Spectrum Quantization and Feedback for Downlink Massive MIMO Systems With Cascaded Precoding

Yinsheng Liu, Geoffrey Ye Li, and Wei Han.

Abstract

In this paper, we investigate the quantization and the feedback of downlink statistical channel information for massive multiple-input multiple-output (MIMO) systems with cascaded precoding. Massive MIMO has gained a lot of attention recently because of its ability to significantly improve the network performance. To reduce the overhead of downlink channel estimation and uplink feedback in frequency-division duplex massive MIMO systems, cascaded precoding has been proposed, where the outer precoder is implemented using traditional limited feedback while the inner precoder is determined by the spatial covariance matrix of the channels. In massive MIMO systems, it is difficult to quantize the spatial covariance matrix because of its large size caused by the huge number of antennas. In this paper, we propose a low-complexity and low-overhead approach based on the quantization and the feedback of the spatial spectrum to construct a codebook composed of quantized covariance matrices. The proposed approach has much smaller leakage than the traditional discrete Fourier transform submatrix based precoding. Simulation results show that the proposed approach can significantly reduce the feedback overhead at the cost of a slight performance degradation.

Index Terms

Massive MIMO, precoding, limited feedback, covariance matrix.

Yinsheng Liu is with School of Computer Science and Information Technology and State Key Laboratory of Rail Traffic Control and Safety, Beijing Jiaotong University, Beijing 100044, China, e-mail: ys.liu@bjtu.edu.cn.

Geoffrey Ye Li is with ITP lab, School of ECE, Georgia Institute of Technology, Atlanta 30313, Georgia, USA, e-mail: liye@ece.gatech.edu.

Wei Han is with Huawei Technologies, Co. Ltd., e-mail: wayne.hanwei@huawei.com.
I. INTRODUCTION

Massive *multiple-input multiple-output* (MIMO) formed by installing a huge number of antennas at the *base station* (BS) can significantly increase the spectrum- and energy-efficiencies of wireless networks [1], [2]. As a promising technique for the next generation cellular systems, it has gained a lot of attention recently [3], [4].

For the best performance gain of downlink transmission, accurate downlink *channel state information* (CSI) is required at the BS for precoding [2]. Downlink CSI at the BS can be obtained by exploiting the channel reciprocity in a time-division duplex system, [2], [4], [5]. However, due to the huge number of antennas in massive MIMO systems, the acquisition of downlink CSI at the BS is not that easy for a *frequency-division duplex* (FDD) system even if most existing cellular networks are operated in FDD mode.

For regular MIMO in FDD systems, downlink CSI can be estimated at the *user equipment* (UE) and fed back to the BS [6], [7]. In massive MIMO systems, however, traditional channel estimation and limited feedback techniques can be hardly used because of the large overhead caused by the huge number of antennas. To address this issue, a closed-loop training technique has been proposed in [8], [9] for downlink CSI acquisition. It can achieve significant performance improvement within only a few iterations by exploiting the spatial correlation of the channels corresponding to different antennas. The channel spatial correlation has also been used in [10], [11] to develop a cascaded precoding, where the precoder is divided into an outer precoder and an inner precoder. The outer precoder is based on a low-dimension effective channel and can be implemented using traditional channel estimation and limited feedback. The inner precoder is determined by the eigen-matrix of the channel spatial covariance matrix. When the antenna number is very large, the eigen-matrix is usually approximated by a *discrete Fourier transform* (DFT) submatrix with each column corresponding to a beam. Therefore, the DFT submatrix can be used as the inner precoder to avoid the need of a precise estimation of the covariance matrix [11], [12], [16]. Apparently, the best performance of the cascaded precoding can be achieved by choosing the strongest beams to form the inner precoder, which will cause a large feedback overhead. To reduce the overhead, the feedback is conducted with respect to
a group of sub-sectors in [12], and each sub-sector can be viewed as a wide beam that covers multiple narrow beams. The wide-beam based approach is also widely used to construct a multi-resolution codebook, which usually has multiple codebook levels with each level corresponding to a different angular resolution [14], [15]. As an alternative, it is assumed in [13] that the spatial covariance matrices for the downlink and the uplink are same, and thus can be estimated through uplink transmission and then used for downlink precoding directly.

Although the eigen-matrix of the downlink spatial covariance matrix can be approximated by the DFT submatrix when the antenna number is very large, the antenna number in practical systems is always finite. In this case, spectrum leakage exists and thus the DFT submatrix based inner precoder fails to capture the channel power efficiently. As a result, the inner precoder based on the quantized covariance matrix is expected to achieve better performance. For covariance matrix quantization, a Lloyd-type algorithm has been proposed in [17] where the codebook is generated by training with a large number of sample covariance matrices. However, the approach there can be hardly used in massive MIMO systems due to the huge size of the covariance matrix. In this paper, we develop a spectrum quantization approach for the quantization of the spatial covariance matrix. The quantization is conducted with respect to the spatial spectrum, which is defined as the Fourier transform of the spatial correlation, to avoid the complicated matrix calculation. We construct a codebook composed of quantized covariance matrices with each one generated from a codeword spectrum. Our analysis shows that the quantized covariance matrix based precoding has much smaller spectrum leakage than the traditional DFT submatrix based one. As a consequence, the BS can not only focus the transmit power on the direction of interest but also yield much smaller interference in undesired directions. For practical application, a multi-codeword feedback strategy has been also proposed to achieve a balance in the width of the codeword spectrum.

For a given codebook, a specially designed training procedure is required such that the UE can estimate the signal-to-noise ratio (SNR) for each codeword and then determine the best one. For SNR estimation, a multi-stage training is designed for the multi-resolution codebook where the angular resolution can be gradually improved at the cost of multiple training symbols
Fig. 1. Cascaded precoding in downlink massive MIMO system with a codebook composed of quantized covariance matrices. UE determines the best quantized covariance matrix and feedback its index to the BS.

[14]. To save the number of training symbols, different subcarriers in orthogonal frequency division multiplexing (OFDM) modulation are used for downlink training in [18] by assuming the subcarriers are within the coherent bandwidth of the channel. In this paper, we propose a specially designed training, which is consisted of a repetitive subcarrier allocation strategy for all codewords over the whole bandwidth and a random outer precoder. By averaging over different subcarriers, the proposed approach can estimate the average SNRs for all codewords with only one training symbol.

The rest of this paper is organized as follows. The system model is presented in Section II. The codebook design is shown in Section III and an insightful analysis is presented in Section IV. Practical issues are discussed in Section V. Simulation results are presented in Section VI, and conclusions are finally drawn in Section VII.

II. SYSTEM MODEL

As in Fig. 1 we consider a cascaded precoding for downlink transmission in an OFDM based massive MIMO system with \( M \) antennas at the BS and a single antenna at the UE of interest [11]. We only consider the single user case in this paper although our approach can be also extended...
to the multiuser case. Denote $v[k] \in \mathbb{C}^{D \times 1}$ to be the outer precoder at the $k$-th subcarrier, where $D$ is the size of the effective spatial channel and is much smaller than the antenna number. Denote $W \in \mathbb{C}^{M \times D}$ to be the inner precoder determined by the spatial covariance matrix. Note that different subcarriers can share the same inner precoder because the spatial covariance matrix is long-term channel statistics, which are independent of the frequency selectivity \cite{13}. UE will determine the index of the best quantized covariance matrix through specially designed training and then feed its index back to the BS.

From Fig. \[1\] the received signal at the $k$-th subcarrier can be expressed as

$$y[k] = h^H[k]Wv[k]x[k] + z[k],$$

where $x[k]$ is the transmit symbol at the $k$-th subcarrier with zero mean and unit variance, $z[k]$ is the corresponding additive Gaussian noise with zero mean and noise power, $N_0$, and $h[k] = (h_0[k], h_1[k], \cdots, h_{M-1}[k])^T$ denotes the spatial channel vector with $h_m[k]$ indicating the channel frequency response at the $k$-th subcarrier on the $m$-th antenna.

In massive MIMO systems, the antennas at the BS are usually tightly placed, leading to high spatial correlation of the channels corresponding to different antennas \cite{19}. To capture the nature of the spatial correlation, a physical channel model in \cite{13}, \cite{20} can be used to describe each propagation path between the antenna at the BS and the UE individually. For a typical \textit{uniform linear array} (ULA),

$$h_m[k] = \sum_{l=0}^{L-1} \alpha_l e^{-j\frac{2\pi k\tau_l}{T}} e^{j2\pi mv_l},$$

where $\alpha_l$ is the attenuation of the $l$-th path with zero mean and $E(\alpha_l\alpha_p^*) = \sigma^2 \delta[l - p]$ with $\delta[\cdot]$ indicating the Kronecker delta function and we assume $\sum_{l=0}^{L-1} \sigma^2 = 1$ for normalization, $\tau_l$ denotes the delay corresponding to the $l$-th path, $T$ denotes the OFDM symbol duration, and $v_l = \frac{d}{\lambda} \sin \theta_l$ denotes the wave number of the $l$-th path with corresponding \textit{angle of departure} (AoD) $\theta_l$ and antenna spacing $d/\lambda$ normalized by the wave length. Accordingly, the spatial
correlation function can be expressed as

\[
r[m] \triangleq E(h_{m+n}[k]h^*_n[k]) = \sum_{l=0}^{L-1} \sigma^2_l e^{j2\pi mvl},
\]
(3)
or in a matrix for as \( \mathbf{R} \triangleq E(\mathbf{h}[k]\mathbf{h}^H[k]) = \{r[m-n]\}_{m,n=0}^{M-1} \). Denote the eigen-value decomposition (EVD) of the covariance matrix, \( \mathbf{R} \), as

\[
\mathbf{R} = \mathbf{U} \Lambda \mathbf{U}^H,
\]
(4)
where \( \mathbf{U} \in \mathbb{C}^{M \times M} \) denotes the eigen-matrix and \( \Lambda = \text{diag}\{\lambda[m]\}_{m=0}^{M-1} \) is a diagonal matrix with \( \lambda[m] \) indicating the \( m \)-th largest eigenvalue.

The outer precoder can be implemented using traditional limited feedback since it has a low dimension. The optimal inner precoder will be \( \mathbf{W} = \mathbf{U}_D \), where \( \mathbf{U}_D \) consists of the first \( D \) columns of the eigen-matrix, \( \mathbf{U} \), corresponding to the \( D \) largest eigenvalues [11]. It is a challenging problem for the inner precoder design due to its large size caused by the huge number of antennas.

Since the optimal inner precoder is determined by the spatial covariance matrix, a good codebook should be designed with respect to the quantization of the covariance matrix. With the specially designed training in this paper, the UE can determine the best codeword from the codebook \( \mathcal{R} = \{\mathbf{R}_0, \mathbf{R}_1, \cdots, \mathbf{R}_{Q-1}\} \), which is composed of \( Q \) quantized covariance matrices with \( Q \ll M \) to reduce the feedback overhead. Each quantized covariance matrix, \( \mathbf{R}_q = \{r_q[m-n]\}_{m,n=0}^{M-1} \), corresponds to a spatial correlation function, \( r_q[m] \). If most power can be captured by the \( D \) largest eigenvalues for each quantized covariance matrix, which can be assured by the codebook design in this paper, then the EVD of the quantized covariance matrix can be approximated by

\[
\mathbf{R}_q = \mathbf{U}_q \mathbf{A}_q \mathbf{U}_q^H,
\]
(5)
where \( \mathbf{U}_q \in \mathbb{C}^{M \times D} \) denotes the eigen-matrix with the columns corresponding to the \( D \) largest eigenvalues, and \( \mathbf{A}_q = \text{diag}\{\lambda_q[m]\}_{m=0}^{D-1} \) is a diagonal matrix with \( \lambda_q[m] \) indicating the \( m \)-th
largest eigenvalue of the $q$-th quantized covariance matrix. Therefore, designing the codebook is equivalent to finding $Q$ quantized covariance matrices.

III. CODEBOOK DESIGN

In this section, we will first introduce the design criterion based on the maximization of the average SNR, and then present the codebook design using spectrum quantization.

A. Maximization of Average SNR

To focus on the codebook design for the inner precoder, we assume an ideal outer precoder in this section, that is

$$v[k] = \frac{W^H h[k]}{\|W^H h[k]\|_2}.$$  

(6)

In this case, the instantaneous SNR at the $k$-th subcarrier can be given by

$$\gamma[k] \triangleq \|W^H h[k]\|_2^2,$$  

(7)

where the noise power has been normalized for simple notation. The inner precoder is determined by the covariance matrix, that is long-term statistics of the channel, and thus it should be designed with respect to the average SNR,

$$\gamma \triangleq E(\gamma[k]) = \text{Tr}(W^HRW),$$  

(8)

rather than the instantaneous SNR.

To achieve the best performance, the UE should choose the quantized covariance matrix to maximize the average SNR,

$$q^* = \arg \max_{\{0,1,\ldots,Q-1\}} \gamma_q,$$  

(9)

where $\gamma_q \triangleq \text{Tr}(U_q^H R U_q)$ is the average SNR for the $q$-th quantized covariance matrix. Then, the index of the optimal quantized covariance matrix, $q^*$, is fed back to the BS, which needs $\log_2 Q$-bit feedback overhead.
B. Spectrum Quantization

As in (9), the codeword is chosen to maximize the average SNR from a predetermined codebook. From Appendix A, we have

$$\lambda_{q,max}^{-1} \text{Tr}(R_q R) \leq \text{Tr}(U_q^H R U_q) \leq \lambda_{q,min}^{-1} \text{Tr}(R_q R),$$

(10)

where $\lambda_{q,min}$ and $\lambda_{q,max}$ denote the minimum and the maximum eigenvalues among $\lambda_q[m]$’s, respectively, and the equation holds when all the eigenvalues are equal.

As the codebook design based on (9) is difficult, we therefore consider the bound expression, $\text{Tr}(R_q R)$, rather than the original average SNR for codebook design. In this case, the criterion in (9) can be rewritten as

$$q^* = \arg \max_{\{0,1,\ldots,Q-1\}} \text{Tr}(R_q R),$$

(11)

Direct calculation yields that

$$\|R_q - R\|_F^2 = \text{Tr}(R_q^H R_q + R^H R - R_q^H R - R^H R_q).$$

(12)

If we assume that $\text{Tr}(R_q^H R_q)$’s are the same for different codewords in the designed codebook, then (11) is equivalent to

$$q^* = \arg \min_{\{0,1,\ldots,Q-1\}} \|R_q - R\|_F^2,$$

(13)

where we have used the identities $R = R^H$ and $R_q = R_q^H$. In other words, the best codeword can be determined by choosing the one nearest to the true covariance matrix in terms of the Frobenius norm.

On the other hand, the spatial spectrum can be defined as the Fourier transform of the spatial correlation in (3) [21], that is

$$R(v) \triangleq \sum_{m=-\infty}^{+\infty} r[m] e^{-j2\pi mv} = \sum_{l=0}^{L-1} \sigma_l^2 \delta(v - v_l),$$

(14)

where $\delta(\cdot)$ denotes the Dirac delta function. Note that the spatial spectrum is always positive.
for a stationary random process. Direct calculation yields that

\[ \|R_q - R\|_F^2 = \sum_{m=-\infty}^{+\infty} |(r[q][m] - r[m])a[m]|^2 \]

\[ = \int_{-\frac{1}{2}}^{+\frac{1}{2}} |R_q(v) * A(v) - R(v) * A(v)|^2 dv \]

\[ = \int_{-\frac{1}{2}}^{+\frac{1}{2}} [R_q(v) * A(v)]^2 - 2[R_q(v) * A(v)]R(v) * A(v) + [R(v) * A(v)]^2 dv, \quad (15) \]

where * denotes the circular convolution, \( R_q(v) \) is the Fourier transform of \( r_q[m] \), and \( A(v) \) is the Fourier transform of \( a[m] \) that is defined as

\[ a[m] = \begin{cases} \sqrt{M - |m|}, & |m| \leq M - 1 \\ 0, & \text{otherwise} \end{cases}. \quad (16) \]

If we assume that the first term in the third equation of (15) is the same for all codewords in the designed codebook, then the criterion in (13) can be rewritten as

\[ q^* = \arg \max_{\{0, 1, \ldots, Q-1\}} \int_{-\frac{1}{2}}^{+\frac{1}{2}} [R_q(v) * A(v)]R(v) * A(v) dv \]

\[ = \arg \max_{\{0, 1, \ldots, Q-1\}} \int_{-\frac{1}{2}}^{+\frac{1}{2}} \int_{-\frac{1}{2}}^{+\frac{1}{2}} [R_q(v) * A(v)]A(v - \tau)R(\tau)d\tau d\tau \]

\[ = \arg \max_{\{0, 1, \ldots, Q-1\}} \int_{-\frac{1}{2}}^{+\frac{1}{2}} [R_q(v) * G(v)]R(v) dv, \quad (17) \]

where we have used the identity \( A(v) = A(-v) \) since \( a[m] = a[-m] \), and

\[ G(v) = A(v) * A(v) = \frac{\sin^2(\pi Mv)}{\sin^2(\pi v)}. \quad (18) \]

If the antenna number is very large in a massive MIMO systems, then \( G(v) \approx \delta(v) \). As a result, the best codeword should be determined by

\[ q^* = \arg \max_{\{0, 1, \ldots, Q-1\}} \int_{-\frac{1}{2}}^{+\frac{1}{2}} R_q(v)R(v) dv, \quad (19) \]
or equivalently,

$$q^* = \arg \min_{\{0,1,\cdots,Q-1\}} \sqrt{1 - \left| \int_{-\frac{1}{2}}^{+\frac{1}{2}} R_q(v)R(v)dv \right|^2}, \quad (20)$$

where the right side of the equation can be viewed as minimizing the chordal distance on a functional space that has infinite dimensions.

In [22], a geometric model has been developed to solve the optimal codebook design for beamforming in regular MIMO systems, where the distance metric has a similar form as (20) except that the latter is defined on the finite dimension space. We can similarly adopt such a model for the codebook design in this paper. As a result, the design of good codebook turns to the following optimization problem,

$$\max_{\mathcal{R}} \min_{0 \leq p,q \leq Q-1} \sqrt{1 - \left| \int_{-\frac{1}{2}}^{+\frac{1}{2}} R_p(v)R_q(v)dv \right|^2}, \quad (21)$$

that is, the codebook should be designed to maximize the minimum chordal distance on a functional space between different codewords.

By viewing each codeword as a point on a Grassmannian manifold, Grassmannian line packing has been used to design a good codebook, assuming that the codebook size is not smaller than the antenna number [23]. However, such an approach cannot be used on a functional space because the dimension of the functional space is infinite and is therefore always larger than the codebook size.

As an alternative, we consider a codebook composed of a group of mutually orthogonal functions. Actually, we can always find such a codebook because the dimension of the functional space is larger than the codebook size. Due to the orthogonality, the chordal distance between any two different codewords will be unit, which has achieved the maximum of (21). In this sense, a codebook composed of mutually orthogonal functions can be viewed as a good codebook.

Although a good codebook can be constructed using a group of orthogonal functions, the best

---

1Although the Grassmannian line packing cannot be used in this paper, the criterion for designing a good codebook in [21] is still valid because the geometric model in [22] can work when the antenna number is larger than the codebook size.
codebook should depend on the knowledge of the AoD distribution, which is usually unknown in advance at the BS. Inspired by the fact that the AoDs are locally distributed [11], [24], we consider the following codebook in this paper,

$$R_q(v) = \begin{cases} \sqrt{Q}, & -\frac{1}{2} + \frac{q}{Q} \leq v \leq -\frac{1}{2} + \frac{q+1}{Q}, \\ 0, & \text{otherwise}, \end{cases}$$

(22)

which divides the whole band into $Q$ equal pieces. The best codeword is then determined by the size of the overlap region between the codeword spectrum and the real spatial spectrum, as in Fig. 2. From [25], the dimension of the outer precoder can be approximated by $D \approx M/Q$. Apparently, reduction of $Q$ will increase the dimension of the outer precoder.

With the codeword spectrum in (22), the corresponding correlation function can be given as

$$r_q[m] = \int_{-\frac{1}{2}}^{+\frac{1}{2}} R_q(v)e^{j2\pi mv}dv = \sqrt{Q} \frac{\pi m}{Q} \sin \left( \frac{\pi m}{Q} \right) e^{j2\pi m(-\frac{1}{2} + \frac{q+0.5}{Q})}.$$  

(23)

Based on $r_q[m]$ in (23), the $q$-th quantized covariance matrix can be also obtained accordingly.
IV. CODEBOOK ANALYSIS

In this section, we first analyze the spectrum leakage for the proposed codebook, and then discuss the average SNR. The SNR loss compared to the ideal inner precoder will be also discussed at the end of this section.

A. Spectrum Leakage

Although the DFT-submatrix based inner precoder can avoid the estimation of covariance matrix, we will show in this subsection that the eigen-matrix based inner precoder yields much smaller spectrum leakage than the DFT-submatrix based one.

Suppose that the $q$-th codeword is fed back to the BS, and thus $U_q$ is used as the inner precoder. From Appendix B, the columns of the eigen-matrix, $U_q$, can be viewed as the orthogonal basis functions of a subspace,

$$
S = \text{span}\{s(v)|v \in V_q\},
$$

(24)

where $V_q = \{v| -\frac{1}{2} + \frac{q}{Q} \leq v \leq -\frac{1}{2} + \frac{q+1}{Q}\}$ and $s(v) = (1, e^{j2\pi v}, \cdots, e^{j2\pi(M-1)v})^T$ denotes the steering vector.

When the antenna number is very large, we have

$$
\frac{1}{M}s^H(v_0)s(v) = 0,
$$

(25)

for any $v_0 \notin V_q$ and $v \in V_q$ [13]. Therefore, $s(v_0)$ should be orthogonal to any element in the subspace $S$ since it is spanned by $\{s(v)|v \in V_q\}$. As a result, $s(v_0)$ should be also orthogonal to the basis functions of $S$, that is

$$
\frac{1}{M}s^H(v)U_q = 0, \text{ for } s(v) \notin S,
$$

(26)

where $0$ denotes a $1 \times D$ row vector.

To evaluate the spectrum leakage, denote $A(v)$ as the spectrum of the transmit signal, which
is defined as the Fourier transform of the transmit signal over different antennas, that is
\[ A(v) = \frac{1}{M} s^H(v) U_q v[k] x[k]. \] (27)

Then, using the relation in (26), we can obtain
\[ A(v) = 0 \text{ for } v \notin V_q, \] (28)

which means the transmit signal to the undesired direction is zero.

In practical systems, the antenna number is finite and therefore the eigen-matrix based approach causes much smaller leakage than the DFT-submatrix based one even though the spectrum leakage cannot be avoided. As a result, more power can be concentrated on the direction of interest, leading to improved performance as shown in the simulation results.

As an example in Fig. 3, we consider the spectra for an \( M = 64 \) antenna case with \( Q = 8 \). The wave numbers of different paths are assumed to be uniformly distributed within \([0, 1/Q]\),
corresponding to an AoD range $[0, \arcsin(2/Q)]$. In this case, the codeword spectrum with $q = 4$ is the desired inner precoder, and the ideal outer precoder with $D = M/Q - 1$ is assumed for both the eigen-matrix based inner precoder and the DFT-matrix based one. The desired codeword spectrum is also shown in the figure for comparison. As expected, the eigen-matrix based inner precoder has much lower spectrum leakage than the DFT-submatrix based precoder. The reduction of spectrum leakage can not only increase the transmit power on the desired direction but also degrades interferences to the undesired directions.

**B. Average SNR**

To gain some insights, we assume that the wave numbers of the paths are integer times of the inverse of the antenna number in the subsequential of this section. In this case,

\[ v_l = \frac{n_l}{M}, \]

where $n_l$’s are arbitrary integers from 0 to $M - 1$. In this case, the spatial correlation matrix in (4) becomes a circular matrix and the eigen-matrix in (4) can be replaced by a DFT matrix,

\[ F = \{ \frac{1}{\sqrt{M}} e^{-j\frac{2\pi mn}{M}} \}_{M-1}^{m,n=0}. \]

Correspondingly, the eigen-matrix for the $q$-th codeword can be replaced by $F_q \in C^{M \times D}$ which is a DFT-submatrix including the $\left[qMQ\right]$-th to the $\left[(q+1)MQ - 1\right]$-th columns of $F$.

Suppose the $q$-th codeword has the largest average SNR, given by

\[ \gamma_q = \text{Tr}(F_q^H \Lambda F_q), \]

\[ = M \sum_{n_l \in N_q} \sigma_{n_l}^2, \]

where $N_q = \{m\lfloor \frac{qM}{Q} \rfloor \leq m \leq \frac{(q+1)M}{Q} - 1 \}$ denotes the range of the wave number covered by the $q$-th codeword spectrum. Apparently, the reduction of $Q$ will increase the width of the codeword spectrum such that more beams can be included for the desired codeword. As a result, the average SNR can be improved by increasing the width of the codeword spectrum because more
channel power can be captured.

C. SNR Loss

The SNR loss denotes the degradation of the average SNR compared to that of the ideal inner precoder where the spatial covariance matrix is perfectly known at the BS. We will show that increasing the width of the codeword spectrum will almost always cause more SNR loss, even though it can improve the average SNR.

Denote $\gamma_q[D]$ and $\gamma_o[D]$ to be the average SNRs for the proposed approach and the ideal inner precoder both with dimension $D$. Then the SNR loss is defined as

$$L[D] = \gamma_o[D] - \gamma_q[D].$$  \hfill (32)

From the last subsection, the average SNR can be increased by widening the codeword spectrum because more channel power can be included. Denote $\Delta_q[D]$ and $\Delta_o[D]$ to be the powers of the newly included beams for the proposed approach and the ideal outer precoder,

$$\gamma_q[D + 1] - \gamma_q[D] = M\Delta_q[D],$$  \hfill (33)

$$\gamma_o[D + 1] - \gamma_o[D] = M\Delta_o[D].$$  \hfill (34)

For a set of sorted beam strengths, $\sigma_0^2 > \sigma_1^2 > \cdots > \sigma_{M-1}^2$, $\Delta_o[D] = \sigma_D^2$ which is exactly the $(D + 1)$-th entry of this set. For the proposed approach, however, the newly included beam is unknown, and thus $\Delta_q[D]$ is random.

To gain insightful results, $\Delta_q[D]$ is assumed to be uniformly distributed within the ordered set above. In this case, we have

$$\Pr(\Delta_o[D] \geq \Delta_q[D]) = \frac{M - D}{M},$$  \hfill (35)

$$\Pr(\Delta_o[D] < \Delta_q[D]) = \frac{D}{M}.$$  \hfill (36)

\footnote{Strictly speaking, this assumption is inaccurate because $D$ paths have been picked out to form the inner precoder. Even though, this assumption can simplify the analysis and we thus adopt it here.}
Since $D \ll M$, we therefore almost always have $\Delta_o[D] > \Delta_q[D]$. It means that the newly included beam for the ideal inner precoder is almost always stronger than that of the proposed approach.

Subtracting (33) from (34),

$$L[D + 1] - L[D] = M (\Delta_o[D] - \Delta_q[D]).$$ (37)

As a result, we have $L[D + 1] > L[D]$ with high probability. In other words, increasing the dimension of the outer precoder, which is equivalent to widening the codeword spectrum, will almost always cause more SNR loss. Therefore, a narrow codeword spectrum is required to save the SNR loss. The above analysis has been also confirmed by the simulation results.

V. PRACTICAL CONSIDERATION

In this section, we will discuss practical issues of the proposed codebook, including a multi-codeword feedback strategy and a specially designed downlink training for SNR estimation.

A. Multi-Codeword Feedback

In Section III, our codebook is designed based on the assumption that the AoDs are within a local range [11], [24], which may not be for some practical channels. On the other hand, the analysis in Section IV shows that widening the codeword spectrum can improve the SNR but also causes more SNR loss compared to the ideal precoder.

To address the above issues and achieve a tradeoff, we can adopt a relatively narrower codeword spectrum by using a larger $Q$ such that the SNR loss compared to the ideal inner precoder can be kept small, and then feed back multiple codewords to the BS so as to maintain the captured channel power unchanged. Actually, when the AoDs are distributed in a much wider range, multiple codewords allow us to obtain the channel power more efficiently so that the SNR can be even improved.

In terms of the feedback strategy, the multi-codeword feedback is actually similar to the narrow DFT beam based inner precoding. They both feed back multiple codewords to the BS.
except that the beam widths are different. For the narrow DFT beam based approach, the DFT matrix divides the wave number domain into $M$ discrete narrow beams, with each narrow beam corresponding to a wave number range of width $\frac{1}{M}$. Ideally, the UE should pick the $D$ strongest narrow beams in order to achieve the best performance, which will cause $D \log_2 M$-bit feedback overhead. On the other hand, each codeword in the proposed approach corresponds to a beam of width $M/Q$. In this sense, the proposed approach can be considered as a generalization of the narrow DFT beam based approach.

B. SNR Estimation

To determine the best codeword, the UE needs to estimate the average SNR for each codeword. In above, we have assumed an ideal outer precoder for the codebook design, which implies that the effective channel is known in advance. In practical systems, the effective channel cannot be determined until the BS determines the inner precoder. As a result, the ideal outer precoder cannot be used for practical downlink training.

We propose a specially designed downlink training such that the UE can obtain SNR estimation for each codeword with only one training symbol. In particular, we adopt a repetitive subcarrier allocation strategy for all the codewords over the whole bandwidth, and an outer precoder with random coefficients.

The subcarrier allocation strategy allows the UE to obtain the average SNR by averaging over instantaneous channels. As in Fig. 4, we divide all $K$ subcarriers into $K/Q$ groups. Every group has $Q$ subcarriers with each subcarrier allocated to one individual codeword. From the figure, the $q$-th codeword will occupy the subcarriers with indexes $pQ + q$'s where $p = 0, 1, \cdots, K/Q - 1$. 

---

**Fig. 4.** Allocation of subcarriers for different codeword matrices.
For the outer precoder, it is randomly generated such that

$$E(v[pQ + q]v^H[p_1 Q + q]) = \delta[p - p_1]I,$$  \hspace{1cm} (38)

where the gain has been normalized for simple notation. There is no particular requirement on the distribution of those random coefficients and we therefore assume they are generated from a quadrature-phase-shift-keying (QPSK) constellation for simplicity. The independency of the precoder coefficients allows to separate different dimensions such that the UE can obtain the channel powers from each dimension individually.

With the subcarrier allocation and the random outer precoder, the estimation of the average SNR for the $q$-th quantized covariance matrix can be given by

$$\hat{\gamma}_q = \frac{Q}{K} \sum_{p=0}^{K-1} |x^*[pQ + q]y[pQ + q]|^2$$

$$= \frac{Q}{K} \sum_{p=0}^{K-1} \left\{ |h^H[pQ + q]U_q v[pQ + q]|^2 + |z[pQ + q]|^2 + 2\Re(h^H[pQ + q]U_q v[pQ + q]x^*[pQ + q]z[pQ + q]) \right\},$$  \hspace{1cm} (39)

where $x[pQ+q]$’s denotes the training symbols. If we assume the number of subcarriers allocated to each codeword is large enough, the sample average in (39) can be approximated by the statistical average. From Appendix C, we have

$$E(|h^H[pQ + q]U_q v[pQ + q]|^2) = \text{Tr}(U_q^H R U_q),$$  \hspace{1cm} (40)

$$E(|z[pQ + q]|^2) = N_0,$$  \hspace{1cm} (41)

$$E(h^H[pQ + q]U_q v[pQ + q]x^*[pQ + q]z[pQ + q]) = 0,$$  \hspace{1cm} (42)

where we have used the identity (38) for (40). Substituting (40) to (42) into (39), the SNR estimation is then reduced to

$$\hat{\gamma}_q = \gamma_q + N_0.$$  \hspace{1cm} (43)
Since the noise power, $N_0$, is constant for different codewords, selection of the codeword using the SNR estimation is then equivalent to that using the true SNR.

The \textit{mean-square-error} (MSE) for the SNR estimation can be defined as

$$\text{MSE} = \frac{1}{Q} \sum_{q=0}^{Q-1} \mathbb{E} \left( \frac{|\hat{\gamma}_q - (\gamma_q + N_0)|^2}{|\gamma_q + N_0|^2} \right).$$ (44)

It is in general difficult to derive a closed form for (44) because the higher order statistics make the calculation intractable. We therefore evaluate the MSE in the simulation as in Section VI.

\section{VI. Simulation Results}

In this section, computer simulation is conducted to verify the proposed approach. In the simulation, we consider an OFDM modulation with 15 KHz subcarrier spacing and a typical ULA with $M = 64$ antennas spaced by the half wave-length. The channel is composed of $L = 20$ paths. An exponential power delay profile is assumed and the delay spreads are uniformly distributed within $[0, \tau_{\text{max}}]$, where $\tau_{\text{max}} = 5 \mu$s indicates the maximum delay spread. In practical systems, the AoDs for different paths can be distributed within a local range or a rather wide range. To take various cases into account, the AoDs for different paths are assumed independently and uniformly distributed within a range that has a random central angle and a random angular spread uniformly distributed within $[-\pi, \pi]$ and $[0, \pi/2]$, respectively. For comparison, we also consider the narrow DFT beam based approach that will cause $D \log_2 M$-bit feedback overhead, and the wide DFT beam based approach with each wide beam covering $M/Q$ narrow beams.

Fig. 5 shows the MSE of the SNR estimation for different numbers of quantized matrices. From the figure, a smaller $Q$, corresponding to a wider codeword spectrum, can achieve better estimation performance. For a smaller $Q$, more subcarriers can be allocated to a codeword such that the average can be conducted with more samples, leading to the improved estimation accuracy.

Fig. 6 shows SNR loss for different sizes of the codeword spectra. A smaller $Q$ suffers from more SNR loss, which coincides with our analysis in Section VI. With a large probability, the newly included beam for the ideal precoder is stronger than that for the proposed approach, and
Fig. 5. MSE of the SNR estimation for different numbers of quantized covariance matrices.

Fig. 6. SNR loss for different sizes of the codeword spectra.
thus the SNR loss is increased for a wider codeword spectrum.

Fig. 7 compares the single-codeword feedback with $Q = 8$ and the multi-codeword feedback where we consider a two codeword case with $Q = 16$. As expected, two-codeword feedback can achieve improved performance because we can not only obtain more channel power but also reduce the SNR loss, at the cost of 5-bit more overhead. On the other hand, the eigen-matrix based approach can concentrate more power on the direction of interest and therefore outperform the DFT-matrix based one. The ideal precoder is also shown in the figure as a benchmark, which shows an about 1.5 dB gap from the proposed approach with two-codeword feedback.

Fig. 8 compares the proposed approach and the narrow DFT beam based one. For the narrow DFT beam, the feedback overhead is $D \log_2 M$. Given $D = 6$, although the narrow DFT beam based approach can achieve 0.7 dB improvement than the proposed approach with two-codeword feedback, it needs 36-bit feedback overhead while the proposed approach only needs 8-bit feedback. To reduce the feedback overhead, the narrow DFT beam based approach has to reduce the dimension number, resulting a significant performance degradation. On the other
Fig. 8. Comparison between the proposed approach and narrow DFT beam based approach.

VII. CONCLUSIONS

In this paper, we have investigated the spectrum quantization approach for the codebook design in massive MIMO systems with cascaded precoding. Our analysis shows that the proposed approach can achieve much smaller spectrum leakage than the traditional DFT-submatrix based approach. Practical issues including the multi-codeword feedback and the downlink training have also been addressed in this paper. Our simulation results show that the proposed approach can significantly reduce the feedback overhead at the cost of a slight performance degradation. Given the same amount of the feedback overhead, the proposed approach can outperform the DFT based approach.
APPENDIX A

For any non-zero vector \( w \in \mathbb{C}^{D \times 1} \), we have

\[
E(|w^H U_q^H h[k]|^2) = E(w^H U_q^H h[k] h^H [k] U_q w) = w^H U_q^H R U_q w. \tag{A.1}
\]

Since \( E(|w^H U_q h[k]|^2) \geq 0 \), \( w^H U_q R U_q w \geq 0 \) holds for any non-zero vector, \( w \). As a result, \( U_q R U_q \) is a semi-positive definite matrix, and thus \( [U_q^H R U_q]_{(m,m)} \geq 0 \) for \( 0 \leq m \leq D - 1 \). Therefore,

\[
\text{Tr}(U_q^H R U_q) = \sum_{m=0}^{D-1} [U_q^H R U_q]_{(m,m)}
\]

\[
= \lambda_{q, \min}^{-1} \sum_{m=0}^{D-1} \lambda_{q, \min} [U_q^H R U_q]_{(m,m)}
\]

\[
\leq \lambda_{q, \min}^{-1} \sum_{m=0}^{D-1} \lambda_{q, \min}^{\frac{1}{2}} [m] [U_q^H R U_q]_{(m,m)} \lambda_{q, \min}^{\frac{1}{2}} [m]
\]

\[
= \lambda_{q, \min}^{-1} \text{Tr}(U_q^H R U_q \Lambda_q^{\frac{1}{2}}) \Lambda_q^{\frac{1}{2}}. \tag{A.2}
\]

where we have used the fact \( \lambda_q[m] > 0 \). Then, using \( \text{Tr}(AB) = \text{Tr}(BA) \) where \( A \) and \( B \) are two arbitrary matrices, we obtain

\[
\text{Tr}(U_q^H R U_q) \leq \lambda_{q, \min}^{-1} \text{Tr}(R_q R). \tag{A.3}
\]

Similarly, we can obtain

\[
\lambda_{q, \max}^{-1} \text{Tr}(R_q R) \leq \text{Tr}(U_q^H R U_q). \tag{A.4}
\]

APPENDIX B

Denote \( S_N \) to be an \( M \times N \) matrix,

\[
S_N = \left\{ s \left( -\frac{1}{2} + \frac{q}{Q} + \frac{n}{QN} \right) \right\}_{n=0}^{N-1}. \tag{B.1}
\]

Then the subspace in (24) can be viewed as the column space of \( S_\infty \). Therefore, the orthogonal basis functions of subspace \( S \) is determined by the eigen-matrix of the EVD for \( \frac{1}{N} S_N S_N^H \) when
$N \to \infty$. On the other hand, when $N \to \infty$, we have

\[
\lim_{N \to \infty} \frac{1}{N} S_N S_N^H = Q \int_{-\frac{1}{2} + \frac{q}{4}}^{\frac{1}{2} + \frac{q}{4}} s(v) s^H(v) dv = Q R_q,
\]

where the second equation can be easily verified using (23). Therefore, the orthogonal basis functions of subspace $S$ is determined by the eigen-matrix of $R_q$, which is exactly $U_q$.

**APPENDIX C**

The left side of (40) can be represented as

\[
\mathbb{E}(|h^H [pQ + q] U_q v[pQ + q]|^2) = \sum_{d_1=0}^{D-1} \sum_{d_2=0}^{D-1} \sum_{m_1=0}^{M-1} \sum_{m_2=0}^{M-1} u_{q,d_1}[m_1] u_{q,d_2}^*[m_2] \cdot \mathbb{E}(h_{m_1}^* [pQ + q] v_{d_1} [pQ + q] h_{m_2} [pQ + q] v_{d_2}^* [pQ + q]),
\]

where $u_{q,d}[m]$ denotes the $m$-th entry on the $d$-th column of matrix $U_q$ and $v_d[pQ + q]$ denotes the $d$-th entry of the outer precoder $v[pQ + q]$. Note that the statistical average in (C.1) is with respect to both the channels and the precoding coefficients. Then, using the relation between the cumulant and the expectation [26], we have

\[
\mathbb{E}(h_{m_1}^* [pQ + q] v_{d_1} [pQ + q] h_{m_2} [pQ + q] v_{d_2}^* [pQ + q]) = \mathbb{E}(h_{m_1}^* [pQ + q] v_{d_2}^* [pQ + q]),
\]

(C.2)

since $v_d[pQ + q]$’s and $h_m[pQ + q]$’s are mutually independent. Meanwhile, we can obtain from (3) and (38) that

\[
\mathbb{E}(h_{m_1}^* [pQ + q] h_{m_2} [pQ + q]) = r[m_2 - m_1],
\]

(C.3)

\[
\mathbb{E}(v_{d_1} [pQ + q] v_{d_2}^* [pQ + q]) = \delta[d_1 - d_2],
\]

(C.4)
we therefore have

\[
E\left(\left|\mathbf{h}^H[pQ + q]\mathbf{U}_q \mathbf{v}[pQ + q]\right|^2\right) = \sum_{d=0}^{D-1} \sum_{m_1=0}^{M-1} \sum_{m_2=0}^{M-1} u_{q,d}[m_1] u^*_{q,d}[m_2] r[m_2 - m_1]
\]

\[
= \sum_{d=0}^{D-1} \mathbf{u}^H_{q,d} \mathbf{R} \mathbf{u}_{q,d},
\]

(C.5)

where \(\mathbf{u}_{q,d}\) denotes the \(d\)-th column of matrix \(\mathbf{U}_q\). Rewrite the second equation in (C.5) in a matrix form will lead to the result in (40).

Using similar approach, we can easily obtain (41) and (42) since the channels, the precoding coefficients, the transmit symbols, and the additive noise are also mutually independent.

REFERENCES


