agent's partition.

The first claim focuses on the transition \(Z_1 \rightarrow Z_2\).

Claim 4.1: If \((x, P) \in Z_1\), then the above algorithm will produce a sequence of states (with positive probability bounded away from 0) that results in a new state \((x', P') \in Z_2\) with the property that \(\phi(x', P') \geq \phi(x, P)\).

Proof: We will prove this claim by showing that for any state \((x, P) \in Z_1\), there exists a resulting state trajectory, selected with positive probability strictly greater than 0, such that the final state \((x', P')\) will satisfy either (i) \((x', P') \in Z_2\) and \(\phi(x', P') \geq \phi(x, P)\) or (ii) \((x', P') \in Z_1\) and \(\phi(x', P') > \phi(x, P)\). Note that due to the finiteness of the space of admissible allocation partition pairs, property (ii) can be repeated at most a finite period of times before property (i) is satisfied which completes the proof.

Let \((x, P) \in Z_1\) and consider any agent \(j \in S(x, P)\) and without loss of generality, suppose \(j \leq i_{\min}(x, P)\). Further, suppose the pair \(p = (j, j + 1)\) is selected to revise their strategy by our algorithm which happens with probability \(1/(n-1)\). For notational simplicity, let \(V^* = V(x, P) = M_1(x; j; P_j)\) and \(P_p = P_j \cup P_{j+1}\). The pair \(p\) fits into one of the following two categories.

- Case #1: Suppose \(i_{\min}(x, P) \in p\) or \(M_3(\emptyset; P) - M_2(\emptyset; P) \leq U_{\min}(x, P)\). Let \(x'\) be the new allocation and \(P'\) represent the new partition selected according to Step 2 and Step 3 of the above algorithm. For comparing the allocation partition pair \((x, P)\) to \((x', P')\), we adopt the shorthand notation that \(U'_j = U_j(x', P')\) and \(U_j(x, P)\). Further, we extend this shorthand notation in the same manner to all terms \(V, U_{\min}, i_{\min}\) and \(\phi\). According to the defined algorithm, it is straightforward to show that

\[
U'_j + U'_{j+1} \geq U_j + U_{j+1} + V - U_{\min}.
\]

(36)

When \(i_{\min} \in p\), (36) follows from the fact that \(U_{\min} = U_k\) for some \(k \in \{j, j + 1\}\). Alternatively, when \(i_{\min} /\in p\), (36) follows straight from the fact that \(M_3(\emptyset; P) - M_2(\emptyset; P) \leq U_{\min}\). Accordingly, we have

\[
\phi' = \sum_{i \in N} U'_i + [V' - U_{\min}]_+ + \sum_{i \in N} U_i + [V - U_{\min}]_+ + [V' - U_{\min}]_+ \geq \phi.
\]

(37)

where (37) follows from (36) and the fact that \(U_i = U'_i\) for all \(i \notin p\). Consequently, we have that \(\phi' \geq \phi\). Note that if \(\phi' = \phi\), then we necessarily have \(V' \leq U_{\min}\), which implies that \((x', P') \in Z_2\). Hence, the new allocation partition pair \((x', P')\) satisfies either \((x', P') \in Z_2\) or \(\phi(x', P') > \phi(x, P)\).

- Case #2: Suppose \(i_{\min} /\in p\) and \(M_3(\emptyset; P_p) - M_2(\emptyset; P_p) > U_{\min}\). Let \(x'\) be the new allocation and \(P'\) represent the new partition selected according to Step 2b and Step 3 of the above algorithm. Since \(M_3(\emptyset; P_p) - M_2(\emptyset; P_p) > U_{\min}\), we have that \(U'_{j} > U_{\min}, U_{j+1} > U_{\min}\) and \(M_1(x_{j+1}; P'_{j+1}) > U_{\min}\), which ensures that \(U_{\min} = U_{\min}\) since \(i_{\min} /\in p\). Accordingly, we have

\[
\phi' = \begin{cases} 
\sum_{i \in N \setminus p} U'_i + \sum_{i \in p} U'_i + [V' - U_{\min}]_+ = \sum_{i \in N \setminus p} U_i + \sum_{i \in p} U'_i + [V' - U_{\min}]_+ \\
\geq \sum_{i \in N \setminus p} U_i + \sum_{i \in p} U'_i + M^1(x_{j+1}, P'_{j+1}) - U_{\min} \\
\geq \sum_{i \in N} U_i + [V - U_{\min}]_+ \end{cases}
\]

(38)

where (38) follows from the fact that

\[
U'_j + U'_{j+1} + M_1(x_{j+1}; P'_{j+1}) \geq U_j + U_{j+1} + V.
\]

Consequently, we have that \(\phi' \geq \phi\). Note that if \(V' \neq M_1(x_{j+1}; P'_{j+1})\), then \(\phi' > \phi\) and we are done. Otherwise, if \(M_1(x_{j+1}; P'_{j+1}) = V'\), then we know that agent \(j + 1 \in B(x'; P')\). Hence, suppose the next iteration that the pair \(p' = (j + 1, j + 2)\) is selected which happens with probability \(1/(n-1)\) and the new allocation partition pair is \((x'', P'')\). If the pair \(p'\) fits into Case #1, then we are done. Otherwise, if pair \(p'\) fits into Case #2, then we know the ensuing allocation partition pair \((x'', P'')\) satisfies \(\phi(x'', P'') \geq \phi(x', P')\). If \(\phi(x'', P'') > \phi(x', P')\), then we are done. Otherwise, we can consider the pair \(p'' = (j + 2, j + 3)\) and repeat the process above. Note that each time we repeat this process, the new pair gets closer to \(i_{\min}\); hence, this process can be repeated at most \(|i_{\min} - j| \leq n\) before Case #1 occurs. This completes the proof.

The second claim focuses on the transition \(Z_2 \rightarrow Z_3\).

Claim 4.2: If \((x, P) \in Z_2\), then the above algorithm will produce a sequence of states (with positive probability bounded away from 0) that results in a new state \((x', P') \in Z_3\) with the property that \(\phi(x', P') \geq \phi(x, P)\).

Proof: Consider any state \((x, P) \in Z_2 \setminus Z_3\). Note that the value of the Lyapunov function associated with this state satisfies \(\phi = \sum_{i \in N} U_i\). Since \((x, P) \in Z_2 \setminus Z_3\), there exists a pair of neighboring agents \(p = (i, i + 1)\) such that \(M_3(\emptyset; P_p) - M_2(\emptyset; P_p) > U_{\min}\). Suppose the pair of agents \(p\) are selected to revise their allocation and partition. The pair \(p\) fits into one of the following two categories.

- Case #1: Suppose \(i_{\min} \in p\). Let \(x'\) be the new allocation and \(P'\) represent the new partition selected according to Step 2a and Step 3 of the above algorithm. Accordingly,
we have that
\[ U_j' + U_{j+1}' \geq U_j + U_{j+1} + V - U_{\min} \]
\[ > U_j + U_{j+1} \]
following the same logic as the proof of Case #1 in Claim 4.1. Accordingly, we have \( \phi' = \sum_{i \in N} U_i' > \sum_{i \in N} U_i = \phi \).

- **Case #2:** Suppose \( i_{\min} \notin p \). Let \( x' \) be the new allocation and \( \mathcal{P}' \) represent the new partition selected according to Step 2a and Step 3 of the algorithm. Since \( \mathcal{M}_3(\emptyset; \mathcal{P}_p) - \mathcal{M}_2(\emptyset; \mathcal{P}_p) > U_{\min} \) and \( i_{\min} \notin p \), we know that
\[ U_j' + U_{j+1}' + |V' - U_{\min}| > U_j + U_{j+1}, \quad (39) \]
which implies that \( \phi' > \phi \). However, note that \( (x', \mathcal{P}') \in Z_1 \). Nonetheless, we know from Claim 4.1 that a new state \( (x'', \mathcal{P}'') \in Z_2 \) will be reached in a finite number of iterations with probability bounded away from 0 with the property that
\[ \phi(x'', \mathcal{P}'') > \phi(x, \mathcal{P}). \]

Note that this argument can be repeated at most a finite number of times due to the finiteness of the allocation partition pairs. This completes the proof.

The third claim focuses on the transition \( Z_3 \rightarrow Z_4 \).

**Claim 4.3:** If \((x, \mathcal{P}) \in Z_3\), then the above algorithm will produce a sequence of states (with positive probability bounded away from 0) that results in a new state \((x', \mathcal{P}') \in Z_4\) with the property that \(\phi(x', \mathcal{P}') \geq \phi(x, \mathcal{P})\).

**Proof:** Consider any state \((x, \mathcal{P}) \in Z_3 \setminus Z_4\). Note that the value of the Lyapunov function associated with this state satisfies \( \phi = \sum_{i \in N} U_i \). Since \((x, \mathcal{P}) \in Z_3 \setminus Z_4\), there exists a pair of neighboring agent \( p = \{i, i + 1\} \) such that
\[ \mathcal{M}_3(\emptyset; \mathcal{P}_p) - \mathcal{M}_2(\emptyset; \mathcal{P}_p) \leq U_{\min} \]
and either (i) \( U_i + U_{i+1} < \mathcal{M}_2(\emptyset; \mathcal{P}_p) \) or (ii) \( \{P_i, P_{i+1}\} \notin \mathcal{P}^*(x_i, x_{i+1}, \mathcal{P}_p) \). Suppose the pair of agents \( p \) are selected to revise their allocation and partition. Let \( x' \) be the new allocation and \( \mathcal{P}' \) represent the new partition selected according to Step 2a and Step 3 of the above algorithm. If (i) is true, then we have that
\[ U_j' + U_{j+1}' > U_j + U_{j+1} \]
which directly implies that \( \phi' > \phi \). Alternatively, if (ii) is true, then we have that \( \phi' > \phi \) which completes the proof.

**Proof of Theorem 4.1:** We now conclude the proof by showing that the efficiency associated with any allocation partition pair \((x, \mathcal{P}) \in Z_4\) satisfies \( G(x) \geq (1/2)G(x^*) \). First, note that if \((x, \mathcal{P}) \in Z_4\) then \( \mathcal{M}_3(\emptyset; \mathcal{P}_p) - \mathcal{M}_2(\emptyset; \mathcal{P}_p) \leq U_{\min} \) for every pair of neighboring agents \( p = \{i, i + 1\} \subset N \). This directly implies that \( \mathcal{M}_1(x; \mathcal{P}) \leq U_{\min} \). Using this fact gives us
\[ G(x) \leq G(x^*) \leq G(x, x^*) \]
\[ = G(x) + \sum_{i=1}^{n} G(x, x_{1:i}) - G(x, x_{1:i-1}) \]
\[ \leq G(x) + n \cdot U_{\min} \leq 2G(x). \]
Consequently, we have that \( G(x) \geq (1/2)G(x^*) \) which completes the proof. \( \square \)

V. CONCLUSION

The paper studies the impact of informational restriction on the efficiency guarantees associated with the emergent behavior in multiagent systems. The take away point of this paper is that many algorithms which adopt stringent information demands on the agents’ control policies, i.e., Lloyd’s algorithm, inherit extremely poor worst case efficiency guarantees. However, the last part of this paper demonstrates that one can overcome these guarantees by allowing the agents to communicate minimally with neighboring agents. Future work will seek to extend these results to broader problem domains.

**REFERENCES**


