Jensen type inequality for extremal universal integrals

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Abstract—The integrals based on non-additive measures, e.g. Choquet, Sugeno play important roles in several practical areas. Universal integral as generalization of Choquet and Sugeno integrals has been recently proposed. Since the Jensen inequality for Lebesgue integral has applications in many areas, in this paper, the corresponding inequality related to the extremal universal integral as generalization of Choquet, Shilkret and seminormed fuzzy integrals is observed.

Keywords. Jensen inequality, Pseudo-operations, Monotone measure, Universal integral.

II. Preliminary notions

In this paper we shall consider the following properties of functions ([8]).

Definition 1: A function \( f : [0, \infty] \to [0, \infty] \) is

i) convex, if
\[
f(\lambda x + (1 - \lambda) y) \leq \lambda f(x) + (1 - \lambda) f(y)
\]
for all \( \lambda \in [0, 1] \) and \( x, y \geq 0 \).

ii) subhomogeneous, if
\[
f(\lambda x) \leq \lambda f(x)
\]
for all \( \lambda \in [0, 1] \) and \( x \geq 0 \).

iii) superadditive, if
\[
f(x + y) \geq f(x) + f(y)
\]
for all \( x, y \in [0, \infty] \).

The relationship between these three classes of functions has been studied in [8], where the authors are focused on functions which are continuous, nonnegative and for which \( f(0) = 0 \). The following implications are valid for functions with these properties but the reverse implications does not hold:

\[(1) \Rightarrow (2) \Rightarrow (3).\]

The following result is very useful ([8]).

Lemma 2: A function \( f : [0, \infty] \to [0, \infty] \) is subhomogeneous iff \( f(\frac{x}{x}) \) is increasing function.

The universal integral is based on monotone measure (see [17]). Let \( X \) be a non-empty set and \( \mathcal{A} \sigma \)-algebra on \( X, \) i.e., \((X, \mathcal{A})\) be a measurable space.

Definition 3: ([17], [25], [33]) A function \( m : \mathcal{A} \to [0, \infty] \) is a monotone measure on a measurable space \((X, \mathcal{A})\) if it satisfies:

i) \( m(\emptyset) = 0 \),
ii) \( m(X) > 0 \),
iii) for all \( A, B \in \mathcal{A} \) if \( A \subseteq B \), then \( m(A) \leq m(B) \).

If a monotone measure \( m \) satisfies \( m(X) = 1 \) then it is called a normed fuzzy measure, see [16], [33], [35].

Recall that a function \( f : X \to [0, \infty] \) is called \( \mathcal{A} \)-measurable if for each \( B \in \mathcal{B}([0, \infty]) \) the preimage \( f^{-1}(B) \in \mathcal{A} \), where \( \mathcal{B}([0, \infty]) \) is the \( \sigma \)-algebra of Borel subsets of \([0, \infty]\). For simplicity, we use the following notations:
• $\mathcal{F}(X,A)$ denote the set of all $\mathcal{A}$-measurable functions $f : X \to [0, \infty]$;

• For each number $a \in [0, \infty]$, $\mathcal{M}_a^{(X,A)}$ is the set of all monotone measures satisfying $m(X) = a$, and denote by

$$\mathcal{M}^{(X,A)} = \bigcup_{a \in [0, \infty]} \mathcal{M}_a^{(X,A)};$$

• $\mathcal{S}$ is the class of all measurable spaces, and

$$\mathcal{D}_{[0,\infty]} = \bigcup_{(X,A) \in \mathcal{S}} \mathcal{M}^{(X,A)} \times \mathcal{F}^{(X,A)}.$$

Using the previous notation the Choquet [9], Sugeno [33] and Shilkret [30] integral, respectively, are given for any measurable space $(X,A)$, for any measurable function $f \in \mathcal{F}(X,A)$ and for any monotone measure $m \in \mathcal{M}^{(X,A)}$, by

$$\text{Ch}(m,f) = \int_0^\infty m \left( \{ x \in X \mid f(x) \geq t \} \right) dt \quad (4)$$

$$\text{Su}(m,f) = \sup_{t \in [0,\infty]} \left( \min \left( t, m \left( \{ x \in X \mid f(x) \geq t \} \right) \right) \right) \quad (5)$$

$$\text{Sh}(m,f) = \sup_{t \in [0,\infty]} \left( t \cdot m \left( \{ x \in X \mid f(x) \geq t \} \right) \right) \quad (6)$$

where the convention $0 \cdot \infty = 0$ is used.

An equivalence relation between pairs $(m_1, f_1), (m_2, f_2) \in \mathcal{D}_{[0,\infty]}$ was introduced in [17].

Definition 4: Two pairs $(m_1, f_1) \in \mathcal{M}(X_1,A_1) \times \mathcal{F}(X_1,A_1)$ and $(m_2, f_2) \in \mathcal{M}(X_2,A_2) \times \mathcal{F}(X_2,A_2)$ satisfying

$$m_1 \left( \{ x \in X \mid f_1(x) \geq t \} \right) = m_2 \left( \{ x \in X \mid f_2(x) \geq t \} \right)$$

for all $t \in [0, \infty]$, will be called integral equivalent, in symbols

$$(m_1, f_1) \sim (m_2, f_2).$$

The universal integral is based on an operation which, in general, is neither commutative nor associative operations.

Definition 5: ([17], [33]) A pseudo-multiplication is a function $\otimes : [0, \infty]^2 \to [0, \infty]$ with the following properties:

i) it is non-decreasing in each component, i.e., for all $a_1, a_2, b_1, b_2 \in [0, \infty]$ with $a_1 \leq a_2$ and $b_1 \leq b_2$ we have $a_1 \otimes b_1 \leq a_2 \otimes b_2$.

ii) there exists a pseudo-multiplication $\otimes : [0, \infty]^2 \to [0, \infty]$ such that for all pairs $(m, c \cdot 1_A) \in \mathcal{D}_{[0,\infty]}$ (where $1_A$ is the characteristic function of the set $A$)

$$\text{I}(m, c \cdot 1_A) = c \otimes m(1_A);$$

iii) for all integral equivalent pairs $(m_1, f_1), (m_2, f_2) \in \mathcal{D}_{[0,\infty]}$ we have

$$\text{I}(m_1, f_1) = \text{I}(m_2, f_2).$$

In the following theorem are given the smallest and the greatest universal integral based on $\otimes$ ([17]).

Theorem 7: Let $\otimes : [0, \infty]^2 \to [0, \infty]$ be a pseudo-multiplication on $[0, \infty]$. Then the smallest universal integral $\text{I}_\otimes$ and the greatest universal integral $\text{I}^\otimes$ based on $\otimes$ are given by

$$\text{I}_\otimes(m,f) = \sup_{t \in [0,\infty]} \left( t \otimes m \left( \{ x \in X \mid f(x) \geq t \} \right) \right),$$

$$\text{I}^\otimes(m,f) = \essup_m f \otimes \sup_{t \in [0,\infty]} m \left( \{ x \in X \mid f(x) \geq t \} \right),$$

where

$$\essup_m f = \sup \{ t \in [0,\infty] \mid m \left( \{ x \in X \mid f(x) \geq t \} \right) > 0 \}.$$

If the pseudo-multiplications are given by Min$(a,b) = \min(a,b)$ and Prod$(a,b) = a \cdot b$, the smallest universal integral reduce to the Sugeno and Shilkret integral, i.e., $\text{Su} = \text{I}_{\text{Min}}$ and $\text{Sh} = \text{I}_{\text{Prod}}$, respectively.

In [17] the authors also consider the universal integral on the unit interval $[0, 1]$. In this case functions $f \in \mathcal{F}(X,A)$ satisfying $\text{Ran}(f) \subseteq [0, 1]$ are observed and then we shall write $f \in \mathcal{F}_{[0,1]}$. The restriction of pseudo-multiplication to $[0,1]^2$ is called a semicopula or a conjunctor, i.e., a binary operation $\otimes : [0,1]^2 \to [0,1]$ which is non-decreasing in both components, has 1 as neutral element and satisfies $a \otimes b \leq \min(a,b)$ for all $(a,b) \in [0,1]^2$ see [7], [11], and universal integrals are restricted to the class

$$\mathcal{D}_{[0,1]} = \bigcup_{(X,A) \in \mathcal{S}} \mathcal{M}^{(X,A)} \times \mathcal{F}^{(X,A)}_{[0,1]}.$$

Specially, for a fixed strict $t$-norm $T$, the corresponding universal integral $\text{I}_T$ is the Sugeno-Weber integral [37]. The smallest and the greatest universal integral on the interval $[0,1]$ related to the semicopula $\otimes$ is given by, respectively:

$$\text{I}_{\otimes}(m,f) = \sup_{t \in [0,1]} \left( t \otimes m \left( \{ x \in X \mid f(x) \geq t \} \right) \right),$$

$$\text{I}^\otimes(m,f) = \essup_m f \otimes \sup_{t \in [0,1]} m \left( \{ x \in X \mid f(x) \geq t \} \right).$$

This type of integral was called seminormed integral in [32].
III. JENSEN’S INEQUALITY FOR THE SMALLEST UNIVERSE integral

Now we shall give Jensen type inequality related to the smallest universe integral. We shall see that, in a special case, the following theorem reduced to the corresponding inequality for Sugeno integral obtained in [28].

Theorem 8: Let \( \varphi : [0, \infty] \rightarrow [0, \infty] \) be continuous and strict increasing function and \( \otimes : [0, \infty]^2 \rightarrow [0, \infty] \) pseudo-multiplication on \([0, \infty]\). Let \( f \in F^{(X,A)} \) and \( m \in M_{1}^{(X,A)} \) be a monotone measure. If
\[
\varphi(x \otimes y) \leq \varphi(x) \otimes y
\]
for all \( x \in [0, m(X)] \) and \( y \in [0, \infty] \), then
\[
\varphi(I_{\otimes}(m,f)) \leq I_{\otimes}(m,\varphi(f)).
\]

Proof. Due to the properties of \( \varphi \) and (7) we have:
\[
\varphi(I_{\otimes}(m,f)) = \varphi\left(\sup_{t \in [0,\infty]} (t \otimes m(\{x \in X \mid f(x) \geq t\}))\right) = \sup_{t \in [0,\infty]} \varphi(t \otimes m(\{x \in X \mid f(x) \geq t\})) \leq \sup_{t \in [0,\infty]} (\varphi(t) \otimes m(\{x \in X \mid f(x) \geq t\})) = \sup_{t \in [0,\infty]} (\varphi(t) \otimes m(\{x \in X \mid \varphi(f(x)) \geq \varphi(t)\}))
\]
On the other hand, for all \( t \in [0,\infty] \) it holds
\[
\varphi(t) \otimes m(\{x \in X \mid \varphi(f(x)) \geq \varphi(t)\}) \leq \sup_{s \in [0,\infty]} (s \otimes m(\{x \in X \mid \varphi(f(x)) \geq s\}),
\]
due to \( \varphi(t) \in [0,\infty] \). Therefore, we have
\[
\sup_{t \in [0,\infty]} (\varphi(t) \otimes m(\{x \in X \mid f(x) \geq t\})) \leq \sup_{s \in [0,\infty]} (s \otimes m(\{x \in X \mid f(x) \geq s\}))
\]
i.e., the inequality (8) holds. \( \Box \)

Example 9: Let the pseudo-multiplication is given by \( x \otimes y = \frac{xy}{x+y} \), with neutral element \( \infty \). If \( \varphi : [0, \infty] \rightarrow [0, \infty] \) is subhomogeneous function such that \( \varphi(x) \leq x \) for every \( x \in [0, \infty] \) then the inequality (7) holds. Namely, since \( \frac{y}{x+y} \leq 1 \) for all \( x \geq 0, y > 0 \) we have:
\[
\varphi(x \otimes y) = \varphi\left(\frac{xy}{x+y}\right) \leq \frac{y}{x+y} \varphi(x) \leq \frac{\varphi(x)y}{\varphi(x)+y} = \varphi(x) \otimes y.
\]

For example, function \( \varphi(x) = \frac{x^2}{x+1}, x \in [0, \infty] \) satisfies the conditions of Theorem 8. Therefore, we have
\[
\frac{(\varphi(I_{\otimes}(m,f)))^2}{\varphi(I_{\otimes}(m,f)) + 1} \leq I_{\otimes}(m,\frac{f^2}{f+1})
\]
for all \( f \in F^{(X,A)} \).

If we assume that \( \varphi(x) \leq x \) for every \( x \in [0, m(X)] \) and \( \otimes = \min \) in Theorem 8, the condition (7) holds. Also, the reverse implication is valid, i.e. if \( \otimes = \min \) and the condition (7) holds then \( \varphi(x) \leq x \) for every \( x \in [0, m(X)] \). Hence, the inequality (8) is Jensen’s inequality for Sugeno integral which is given in [28].

Corollary 10: Let \( \varphi : [0, \infty] \rightarrow [0, \infty] \) be continuous and strict increasing function such that \( \varphi(x) \leq x \) for every \( x \in [0, m(X)] \), then:
\[
\varphi(Su(m,f)) \leq Su(m,\varphi(f)).
\]
for all \( f \in F^{(X,A)} \).

Example 11: ([28]) If \( \varphi(x) = x^\lambda, \lambda \geq 1 \), then \( \varphi(x) \leq x \) for all \( x \in [0, 1] \). We obtain:
\[
(Su(m,f))^\lambda \leq Su(m,f^\lambda)
\]
for all \( \lambda \geq 1 \), \( f \in F^{(X,A)} \) and \( m \in M_{1}^{(X,A)} \).

If the pseudo-multiplication is given by \( \text{Prod}(x,y) = x \cdot y \), then we get Jensen’s inequality for Shilkret integral.

Corollary 12: Let \( \varphi : [0, \infty] \rightarrow [0, \infty] \) be continuous, strict increasing and subhomogeneous function. If \( f \in F^{(X,A)} \) and \( m \in M_{1}^{(X,A)} \) then:
\[
\varphi(Sh(m,f)) \leq Sh(m,\varphi(f)).
\]

Example 13: i) Let \( \varphi(x) = e^x - 1 \), then
\[
e^{Sh(m,f)} - 1 \leq Sh(m,e^f - 1),
\]
for all \( f \in F^{(X,A)} \) and \( m \in M_{1}^{(X,A)} \).

ii) Let \( \varphi(x) = x^2 + x \), then
\[
(Sh(m,f))^2 + Sh(m,f) \leq Sh(m,f^2 + f),
\]
for all \( f \in F^{(X,A)} \) and \( m \in M_{1}^{(X,A)} \).

Obviously, if we consider a monotone measure \( m \in M_{1}^{(X,A)} \), functions \( f \in F_{[0,1]}^{(X,A)} \) and a general semicopula, the inequality (8) holds with the same assumptions for the function \( \varphi : [0, 1] \rightarrow [0, 1] \) as in Theorem 8. In that case we have Jensen inequality for a seminor integral.

Example 14: Let we observe Łukasiewicz \( t \)-norm, \( T_L(x,y) = \max(0, x+y-1) \). We shall prove that if \( \varphi : [0,1] \rightarrow [0,1] \) is subhomogeneous function such that \( \varphi(x) \geq x \) for every \( x \in [0,1] \) and \( \varphi(0) = 0 \) then holds
\[
\varphi(T_L(x,y)) \leq T_L(\varphi(x), y)
\]
for all \( x, y \in [0,1] \). Let \( x+y-1 > 0 \), then \( T_L(x,y) = x+y-1 \). Since \( x+y-1 \leq x \), due to Lemma 2 it holds
\[
\frac{\varphi(x+y-1)}{x+y-1} \leq \frac{\varphi(x)}{x},
\]
i.e.,
\[
\frac{x}{x + y - 1} \varphi(x + y - 1) \leq \varphi(x).
\] (10)

From \(\varphi(x) \geq x\) it follows
\[
\frac{\varphi(x + y - 1)}{x + y - 1} \geq 1.
\]

Since \(y - 1 \leq 0\) the following holds
\[
\frac{y - 1}{x + y - 1} \varphi(x + y - 1) \leq y - 1.
\] (11)

By adding the inequalities (10) and (11) we obtain
\[
\varphi(x + y - 1) \leq \varphi(x) + y - 1,
\]
i.e., (9) holds. On the other hand, since \(T_L(x, y) = 0\) for \(x + y - 1 \leq 0\), (9) obviously holds.

For example, function \(\varphi(x) = xe^x\), \(x \in [0, 1]\), satisfies the condition (9). Therefore, we have
\[
I_{T_L}(m, f) \leq (m, f) \leq I_{T_L}(m, f)
\]
for all \(f \in F([0, 1])\) and \(m \in M_1(X, A)\).

IV. JENSEN’S INEQUALITY FOR THE GREATEST UNIVERSAL INTEGRAL

The following theorem is valid for the greatest universal integral.

**Theorem 15:** Let \(\varphi : [0, \infty] \to [0, \infty]\) be continuous and strict increasing function and \(\varphi\) pseudo-multiplication on \([0, \infty]\). Let \(f \in F(X, A)\) and \(m \in M(X, A)\) be a monotone measure. If
\[
\varphi(x \otimes y) \leq \varphi(x) \otimes y
\]
for all \(y \in [0, m(X)]\) and \(x \in [0, \infty]\), then
\[
\varphi(I^\otimes(m, f)) \leq I^\otimes(m, \varphi(f)).
\]

**Proof.** Due to properties of function \(\varphi\), the pseudo-multiplication and (12) it holds:

\[
\varphi \left( \left( \sup_m \right) \otimes \sup_{t \in [0, \infty]} \{m \{\{x \in X \mid f(x) \geq t\}\} \right) \right)
\]
\[
\leq \varphi \left( \left( \sup_m \right) \otimes \sup_{t \in [0, \infty]} \{m \{\{x \in X \mid f(x) \geq t\}\} \right) \right)
\]
\[
= \sup_m \varphi(f) \otimes \sup_{t \in [0, \infty]} \{m \{\{x \in X \mid \varphi(f(x)) \geq \varphi(t)\}\} \}
\]
\[
\leq \sup_m \varphi(f) \otimes \sup_{t \in [0, \infty]} \{m \{\{x \in X \mid \varphi(f(t)) \geq t\}\} \}
\]
which completes the proof. \(\square\)

**Example 16:** Let \(\otimes = \min\) and \(\varphi(x) = \arctan x\), then
\[
\arctan I^\otimes(m, f) \leq I^\otimes(m, \arctan f)
\]
for all \(f \in F(X, A)\) and monotone measure \(m \in M(X, A)\).

V. CONCLUSION

We have proved Jensen type inequalities for the smallest and the greatest universal integral. The relationship between obtained results and the known Jensen inequality for Sugeno integral is specified. Also, obtained inequalities cover the Shilkret integral and seminormed fuzzy integrals. The future work will be the investigation of the corresponding inequality for more general universal integral and applications of obtained results.

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