Loop tuning with specification on gain and phase margins via modified second-order sliding mode control algorithm

Abstract - The modified second-order sliding mode algorithm is used for controller tuning. Namely the modified suboptimal algorithm based test (modified SOT) and non-parametric tuning rules for proportional-integral-derivative (PID) controllers are presented in the paper. In the developed method of test and tuning, the idea of coordinated selection of the test parameters and the controller tuning parameters is introduced. The proposed approach allows for the formulation of simple non-parametric tuning rules for PID controllers that provide desired amplitude or phase margins exactly. In the modified SOT, the frequency of the self-excited oscillations can be generated equal to either the phase crossover frequency or the magnitude crossover frequency of the open-loop system frequency response (including a future PID controller) – depending on the tuning method choice. The first option will provide tuning with specification on gain margin, and the second option will ensure tuning with specification on phase margin. Tuning rules for a PID controller and simulation examples are provided.

Keywords: controller tuning, second-order sliding mode, gain and phase margin specification.
I. INTRODUCTION

Despite the apparent success of advanced process control, the PID control still remains the main type of control in the process industries. PID controllers are implemented as either stand-alone controllers or configurable software modules within the distributed control systems (DCS). The DCS software is constantly evolving providing a number of new features, among which the controller autotuning functionality is one of most useful. It is found in the latest releases of such popular DCS as Honeywell Experion PKS® and Emerson DeltaV®. The practice of the use of tuning algorithms shows that simple non-parametric methods (the ones that are not based on any underlying models) such as Ziegler-Nichols’s closed-loop tuning method [1] and Astrom-Hagglund’s relay feedback test (RFT) [2] provide a satisfactory and consistent performance in the conditions characterized by the presence of measurement noise and varying disturbances, when deterioration of more sophisticated methods becomes significant. However, the use of only ultimate gain and frequency cannot ensure sufficient accuracy of tuning. The cause of the relatively low accuracy of [1], [2] and other non-parametric methods is well known, which is the use of only two measurements of the test over the process [3]. Therefore, a trade-off between the accuracy and reliability of tuning (which also translates into accuracy) is apparent.

There is one more factor that also contributes to the issue of accuracy. This is a popular notion that states that the most important test point in the closed-loop test is the one in which the phase characteristic of the process is $-180^\circ$ (frequency $\omega_c$). However, this approach does not account for the change of frequency $\omega_c$ due to the controller introduction, which is the factor that contributes to the deterioration of tuning accuracy.

The recently developed second- and higher-order sliding mode control algorithms find applications in the problems of estimation, identification and control [4]-[8]. The present paper aims to develop two nonparametric tuning methods, that are based on the modified second-order sliding order control algorithm presented in [4]-[5], in which the above-noted problem of the cross-over frequency shift is
compensated for by the algorithm construction. Respective tuning rules are also proposed that allows for increasing the performance of tuning based on the RFT [2].

The paper is organized as follows. At first the problem of selection of the frequency of self-excited oscillations for the test is considered. After that a modified suboptimal algorithm-based test (SOT) that provides generation of the oscillations at a given point of the phase response of the process is proposed. Then, nonparametric tuning rules that ensure the fulfillment of the requirements to the gain margin of the system with controller are derived, and the performance of the test and tuning is analyzed. After that nonparametric tuning rules that ensure the fulfillment of the requirements to the phase margin of the system with controller are derived, and the performance of the test and tuning is analyzed. Finally, examples are given.

II. MOTIVATION FOR MODIFICATION OF RELAY FEEDBACK TEST

It has been a popular notion that the most important point on the frequency response of the system is the point where the phase characteristic of the process is equal to \(-180^\circ\) (frequency \(\omega_n\)). We shall also refer to this point as the phase cross-over frequency. However, this point remains the most important one only in the system with the proportional controller, when introduction of the controller does not change the value of \(\omega_c\). This circumstance is often neglected, and the principle is applied to all types of PID control.

We consider the following motivating example and analyze how the introduction of the controller may affect the results of identification and tuning.

Example 1. Let us assume that the process is given by the following transfer function (which was used in a number of works as a test process):

\[
W_p(s) = e^{-2s} \frac{1}{(2s + 1)^5},
\]

Find the first order plus dead time (FOPDT) approximating model \(\hat{W}_p(s)\) to the process (1) based on matching the values of the transfer functions at frequency \(\omega_c\).
\[ \hat{W}_p(s) = \frac{K_p e^{-\tau s}}{T_p s + 1}, \]  

(2)

where \( K_p \) is the process static gain, \( T_p \) is the time constant, and \( \tau \) is the dead time. Let us apply methods [1] to the tuning of process (1) and note that both (1) and (2) should produce the same ultimate gain and ultimate frequency in the Ziegler-Nichols closed-loop test [1] or the same values of the amplitude and the ultimate frequency in the RFT [2]. (Note: strictly speaking, the values of the ultimate frequency in tests [1] and [2] are slightly different, as the frequency of the oscillations generated in the RFT does not exactly correspond to the phase characteristic of the process \(-180^\circ\); this fact follows from the relay systems theory [9] - [11]; however, we shall use the describing function method and disregard the errors of the method). Obviously, this problem has infinite number of solutions, as there are three unknown parameters of (2) and only two measurements obtained from the test. Assume that the value of the process static gain is known: \( K_p=1 \), and determine \( T_p \) and \( \tau \). These parameters can be found from equation

\[ \hat{W}_p(j \omega) = W_p(j \omega), \]

where \( \omega \) is the phase cross-over frequency for both transfer functions. Therefore, \( \arg W_p(j \omega) = -\pi \).

The value of \( \omega \) is 0.283, which gives \( W_p(j \omega) = (-0.498, j0) \), and the FOPDT approximation is, therefore, as follows (found via solution of the set of two algebraic equations):

\[ \hat{W}_p(s) = \frac{e^{-7.393s}}{6.153s + 1}. \]  

(3)

The Nyquist plots of the process (1) and its approximation (3) are depicted in Fig. 1 (the meaning of frequencies \( \Omega_1 \) and \( \Omega_2 \) is explained below). The point of intersection of the two plots (denoted as \( \Omega_0 \)) is also the point of intersection with the real axis. Also \( \Omega_0 = \omega \) for both process dynamics (1) and (3), and therefore \( \hat{W}_p(j \Omega_0) = W_p(j \Omega_0) \). If the designed controller is of proportional type then the gain margins for processes (1) and (3) are the same. However, if the controller is of PI type then the stability
margins for (1) and (2) are different. We illustrate that below. Design the PI controller given by the following transfer function:

\[ W_c(s) = K_c \left(1 + \frac{1}{T_c s}\right), \tag{4} \]

where \( K_c \) is the proportional gain, \( T_c \) is the integral time constant of the controller, using the Ziegler-Nichols tuning rules [1]. This results in the following transfer function of the controller:

\[ W_c(s) = 0.803 \left(1 + \frac{1}{17.76s}\right), \tag{5} \]

Fig. 1. Nyquist plots for process (1) and FOPDT approximation (3)

Fig. 2. Nyquist plots for open-loop system with PI controller and process
The Nyquist plots of the open-loop systems containing the process (1) or its approximation (3) and the controller (5) are depicted in Fig. 2. It follows from the frequency-domain theory of linear systems and the used tuning rules that the mapping of point $\Omega_0$ in Fig. 1 into point $\Omega_0$ in Fig. 2 is done via clockwise rotation of vector $\tilde{W}_p(j\Omega_0)$ by the angle $\psi = \arctan\left(1/(0.8 \cdot 2\pi)\right) = 11.25^\circ$ and multiplication of its length by such value, so that its length becomes equal to 0.408. This is possible due to the serial connection of the controller and the process and the possibility of treating their frequency response (at $\Omega_0$) as vectors. However, for the open-loop system containing the PI controller, the points of intersection of the Nyquist plots of the system and of the real axis are different for the system with process (1) and with process approximation (3). They are shown as points $\Omega_1$ and $\Omega_2$ in Fig. 2. The points of frequencies $\Omega_1$ and $\Omega_2$ on the Nyquist plots of the original process and its approximation, respectively, are also shown in Fig. 1. Therefore, the stability margins of the systems containing a PI controller are not the same any more. It is revealed as different points of intersection of the plots and of the real axis in Fig. 2. In fact the position of vector $\tilde{W}_{ol}(j\Omega_0) = \tilde{W}_c(j\Omega_0)\tilde{W}_p(j\Omega_0)$ is fixed, but this vector does not reflect the stability of the system. As one can see in Fig. 2, the gain margin of the system containing the FOPDT approximation of the process is higher than the one of the system with the original process.

The considered example illustrates a fundamental problem of all methods of identification-tuning based on the measurements of process response in the critical point ($\Omega_0$). This problem is the shift of the critical point due to the introduction of the controller. The question that follows from the above analysis is whether the test point can be selected in a different way, so that the introduction of the controller would be accounted for in the test itself. And if this is possible then what kind of test it should be to ensure the measurements in the desired test point.
We address the first question now. Assume that we can design a certain test, so that we can generate the test frequency at the desired phase lag of the process \( \arg W_p(j\Omega_0) = \varphi \), where \( \varphi \) is a given quantity, and measure \( W_p(j\Omega_0) \) in this point. Consider the following example.

**Example 2.** Let the plant be the same as in Example 1. Assume that the introduction of the controller will be equivalent to the mapping similar to the mapping described above – the vector of the frequency response of the open-loop system in the point \( \Omega_0 \) will be a result of clockwise rotation of the vector \( \hat{W}_p(j\Omega_0) \) by a known angle and multiplication by a certain known factor:

\[
\hat{W}_{ol}(j\Omega_0) = \hat{W}_c(j\Omega_0)\hat{W}_p(j\Omega_0).
\]

Also assume that the controller will be the same as in Example 1 (for illustrative purpose - because the tuning rules are not formulated yet). Therefore, let us find the values of \( T_p \) and \( \tau \) for the transfer function (2) (we still assume \( K_p=1 \)) that ensure that the equality \( \hat{W}_p(j\Omega_0) = W_p(j\Omega_0) \) holds, where \( \arg W_p(j\Omega_0) = -180^\circ + 11.25^\circ = -168.75^\circ \) (the angle is selected considering the subsequent clockwise rotation by 11.25\(^\circ\)). Therefore, \( \Omega_0 = 0.263 \), and \( W_p(j\Omega_0) = (-0.532, -j0.103) \). The corresponding FOPDT approximation of the process is

\[
\hat{W}_p(s) = \frac{e^{-7.293s}}{5.897s + 1},
\]

Application of controller (5) shifts the point \( \Omega_0 \) of intersection of \( W_p(j\Omega_0) \) and \( \hat{W}_p(j\Omega_0) \) to the real axis. This point remains to be the point of intersection of the two Nyquist plots. Therefore, the gain margin of both systems: with the original process and with the approximated process are the same. Consider now the problem of the design of the test that can provide matching the points of the actual and approximating processes in the point corresponding to a specified phase lag.

**III. Modified suboptimal algorithm based test**

Consider the following discontinuous control:
\[ u(t) = \begin{cases} h & \text{if } \sigma(t) \geq \Delta_1 \text{ or } (\sigma(t) > -\Delta_2 \text{ and } u(t-) = h) \\ -h & \text{if } \sigma(t) \leq \Delta_2 \text{ or } (\sigma(t) < \Delta_1 \text{ and } u(t-) = -h) \end{cases} \]  

(7)

where \( \Delta_1 = \beta \sigma_{\max} \), \( \Delta_2 = -\beta \sigma_{\min} \), \( \sigma_{\max} \) and \( \sigma_{\min} \) are last “singular” points of the error signal (Fig. 3) corresponding to the last maximum and minimum values of \( \sigma(t) \) after crossing the zero level, \( u(t-) = \lim_{\varepsilon \to 0, \varepsilon > 0} u(t-\varepsilon) \) is the control value at the time immediately preceding current time \( t \), \( h \) is the amplitude of the relay, \( \beta \) is a positive constant. Initial value of \( \sigma_{\max} \) or \( \sigma_{\min} \) can be assigned as \( \sigma(0) \) (the choice of either \( \sigma_{\max} \) or \( \sigma_{\min} \) depends on the sign of \( \sigma(0) \)). The algorithm (7) is similar to the so-called “generalized sub-optimal” algorithm used for generating a second-order sliding mode in systems of relative degree two [4], [5]. The difference between the two is that the generalized sub-optimal algorithm involves an advance switching of the relay (in the algorithm formulation (7), parameter \( \beta < 0 \)), which is aimed at driving the system states to zero, but the proposed algorithm is aimed at generating a limit cycle and normally involves a lagged switching of the relay (parameter \( \beta > 0 \); however, the advance switching is possible too), in which the lag value is coordinated with controller tuning rules.

![Fig. 3. Relay feedback test](image)

![Fig. 4. Finding periodic solution](image)
Let the reference signal \( r(t) \) be zero in Fig. 3. We show now that in the steady mode, the motions in the system Fig. 3, where the control is given by (7) are periodic. Apply the describing function (DF) method [12] to the analysis of motions in Fig. 3. DF is an approximate method but – as confirmed by the practice of applications of RFT in process control – provides the model of satisfactory accuracy. Assume that the steady mode periodic, and prove that this is a valid assumption by finding parameters of this periodic motion. If the motions in the system are periodic then \( \sigma_{\text{max}} \) and \( \sigma_{\text{min}} \) represent the amplitude of the oscillations: \( a_0 = \sigma_{\text{max}} = -\sigma_{\text{min}} \), and the equivalent hysteresis value of the relay is \( \Delta = \Delta_1 = \Delta_2 = \beta \sigma_{\text{max}} = -\beta \sigma_{\text{min}} \). The DF of the hysteretic relay is given as the following function of amplitude \( a \) [12]:

\[
N(a) = \frac{4h}{\pi a} \sqrt{1 - \left(\frac{\Delta}{a}\right)^2} - j \frac{4h\Delta}{\pi a^2}, \quad a > \Delta
\]  

(8)

However, system Fig.3 with control (7) is not a conventional relay system. This system has the hysteresis value that is unknown \( a \)-priori and depends on the amplitude value: \( \Delta = \beta a \). Therefore, (8) can be rewritten as follows:

\[
N(a) = \frac{4h}{\pi a} \left(\sqrt{1 - \beta^2} - j\beta\right),
\]  

(9)

The RFT will generate oscillations in the system under control (7). We shall further refer to that test as to the “modified relay feedback test”. Parameters of the oscillations can be found from the harmonic balance equation:

\[
W_p(j\Omega_0) = -\frac{1}{N(a_0)},
\]  

(10)

where \( a_0 \) and \( \Omega_0 \) are the amplitude and the frequency of the periodic motions. The negative reciprocal of the DF is given as follows:

\[
-\frac{1}{N(a)} = -\frac{\pi a}{4h} \left(\sqrt{1 - \beta^2} + j\beta\right)
\]  

(11)
Finding a periodic solution in system Fig.3 with control (7) has a simple graphic interpretation (Fig. 4) as finding the point of intersection of the Nyquist plot of the process and of the negative reciprocal of the DF, which is a straight line that begins in the origin and makes a counterclockwise angle \( \psi = \arcsin \beta \) with the negative part of the real axis. The condition of the existence of a periodic solution is the location of the Nyquist plot of the process in the third quadrant of the complex plane (Fig. 4). This condition is always satisfied in practice because of the inevitable existence of small delays in process dynamics. In addition, the condition of orbital stability of the periodic solution must be satisfied, which may take place in the case of both stable and unstable process.

In the problem of analysis, frequency \( \Omega_0 \) and amplitude \( a_0 \) are unknown variables and are found from the complex equation (10). In the problems of identification and tuning, \( \Omega_0 \) and \( a_0 \) are measured from the modified SOT, and on the basis of the measurements obtained either parameters of the underlying model are calculated (for parametric tuning) or tuning parameters are calculated immediately from \( \Omega_0 \) and \( a_0 \) (for non-parametric tuning).

Reviewing again Example 2, we can note that if, for example, Ziegler-Nichols tuning rules were to be applied, and the subsequent transformation via introduction of the PI controller involving clockwise rotation by angle \( \psi = \arctan \frac{1}{0.8 \cdot 2\pi} = 11.25^\circ \) was going to be applied, then parameter \( \beta \) of the controller for the modified SOT should be \( \beta = \sin 11.25^\circ = 0.195 \). The modified SOT also allows for the exact design of the gain margin (assuming the DF method provides an exact model). Since the amplitude of the oscillations \( a_0 \) is measured from the test, the process gain at frequency \( \Omega_0 \) can be obtained as follows:

\[
|W_p(j\Omega_0)| = \frac{\pi a_0}{4h},
\]

which after introduction of the controller will become the process gain at the critical frequency.
Let the PID controller transfer function be $W_c(s) = K_c \left( 1 + \frac{1}{T_{ic}s} + T_{dc}s \right)$. Define the following tuning rules format of a PID controller:

$$K_c = c_1 \frac{4h}{\pi \omega_0}, \quad T_{ic} = c_2 \frac{2\pi}{\Omega_0}, \quad T_{dc} = c_3 \frac{2\pi}{\Omega_0},$$

(13)

where $c_1$, $c_2$, and $c_3$ are constant parameters that define the tuning rule, to which we shall refer as to the **homogeneous tuning rules**, because the tuning parameters are homogeneous functions of critical gain (given by $\frac{4h}{\pi \omega_0}$) and critical period (inverse critical frequency). The idea behind the tuning rules (13) is the possibility of scaling of tuning parameters for processes having different time scales. It can be noted that if the tuning rules are given by (13) then the closed-loop system characteristics become invariant to the time constants of the process, so that if all time constants of the process were increased by the factor $\rho$ then the critical frequency would decrease by the same factor $\rho$, and the product of every time constant by the critical frequency would remain unchanged. The use of the **homogeneous tuning rules** along with the modified SOT will allow us to fully utilize the features of non-parametric tuning methods. If homogeneous tuning rules (13) are used then the frequency response of the PID controller at the frequency $\Omega_0$ becomes

$$W_c(j\Omega_0) = c_1 \frac{4h}{\pi \omega_0} \left( 1 - j \frac{1}{2\pi c_2} + j2\pi c_3 \right).$$

(14)

Therefore, if the tuning rules are established through the choice of parameters $c_1$, $c_2$, $c_3$, and the test provides self-excited oscillations of the frequency $\Omega_0$, which will be equal to the phase cross-over frequency $\omega_\pi$ of the open-loop system (including the controller), then the controller phase lag at the frequency $\omega_\pi = \Omega_0$ will depend only on the values of $c_2$ and $c_3$:

$$\phi_c(\omega_\pi) = \arctan \left( 2\pi c_3 - \frac{1}{2\pi c_2} \right),$$

(15)

which directly follows from formula (14) if $\omega_\pi = \Omega_0$. 

IV. Non-parametric Tuning Rules for Specification on Gain Margin

It seems that the most appealing application of the presented test is a non-parametric tuning. However, given a large variety of possible process dynamics, it is difficult to formulate certain universal rules for tuning. In practice of process control, tuning rules that provide a less aggressive response than the one provided by IAE, ITAE criteria or Ziegler-Nichols formulas (and others) are widely used. This approach is motivated by the consideration of safety, which chosen versus to high performance. This trend is reflected in the review of the modern PID control given in [13].

We derive now the relationship that would allow us to tune PID controllers with specification on gain margin for the open-loop system. Let the specified gain margin be $\gamma_m > 1$ (in absolute values). Then taking absolute values of both sides of (14) and considering (12) we obtain the following equation:

$$\gamma_m c_1 \sqrt{1 + \left(2 \pi c_3^2 - \frac{1}{2 \pi c_2^2}\right)^2} = 1,$$

which is a constraint that complementary to the tuning rules (13). To provide the specified gain margin, the modified SOT must be carried out with parameter

$$\beta = -\sin \varphi_x (\Omega_0) = -\sin \arctg \left(2 \pi c_3^2 - \frac{1}{2 \pi c_2^2}\right)$$

In the example considered above, if we keep parameter $c_2$ the same as [1]: $c_2 = 0.8$, then to obtain, for example, gain margin $\gamma_m = 2$ the tuning parameter $c_1$ for the modified SOT should be selected as $c_1 = 0.49$, and parameter $\beta$ for the test should be selected in accordance with (17) as $\beta = 0.195$. For any process, the system will have gain margin $\gamma_m = 2$ (6dB) exactly (within the framework of the filtering hypothesis of the DF method). Therefore, the modified SOT with parameter $\beta$ calculated as (17) and tuning rules (13) satisfying the constraint (16) can ensure the desired gain margin. However, (16) is an equation containing three unknown variables, which gives one a freedom to vary parameters $c_1, c_2, c_3$. We do not consider in this paper the problem of optimal selection of these parameters. However, some simple tuning rules can easily be obtained with the utilization of some characteristics of the tuning rules of [1].
In particular, we assume that the controller should provide the same phase response on the frequency of oscillation of the modified SOT $\varphi_c(\Omega_0)$ as the phase response of the corresponding controller at the critical frequency of conventional RFT tuned in accordance with the rules [1]. As a result, we can use values of parameters $c_2, c_3$ equal to the corresponding values of rules [1]. Tuning rules for gain margin $\gamma_m=2$ in the format of values of parameters $c_1, c_2, c_3$ along with the values of $\varphi_c(\Omega_0)$ and parameter $\beta$ for the test are given in Table 1.

One should note the difference between the values of the critical frequency of the conventional RFT and the frequency of oscillations in the modified SOT (except for the proportional controller). Therefore, even if the coefficients $c_2, c_3$ of Table 1 have the same values as corresponding coefficients of [1], they will actually produce different values of controller parameters $T_{ic}$ and $T_{dc}$. In fact, due to the negative value of $\varphi_c(\Omega_0)$ for the PI controller (and consequently, lower frequency of oscillations of the modified SOT), one would get higher value of $T_{ic}$ computed through the modified SOT and data of Table 1. And vice versa, due to the positive value of $\varphi_c(\Omega_0)$ for the PID controller, one would get lower values of $T_{ic}$ and $T_{dc}$ computed through the modified SOT and data of Table 1.

V. NON-PARAMETRIC TUNING RULES FOR SPECIFICATION ON PHASE MARGIN

We derive now the relationship that would allow us to tune PID controllers with specification on phase margin for the open-loop system. Using the same format of the homogeneous tuning rules (13), and considering that if the parameter $\beta$ of the modified SOT is calculated from the sum of $\varphi_c(\Omega_0)$ and the phase margin $\phi_m$ as:

$$\beta = \sin(\phi_m - \varphi_c(\Omega_0)) = \sin(\phi_m + \arctg\left(\frac{1}{2\pi c_2 - 2\pi c_3}\right)),$$

we formulate the constraint for the tuning rules ensuring $\phi_m$ as follows:
The graphical interpretation of modified SOT and tuning with specification on phase margin are presented in Fig. 5.

Indeed, if tuning rules (13) are subject to constraint (19) then at frequency \( \Omega_0 \) of the modified SOT: (a) the absolute value of the open-loop frequency response, in accordance with (12), (14), is

\[
|W_{ol}(j\Omega_0)| = |W_c(j\Omega_0)| |W_p(j\Omega_0)| = \frac{\pi a}{4h} c_1 \sqrt{1 + \left( \frac{2\pi c_3 - \frac{1}{2\pi c_2}}{2\pi c_3 - \frac{1}{2\pi c_2}} \right)^2} = c_1 \sqrt{1 + \left( \frac{2\pi c_3 - \frac{1}{2\pi c_2}}{2\pi c_3 - \frac{1}{2\pi c_2}} \right)^2} = 1,
\]

that constitutes the magnitude cross-over frequency, and (b) the phase of the open-loop frequency response is \( \arg W_{ol}(j\Omega_0) = \arg W_c(j\Omega_0) + \arg W_p(j\Omega_0) = -180^\circ + (\phi_m - \varphi_c(\Omega_0)) + \varphi_c(\Omega_0) = -180^\circ + \phi_m \), which shows that the specification on the phase margin is satisfied. Again, assuming that the controller at frequency \( \Omega_0 \) of the modified SOT should provide the same phase response as at critical frequency of RFT (which, of course, does not give optimal tuning rules and the tuning rules are provided as a possible option with an illustrative purpose), we can obtain the following values of parameters \( c_1, c_2, c_3 \) (see Table 2 for \( \phi_m = 45^\circ \)). Like in the tuning with specification on gain margin, one should note the
difference between the values of the critical frequency of the conventional RFT and the frequency of oscillations in the modified SOT, which will result in different values of the controller parameters.

It is noted above that the derived relationships (16), (19) between the coefficients $c_1$, $c_2$, $c_3$ do not provide tuning rules yet. The problem of development of optimal tuning rules satisfying constraints (16) or (19) still remains unsolved, and different plant/process dynamics will require different optimal tuning rules. Tables 1 and 2 provide examples of not optimal but satisfactory tuning rules that were generated on the basis of [1] by keeping the same values of $c_2$, $c_3$ and finding $c_1$ that would satisfy (16) or (19). Optimal tuning rules may be produced for a particular class of plant/process dynamics via solving the problem of parametric optimization for $c_1$, $c_2$, $c_3$ with a certain criterion (IAE, ITAE or other) and constrains (16), (19). The solution of this problem is still ahead.

The controller tuning can be described as the following step-by-step algorithm:

A) The type of controller (P, PI or PID) and the tuning rules (for example, given by either Table 1 or Table 2, or generated in a similar way for other values of gain/phase margin) are selected.

B) The modified SOT with parameter $\beta$ corresponding to the selection made at step A is carried out.

C) The values of frequency $\Omega_0$ and amplitude $a_0$ of the self-excited oscillations in the system are measured.

D) Tuning parameters of the controller of the controller are calculated per (13).

VI. Example

Example 3. Consider the process transfer function (1) that was used in Example 1. (a) Apply the modified SOT with amplitude $h=1$, parameter $\beta = 0.195$ and $c_1$, $c_2$ values from Table 1 for tuning a PI controller with specification on gain margin $\gamma_m=2$. (b) After that use the modified SOT with amplitude $h=1$, parameter $\beta = 0.659$ and $c_1$, $c_2$ values from Table 2 for tuning a PI controller with specification on phase margin $\phi_m = 45^\circ$. The controller tuning that is done according to the presented method produces the following results. (a) The modified SOT gives $\Omega_0 = 0.263$ and $a_0=0.691$; for tuning with
specification on gain margin the controller parameters calculated per (13) are $K_c = 0.903$, $T_{ic} = 19.11$; 
(b) The modified SOT gives $\Omega_0 = 0.187$ and $a_0 = 0.918$; for tuning with specification on gain margin the controller parameters calculated per (13) are $K_c = 1.359$, $T_{ic} = 26.88$. The frequency response of the open-loop systems is presented in Fig.6. One can see that, indeed, the gain margin is two for option “a” (solid line), and the phase margin is $45^\circ$ for option “b” (dotted line).

Fig. 6. Nyquist plots of open-loop systems for Example 3.

VII. CONCLUSION
A modified SOT and two methods of non-parametric tuning of a PID controller based on this test are presented in the paper. These methods provide either specified gain margin or specified phase margin of the system with controller exactly. Examples that demonstrate the proposed method and illustrate the conclusions are given. Despite providing specified gain/phase margins, the proposed method leaves a room for optimization of the tuning rules, which is the subject of future research.

REFERENCES


### Table 1. Tuning rules for gain margin $\gamma_m=2$

<table>
<thead>
<tr>
<th>Controller</th>
<th>$c_1$</th>
<th>$c_2$</th>
<th>$c_3$</th>
<th>$\varphi_c(\Omega_0)$</th>
<th>$\beta$</th>
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<tr>
<td>P</td>
<td>0.50</td>
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<td>0.80</td>
<td>0</td>
<td>-11.2°</td>
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<td>PID</td>
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<td>0.50</td>
<td>0.12</td>
<td>23.5°</td>
<td>-0.399</td>
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### Table 2. Tuning rules for phase margin $\phi_m = 45^\circ$

<table>
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<tr>
<th>Controller</th>
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<th>$c_2$</th>
<th>$c_3$</th>
<th>$\varphi_c(\Omega_0)$</th>
<th>$\beta$</th>
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<td>23.5°</td>
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