On the Lattice of Equational Classes of Boolean Functions and Its Closed Intervals

Miguel Couceiro

Abstract. Let $A$ be a finite set with $|A| \geq 2$. The composition of two classes $I$ and $J$ of operations on $A$, is defined as the set of all composites $f(g_1, \ldots, g_n)$ with $f \in I$ and $g_1, \ldots, g_n \in J$. This binary operation gives a monoid structure to the set $E_A$ of all equational classes of operations on $A$.

The set $E_A$ of equational classes of operations on $A$ also constitutes a complete distributive lattice under intersection and union. Clones of operations, i.e. classes containing all projections and idempotent under class composition, also form a lattice which is strictly contained in $E_A$. In the Boolean case $|A| = 2$, the lattice $E_A$ contains uncountably many $(2^{|A|})$ equational classes, but only countably many of them are clones.

The aim of this paper is to provide a better understanding of this uncountable lattice of equational classes of Boolean functions, by analyzing its “closed” intervals $[C_1, C_2]$, for idempotent classes $C_1$ and $C_2$. For $|A| = 2$, we give a complete classification of all closed intervals $[C_1, C_2]$ in terms of their size, and provide a simple, necessary and sufficient condition characterizing the uncountable closed intervals of $E_A$.

1. Introduction

The characterization of the classes of operations on a set $A$, definable by means of functional equations, was first obtained in the Boolean case $A = \{0, 1\}$ by Ekin, Foldes, Hammer and Hellerstein in [4], and in a different framework by Pippenger [9]. This result was extended in [1] to arbitrary non-empty underlying sets $A$, where it was shown that these equational classes are essentially those classes $K$ satisfying $KO_A = K$, where $O_A$ denotes the class containing only projections on $A$. From this characterization it follows that the set $E_A$ of equational classes on $A$ constitutes a complete distributive lattice which properly contains the set of all clones on $A$.

In fact, the classification of operations into equational classes is much finer than the classification into clones. For example, in the Boolean case $|A| = 2$, there are uncountably many equational classes on $A$ (see e.g. [9]), but only countably many of them are clones (see [10]).

Thus it seems effortless to achieve a complete description of the lattice $E_A$, even in the case $|A| = 2$. Nevertheless, the set of equational classes, which are...
idempotent under class composition, induces a subdivision of \( E_A \) into sublattices \([C_1, C_2]\), which are in addition closed under class composition.

In this paper, we study the closed intervals of \( E_A \). The distribution of the equational classes into these intervals is not uniform: some intervals are countable, while others are uncountable. Thus it is natural to ask which are the uncountable intervals of \( E_A \).

We answer this question for \( A = B = \{0, 1\} \). In the next section, we provide definitions and terminology, as well as some preliminary results, used in the sequel. In Section 3, we introduce the lattice \( E_A \) of equational classes of operations on \( A \), and present some facts and general results concerning this lattice and its intervals \([C_1, C_2]\), for idempotent classes \( C_1 \) and \( C_2 \). In particular, we show that an interval \([C_1, C_2]\) is uncountable if and only if there is an infinite antichain of operations in \( C_2 \setminus C_1 \) with respect to the pre-order \( \preceq \) defined on Section 2.

In Section 4, we focus on the lattice \( E_3 \) of equational classes of Boolean functions. In view of the above characterization, we determine which intervals of \( E_3 \) contain only finite antichains, and provide infinite antichains of Boolean functions for the remaining closed intervals (Subsection 4.2). The classification of the closed intervals of \( E_3 \) in terms of size, is then presented in Subsection 4.3. Using this classification, we derive in Subsection 4.4 a simpler, necessary and sufficient condition characterizing the uncountable closed intervals of \( E_3 \).

2. Basic notions and preliminary results

Throughout the paper, let \( A \) be a finite set with \(|A| \geq 2\). An operation on \( A \) is a map \( f : A^n \to A \), where \( n \) is a positive integer called the arity of \( f \). If \( A = B = \{0, 1\} \), then \( f \) is called a Boolean function. The essential arity of an \( n \)-ary operation \( f : A^n \to A \) is the cardinality of the set of indices

\[
I = \{1 \leq i \leq n : \text{there are } a_1, \ldots, a_i, b, a_{i+1}, \ldots, a_n \text{ with } a_i \neq b \text{ and } f(a_1, \ldots, a_{i-1}, a_i, b, a_{i+1}, \ldots, a_n) = f(a_1, \ldots, a_{i-1}, b, a_{i+1}, \ldots, a_n)\}
\]

For each \( i \in I \), the \( i \)th variable of \( f \) is said to be essential. A variable \( x_i \) of \( f \) is called dummy if \( i \notin I \). By definition it follows that constant operations have only dummy variables. Operations of essential arity at most 1 are usually called quasi-monadic.

An operation \( f : A^n \to A \) is said to be idempotent, if \( f(x_1, \ldots, x_1) = x_1 \).

For any maps \( g_1, \ldots, g_n : A^m \to A \) and \( f : A^n \to A \), their composition is defined as the map \( f(g_1, \ldots, g_n) : A^m \to A \) given by \( f(g_1, \ldots, g_n)(a) = f(g_1(a), \ldots, g_n(a)) \), for every \( a \in A^m \). An \( n \)-ary operation \( f : A^n \to A \) is said to be associative if for any \( 2n-1 \)-ary projections \( p_k, 1 \leq k \leq 2n-1 \), and every pair of indices \( 1 \leq i < j \leq n \), we have

\[
f(p_1, \ldots, p_{i-1}, f(p_{i+1}, \ldots, p_{i+n-1}), p_{i+n}, \ldots, p_{2n-1}) =
= f(p_1, \ldots, p_{j-1}, f(p_{j+1}, \ldots, p_{j+n-1}), p_{j+n}, \ldots, p_{2n-1})
\]

Note that if \( f \) is associative of essential arity \( n \geq 2 \), then it does not have dummy variables. Also, by definition it follows that each member of the family \((f^k)_{k \geq 0}\) given by the recursion

\[
(1) \quad f^0 = f(x_1, \ldots, x_1) \text{ and } f^1 = f,
\]
(2) \( f^k = f^{k-1}(x_1, \ldots, x_{(k-1)(n-1)}, f(x_{(k-1)(n-1)+1}, \ldots, x_{(n-1)+1})) \)

is also associative. This notion of associativity for \( n \)-ary operations plays a fundamental role in the generalization of groups to \( n \)-groups (polyadic groups). For an early reference see e.g. [11], and for a bibliographic survey see [6].

If \( I, J \subseteq \bigcup_{n \geq 1} A^n \), then the class composition \( IJ \) is defined as the set

\[ IJ = \{ f(g_1, \ldots, g_n) \mid n, m \geq 1, f \text{ \( n \)-ary in } I, g_1, \ldots, g_n \text{ \( m \)-ary in } J \}. \]

If \( I \) is a singleton, \( I = \{ f \} \), then we write \( fJ \) instead of \( \{ f \}J \). Note that class composition is monotone, i.e. if \( I_1 \subseteq I_2 \) and \( J_1 \subseteq J_2 \), then \( I_1J_1 \subseteq I_2J_2 \).

Let \( O_A \) denote the class containing only projections on \( A \). An \( m \)-ary operation \( g \) on \( A \) is said to be obtained from an \( n \)-ary operation \( f \) on \( A \) by simple variable substitution, denoted \( g \preceq_V f \), if there are \( m \)-ary projections \( p_1, \ldots, p_n \in O_A \) such that \( g = f(p_1, \ldots, p_n) \). In other words,

\[ g \preceq_V f \quad \text{if and only if} \quad gO_A \subseteq fO_A. \]

Thus \( \preceq_V \) constitutes a pre-order (reflexive and transitive) on \( \bigcup_{n \geq 1} A^n \). If \( g \preceq_V f \) and \( f \preceq_V g \), then \( g \) and \( f \) are said to be equivalent. Note that if \( g \not\preceq_V f \) but \( f \not\preceq_V g \), then the essential arity of \( g \) is less than the essential arity of \( f \), and hence, every descending chain with respect to \( \preceq_V \) must be finite.

If \( g \not\preceq_V f \) and \( f \not\preceq_V g \), then \( g \) and \( f \) are said to be incomparable. By an antichain of operations we simply mean a set of pairwise incomparable operations with respect to \( \preceq_V \).

We say that an operation \( g \) is quasi-associative if there is an associative operation \( f \) such that \( g \preceq_V f \). Clearly, every associative operation is also quasi-associative, but there are quasi-associative operations which are not associative. By definition we have:

**Proposition 1.** Let \( f \) be an associative operation. If \( g \preceq_V f \) is not associative, then it is obtained from \( f \) by addition of inessential variables.

We refer to operations which are not quasi-associative as non-associative.

A class \( K \subseteq \bigcup_{n \geq 1} A^n \) of operations on \( A \), is said to be closed under simple variable substitutions if each operation obtained from a operation \( f \) in \( K \) by simple variable substitution is also in \( K \). In other words, the class \( K \) is closed under simple variable substitutions if and only if \( KO_A = \bigcup_{f \in K} fO_A \subseteq K \). Clearly, this condition is equivalent to \( KO_A = K \). We denote by \( V_A \) the set of all classes of operations on \( A \) closed under simple variable substitutions.

Recall that a monoid with universe \( M \) is an algebraic structure \( \langle M, \cdot \rangle \) with an associative operation \( \cdot : M^2 \rightarrow M \), and an identity element, usually denoted by \( 1_M \). In other words, a monoid is a semigroup with an identity element. If \( \leq \) is a partial order on \( M \), and if for every \( x, y, z, w \in M \) the following condition holds

\[ x \leq y, \quad \text{then} \quad z \cdot x \cdot w \leq z \cdot y \cdot w, \]

then \( \langle M, \cdot \rangle \) is called a partially ordered monoid.
Theorem 1. The set $V_A$ constitutes a partially ordered monoid with respect to
class composition, with $O_A$ as its identity.

To prove Theorem 2, we need the following

**Associativity Lemma.** (In [1, 2]:) Let $A$ be a finite set with $|A| \geq 2$, and let $I$, $J$, and $K$ be classes of operations on $A$. The following hold:

(i) $(IJ)K \subseteq I(JK)$;

(ii) If $J$ is closed under simple variable substitutions, then $(IJ)K = I(JK)$.

**Proof of Theorem 1.** By the characterization of the equational classes given in Theorem 1, and using the Associativity Lemma, it follows that class composition is associative on $V_A$. Clearly, for every $K \in V_A$, $O_A K = K$ and $K O_A = K$. Since the members of $V_A$ are closed under variable substitutions, again by making use of the Associativity Lemma it follows that $(K_1 K_2) O_A = K_1 (K_2 O_A) = K_1 K_2$. Furthermore, class composition is order-preserving, and the proof of Theorem 2 is complete. \(\square\)

An idempotent of a monoid $M$ is an element $e$ of $M$ such that $e \cdot e = e$.

**Fact 1.** The idempotents of $V_A$ containing $O_A$ are exactly the clones on $A$. Moreover, $O_A$ is the smallest clone on $A$ and each clone is closed under simple variable substitutions.

**Proposition 2.** If $C_1, C_2 \in V_A$ are idempotents such that $C_1 \subseteq C_2$, then

$$[C_1, C_2] = \{K \in V_A : C_1 \subseteq K \subseteq C_2\}$$

is a semigroup.

**Proof.** The proof of Proposition 1 follows from the fact that if $C_1 \subseteq K_1, K_2 \subseteq C_2$, for idempotents $C_1 \subseteq C_2$, then $C_1 \subseteq K_1 K_2 \subseteq C_2$. \(\square\)

Note that not all intervals $[K_1, K_2]$, for arbitrary $K_1, K_2 \in V_A$ are closed under class composition. We refer to the sets $[C_1, C_2]$, for idempotents $C_1$ and $C_2$, as closed intervals. If $C_1$ is covered by $C_2$, i.e., if for every idempotent $C$ such that $C_1 \subseteq C \subseteq C_2$ we have $C = C_1$ or $C = C_2$, then we say that the interval $[C_1, C_2]$ is minimal.

3. The lattice of equational classes of operations on $A$

A functional equation (for operations on $A$) is a formal expression

$$h_1(f(g_1(v_1, \ldots, v_p)), \ldots, f(g_m(v_1, \ldots, v_p))) =$$

$$= h_2(f(g'_1(v_1, \ldots, v_p)), \ldots, f(g'_t(v_1, \ldots, v_p))) \quad (1)$$

where $m, t, p \geq 1$, $h_1 : A^m \to A$, $h_2 : A^t \to A$, each $g_i$ and $g'_j$ is a map $A^p \to A$, the $v_1, \ldots, v_p$ are $p$ distinct symbols called vector variables, and $f$ is a distinct symbol called function symbol.
For $n \geq 1$, we denote by $n$ the set $n = \{1, \ldots, n\}$, so that an $n$-vector ($n$-tuple) $v$ in $A^n$ is a map $v: n \rightarrow A$. For an $n$-ary operation on $A$, $f : A^n \rightarrow A$, we say that $f$ satisfies the equation (1) if, for all $v_1, \ldots, v_p \in A^n$, we have

$$h_1(f(g_1(v_1, \ldots, v_p)), \ldots, f(g_m(v_1, \ldots, v_p))) = h_2(f(g'_1(v_1, \ldots, v_p)), \ldots, f(g'_r(v_1, \ldots, v_p)))$$

A class $K$ of operations on $A$ is said to be defined, or definable, by a set $E$ of functional equations, if $K$ is the class of all those operations which satisfy every member of $E$. We say that a class $K$ is equational if it is definable by some set of functional equations. We denote by $E_A$ the set of all equational classes of operations on $A$.

The following result was first obtained by Ekin, Foldes, Hammer and Hellerstein [EFHH] for the Boolean case $A = \mathbb{B} = \{0, 1\}$.

**Theorem 2.** (In [1]:) The equational classes of operations on $A$ are exactly those classes that are closed under simple variable substitutions.

In other words, the sets $E_A$ and $V_A$ are exactly the same. By definition of class composition, it follows that

$$(K_1 \cup K_2)O_A = K_1O_A \cup K_2O_A$$

for every $K_1, K_2 \subseteq \bigcup_{n \geq 1} A^{n^2}$. From these facts and using Theorem 2, we obtain:

**Fact 2.** The set $E_A$ of all equational classes of operations on $A$ constitutes a complete distributive lattice under intersection and union, with $\emptyset$ and $\bigcup_{n \geq 1} A^{n^2}$ as minimal and maximal elements, respectively.

The set $E_A$ constitutes a closure system, and thus each equational class can be described by a set of “generators”. In fact, by making use of Theorem 1, we see that the smallest equational class on $A$ containing a set $K \subseteq \bigcup_{n \geq 1} A^{n^2}$ is the class composition $KO_A$. The equational class $KO_A$ is said to be generated by $K$. If $K$ is a finite set of operations, then we say that $KO_A$ is finitely generated.

**Theorem 3.** Let $A$ be a finite set, and let $C$ be an idempotent of $E_A$. Then $C$ is a finitely generated equational class if and only if $C$ contains only quasi-monadic operations. Furthermore, only finitely many equational classes in $E_A$ are finitely generated.

**Proof.** Note that for each finite $A$, there are only finitely many quasi-monadic operations (up to equivalence), and thus the equational classes containing only quasi-monadic operations must be finitely generated. In particular, the equational classes on $A$ which are idempotent and containing only quasi-monadic operations are finitely generated.

To see that these are indeed the only equational classes on $A$ which are idempotent and finitely generated, let $C$ be an idempotent equational class containing an operation $f$ of essential arity $n > 1$. Now, if $C$ were finitely generated, then there would be an integer $N \geq n$, and an $N$-ary generator $f_N$ of essential arity $N$, such that every operation in $C$ has essential arity at most $N$. But the $2N - 1$-ary operation

$$f'_N(x_1, \ldots, x_{2N-1}) = f_N(x_1, \ldots, x_{N-1}, f_N(x_N, \ldots, x_{2N-1}))$$
has essential arity equal to $2N - 1$ and since $C$ is idempotent, it must be in $C$, which constitutes a contradiction. Thus indeed $C$ cannot be finitely generated.

The last claim follows from the fact that there are only finitely many pairwise incomparable quasi-monadic operations on a finite set. □

By reasoning as in the proof of Theorem 3, it is not difficult to verify that the following also holds:

**Theorem 4.** Let $A$ be a finite set, and let $C_1, C_2 \in E_A$ be two idempotent classes such that $C_1 \subseteq C_2$. Then the interval $[C_1, C_2] \subseteq E_A$ is finite if and only if $C_2 \setminus C_1$ contains only quasi-monadic operations.

The following theorem provides a necessary and sufficient for a closed interval to contain uncountably many equational classes.

**Theorem 5.** Let $C_1$ and $C_2$ be two idempotent classes such that $C_1 \subseteq C_2$. Then there are uncountably many ($2^{\aleph_0}$) equational classes in $[C_1, C_2]$ if and only if $C_2 \setminus C_1$ contains an infinite (countable) antichain of operations.

**Proof.** Note that the set of all subsets of an infinite (countable) set is uncountable. Also, distinct subsets of pairwise incomparable functions generate distinct equational classes and thus, if $F = \{f_i\}_{i \in I}$ is an infinite antichain operations in $C_2 \setminus C_1$, then

$$E = \{SC \cup C_1 : S \subseteq F\}$$

is an uncountable ($2^{\aleph_0}$) set of equational classes in $[C_1, C_2]$.

To see that the converse also holds, observe first that for each equational class $K \in [C_1, C_2]$, the relative complement $K^{C_2}_{C_1}$ of $K$ in $[C_1, C_2]$, given by

$$K^{C_2}_{C_1} = C_1 \cup [(\bigcup_{n \geq 1} A^n \setminus K) \cap C_2]$$

is completely determined by maximal antichains of its minimal (under $\preceq_V$) operations, because there are no infinite descending chains with respect to $\preceq_V$.

Now suppose that every antichain in $C_2 \setminus C_1$ is finite. Then it follows from the above observation that there are only countably many relative complements of equational classes in $[C_1, C_2]$, and thus there are only countably many equational classes in $[C_1, C_2]$, and the proof of the theorem is complete. □

4. The closed intervals of the lattice of equational classes of Boolean functions

4.1. Preliminaries. We denote by $\Omega = \bigcup_{n \geq 1} \mathbb{B}^n$ the set of all Boolean functions. The set $\mathbb{B}^n$ is a Boolean lattice (distributive and complemented) of $2^n$ elements under the component-wise order of vectors

$$(a_1, \ldots, a_n) \preceq (b_1, \ldots, b_n)$$

if and only if $a_i \leq b_i$, for all $1 \leq i \leq n$.

In this way, all operations on the Boolean lattice $\mathbb{B}$ are generalized to $\mathbb{B}^n$ by means of component-wise definitions. For example, the complement of a vector $a = (a_1, \ldots, a_n)$ is also defined component-wise by $\bar{a} = (1 - a_1, \ldots, 1 - a_n)$. We denote the all-zero-vector and the all-one-vector by $0 = (0, \ldots, 0)$ and $1 = (1, \ldots, 1)$,
The set $B^n$ is also a Boolean lattice of $2^{2^n}$ elements under the point-wise ordering of functions, i.e.

$$f \leq g \text{ if and only if } f(a) \leq g(a), \text{ for all } a \in \mathbb{B}^n.$$ 

The functions (of any arity) having constant value 0 and 1 are denoted by 0 and 1, respectively. The complement of an $n$-ary Boolean function $f$ is the function $\overline{f}$ defined by $\overline{f}(a) = 1 - f(a)$, for all $a \in \mathbb{B}^n$. The dual of a class $K$ of Boolean functions is defined as the set $K^d = \{f^d : f \in K\}$. We use $\overline{K}$ to denote the class given by $\overline{K}^d = K$. 

**Fact 3.** If $K$ is an equational class, then $\overline{K}$, $K^d$ and $K^c = \{f^c : f \in K\}$ are also equational classes. In fact, $K \mapsto \overline{K}$, $K \mapsto K^d$ and $K \mapsto K^c$ are lattice automorphisms on the set $E_B$ of all equational classes of Boolean functions.

It is well known that every Boolean function $f$ can be represented in the language of Boolean lattices by a DNF expression (disjunctive normal form), i.e. by an expression of the form

$$\bigvee_{i \in I} \left( \bigwedge_{j \in P_i} x_j \bigwedge_{j \in N_i} \overline{x}_j \right),$$

where $I$ is a finite, possibly empty, set of indices and each variable appears at most once in each conjunct. We regard empty disjunctions and empty conjunctions as representing constant functions 0 and 1, respectively. It is easy to verify that if

$$f = \bigvee_{i \in I} \left( \bigwedge_{j \in P_i} x_j \bigwedge_{j \in N_i} \overline{x}_j \right),$$

then the dual $f^d$ of $f$ is represented by

$$f^d = \bigwedge_{i \in I} \left( \bigvee_{j \in P_i} x_j \bigvee_{j \in N_i} \overline{x}_j \right).$$

Expressions of the form (1) are called CNF (conjunctive normal form) representations.

Since Stone [14], it is well-known that any Boolean lattice can be viewed as a Boolean ring (i.e. a commutative ring in which every element is idempotent under product) by defining multiplication and addition by

$$x \cdot y = x \land y \text{ and } x \oplus y = (\overline{x} \land y) \lor (x \land \overline{y}).$$

Thus both $\mathbb{B}^n$ and $\mathbb{B}^{2^n}$ can also be treated as Boolean rings by making use of the above algebraic translations. It is not difficult to see that each $n$-ary Boolean function $f$ can be represented in this Boolean ring language by a multilinear polynomial in $n$ indeterminates over $\mathbb{B}$, called its Zhegalkin polynomial or Reed-Muller polynomial

$$f = \Sigma_{j \in I}(c_j \cdot \prod_{i \in I} x_i).$$

Unlike DNF and CNF representations, the Zhegalkin polynomial representation of a Boolean function is unique (up to permutation of terms and permutation
of variables in the terms). For further normal form representations of Boolean functions, see [3].

Recall that (Boolean) clone is a class $C \subseteq \bigcup_{n \geq 1} \mathbb{B}^n$ idempotent under class composition and containing all projections. In the Boolean case, the only idempotent classes which are not clones are exactly the empty class $\emptyset$, the class $C_0$ of constant 0 functions, the class $C_1$ of constant 1 functions, and the class $C$ containing all constants.

The clones of Boolean functions form an algebraic lattice by defining the meet as the intersection of clones and the join as the smallest clone containing the union. This lattice is known as Post Lattice (see Figure 1), named after Emil Post who first described and classified in [10] the set of all Boolean clones (for recent and shorter proofs of Post’s classification see [12], [15], [16]; for general background see [7] and [8]). We make use of notations and terminology appearing in [5] and in [7].

- $\Omega$ denotes the class $\bigcup_{n \geq 1} \mathbb{B}^n$ of all Boolean functions;
- $T_0$ and $T_1$ denote the classes of 0- and 1-preserving functions, respectively, i.e.,
  $$T_0 = \{ f \in \Omega : f(0, \ldots, 0) = 0 \}, \quad T_1 = \{ f \in \Omega : f(1, \ldots, 1) = 1 \};$$
- $T_c$ denotes the class of constant-preserving functions, i.e., $T_c = T_0 \cap T_1$.
- $M$ denotes the class of all monotone functions, i.e.,
  $$M = \{ f \in \Omega : f(a) \leq f(b), \text{ whenever } a \preceq b \};$$
- $M_0 = M \cap T_0$, $M_1 = M \cap T_1$, $M_c = M \cap T_c$;
- $S$ denotes the class of all self-dual functions, i.e.,
  $$S = \{ f \in \Omega : f^{\mathbb{D}} \};$$
- $S_c = S \cap T_c$, $SM = S \cap M$;
- $L$ denotes the class of all linear functions, i.e.,
  $$L = \{ f \in \Omega : f = c_0 1 + c_1 x_1 + \cdots + c_n x_n \text{ for some } n \text{ and } c_0, \ldots, c_n \in \mathbb{B} \};$$
- $L_0 = L \cap T_0$, $L_1 = L \cap T_1$, $LS = L \cap S$, $L_c = L \cap T_c$;

Let $a \in \{0, 1\}$. A set $A \subseteq \{0, 1\}^n$ is said to be $a$-separating if there is $i, 1 \leq i \leq n$, such that for every $(a_1, \ldots, a_n) \in A$ we have $a_i = a$. A function $f$ is said to be $a$-separating if $f^{-1}(a)$ is $a$-separating. The function $f$ is said to be $a$-separating of rank $k \geq 2$ if every subset $A \subseteq f^{-1}(a)$ of size at most $k$ is $a$-separating.

- For $m \geq 2$, $U_m$ and $W_m$ denote the classes of all 1- and 0-separating functions of rank $m$, respectively;
- $U_\infty$ and $W_\infty$ denote the classes of all 1- and 0-separating functions, respectively, i.e.,
  $$U_\infty = \bigcap_{k \geq 2} U_k \text{ and } W_\infty = \bigcap_{k \geq 2} W_k;$$
- $T_c U_m = T_c \cap U_m$ and $T_c W_m = T_c \cap W_m$, for $m = 2, \ldots, \infty$;
- $MU_m = M \cap U_m$ and $MW_m = M \cap W_m$, for $m = 2, \ldots, \infty$;
- $M_0 U_m = M_0 \cap U_m$ and $M_1 W_m = M_1 \cap W_m$, for $m = 2, \ldots, \infty$;
- $\Lambda$ denotes the class of all conjunctions and constants, i.e.,
  $$\Lambda = \{ f \in \Omega : f = 0, 1, x_1 \land \cdots \land x_n \text{ for some } n \geq 1 \text{ and } i_j \text{'s} \};$$
- $\Lambda_0 = \Lambda \cap T_0$, $\Lambda_1 = \Lambda \cap T_1$, $\Lambda_c = \Lambda \cap T_c$;
- $V$ denotes the class of all disjunctions and constants, i.e.,
  $$V = \{ f \in \Omega : f = 0, 1, x_1 \lor \cdots \lor x_n \text{ for some } n \geq 1 \text{ and } i_j \text{'s} \};$$
Figure 1. Post Lattice.
• \( V_0 = V \cap T_0, \ V_1 = V \cap T_1, \ V_e = V \cap T_e; \)
• \( \Omega(1) \) denotes the class of all quasi-monadic functions, i.e. variables, negated variables, and constants;
• \( I^* \) denotes the class of all variables and negated variables;
• \( I \) denotes the class of all variables and constants;
• \( I_0 = I \cap T_0, \ I_1 = I \cap T_1; \)
• \( I_e \) denotes the smallest clone containing only variables, i.e., \( I_e = I \cap T_e. \)

Since there are essentially 4 quasi-monadic Boolean functions, namely \( \{x_1, \bar{x}_1, 0, 1\} \), and since \( \Omega(1) = \{x_1, \bar{x}_1, 0, 1\} \), and since \( \Omega(1) = \{x_1, \bar{x}_1, 0, 1\} \), we have:

**Theorem 6.** There are exactly 16 equational classes in \([\emptyset, \Omega(1)]\).

![Figure 2. Lattice of equational classes containing only quasi-monadic functions.](image)

Looking at Figure 1, we see that the Post Lattice is co-atomic, that is, every clone is contained in a maximal clone (co-atom). In fact, for any finite set \( A \), the lattice of clones on \( A \) is co-atomic, and the number of maximal clones (co-atoms) is known to be finite (see [13]). This is not the case in the lattice of equational classes.

**Theorem 7.** The lattice \( E_\emptyset \) has no co-atoms.

*Proof.* For a contradiction, suppose that \( E_\emptyset \) has a co-atom, say \( M \). Let \( f \in \Omega \setminus M \). If

- \( f = x \), then \( M \cap L_c = \emptyset, \)
- \( f = \bar{x} \), then \( M \cap (L \setminus L_c) = \emptyset, \)
- \( f = 0 \), then \( M \cap (L_0 \setminus L_c) = \emptyset, \)
- \( f = 1 \), then \( M \cap (L_1 \setminus L_c) = \emptyset. \)
and thus $\mathcal{M} \subseteq \mathcal{M} \cup fI_e \subseteq \mathcal{M} \cup \{f, f\}'I_e \subseteq \Omega$, for a suitable $f'$ in e.g. $\{x_1 + x_2 + x_3, x_1 + x_2 + x_3 + 1, x_1 + x_2, x_1 + x_2 + 1\}$, contradicting our assumption.

So let $f \neq x, \bar{x}, 0, 1$ be of essential arity $n \geq 2$. Without loss of generality, assume that $f$ has no dummy variables. Now consider $f' = x + y + f$, where $x$ and $y$ are not essential variables of $f$. Obviously, $f' \not\leq \mathcal{V} f$. Furthermore, $f' \notin \mathcal{M}$, otherwise, by identifying $x = y$ we would have $f \in \mathcal{M}$. Hence, $\mathcal{M} \subseteq \mathcal{M} \cup fI_e \subseteq \Omega$, which yields the desired contradiction. \hfill $\square$

4.2. Antichains of Boolean functions. The following is a particular case of Proposition 3.4 in [9].

**Lemma 1.** The family $(f_n)_{n \geq 4}$ of 0-preserving Boolean functions, given by

$$f_n(x_1, \ldots, x_n) = \begin{cases} 1 & \text{if } \# \{i : x_i = 1\} \in \{1, n - 1\} \\ 0 & \text{otherwise.} \end{cases}$$

constitutes an (infinite) antichain of Boolean functions, i.e. if $m \neq n$, then $f_m \not\leq \mathcal{V} f_n$ and $f_m \not\leq \mathcal{V} f_m$.

**Lemma 2.** The family $(g_n)_{n \geq 4}$ of constant-preserving Boolean functions, given by

$$g_n(x_1, \ldots, x_n) = \begin{cases} 0 & \text{if } \# \{i : x_i = 1\} \in \{1, n\} \\ 1 & \text{otherwise.} \end{cases}$$

constitutes an (infinite) antichain of Boolean functions.

**Proof.** To prove the lemma, it is enough to show that if $m \neq n$, then $g_m \not\leq \mathcal{V} g_n$.

By definition, $g_m$ and $g_n$ cannot have dummy variables, and hence, $g_m \not\leq \mathcal{V} g_n$, whenever $m > n$. Suppose that $m < n$. Note that for every $t \geq 4$, $g_t$ is constant with value 1 on all $t$-tuples with at least two zeros and at least one 1. For a contradiction, suppose that $g_m \preceq \mathcal{V} g_n$, i.e. there are $m$-ary projections $p_1, \ldots, p_n \in I_e$ such that $g_m = g_n(p_1, \ldots, p_n)$. Since every variable of $g_m$ is essential in $g_n$ and $m < n$, it follows that there are at least two indices $1 \leq i, j \leq n$ such that $p_i = p_j$. Also, since $4 \leq m$, there is at least one index $1 \leq k \leq n$ such that $p_k \neq p_1 = p_j$. Now, consider the set $P$ of all $m$-tuples $(a_1, \ldots, a_m)$ such that $p_1(a_1, \ldots, a_m) = p_j(a_1, \ldots, a_m) = 0$, and $p_k(a_1, \ldots, a_m) = 1$. Clearly, $g_m$ is not constant because $P$ contains an $m$-tuple with exactly one 0, and an $m$-tuple with two 0’s, but $g_n$ is constant with value 1 on all $n$-tuples of the form $(p_1(a_1, \ldots, a_m), \ldots, p_n(a_1, \ldots, a_m))$, for $(a_1, \ldots, a_m) \in P$, because all $n$-tuples of this form have at least two 0’s and at least one 1, yielding the desired contradiction. \hfill $\square$

**Lemma 3.** Let $(f_n)_{n \geq 4}$ and $(g_n)_{n \geq 4}$ be the families of Boolean functions given above, and consider the families $(u_n)_{n \geq 4}$ and $(t_n^u)_{n \geq 4}$ defined by

$$u_n(x_0, x_1, \ldots, x_n) = x_0 \land f_n(x_1, \ldots, x_n)$$

$$t_n^u(x_0, x_1, \ldots, x_n) = x_0 \land g_n(x_1, \ldots, x_n)$$

Each of $(u_n)_{n \geq 4}$ and $(t_n^u)_{n \geq 4}$ constitutes an (infinite) antichain of Boolean functions.
Proof. We only prove that the lemma holds for the family \((u_n)_{n \geq 4}\). The remaining claim can be shown to hold, by proceeding similarly.

We show that if \(m \neq n\), then \(u_m \not\leq V u_n\). By definition, \(u_m\) and \(u_n\) cannot have dummy variables. Therefore, \(u_m \not\leq V u_n\), whenever \(m > n\).

So assume that \(m < n\), and for a contradiction, suppose that \(u_m \leq V u_n\), i.e. there are \(m\)-ary projections \(p_0, \ldots, p_n \in I_e\) such that \(u_m = u_n(p_0, \ldots, p_n)\). Note that for every \(m \geq 4\), \(u_m(1, x_1, \ldots, x_m) = f_m(x_1, \ldots, x_m)\) and \(u_m(0, x_1, \ldots, x_m)\) is the constant 0.

Now, if \(p_0(x_0, \ldots, x_m) = x_0\), then by taking \(x_0 = 1\) we would conclude that \(f_m \leq V f_n\), contradicting Lemma 1. Hence, \(p_0(x_0, \ldots, x_m) \neq x_0\), say \(p_0(x_0, \ldots, x_m) = x_i\) for \(1 \leq i \leq m\). But then

\[
\begin{align*}
u_m(x_0, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_m) &
\neq \\
u_n(0, p_1(x_0, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_m), \ldots, p_n(x_0, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_m)) &
= 0
\end{align*}
\]

which contradicts our assumption \(u_m \not\leq V u_n\). \qedsymbol

A hypergraph is an ordered pair \(G = (V, E)\), where \(V = V(G)\) is a non-empty finite set (called the set of vertices of \(G\)), and \(E = E(G)\) is a set of subsets of \(V\) (called the set of hyperedges of \(G\)). Without loss of generality, we assume that our hypergraphs \(G\) have set of vertices \(V(G) = n = \{1, \ldots, n\}\), for some positive integer \(n\). Examples of hypergraphs are the complete graphs \(K_n\), \(n \geq 2\), whose set of vertices is \(V(K_n) = n\) and whose set of hyperedges is the set of all 2-element subsets of \(V(K_n)\), i.e. \(E(K_n) = \{\{i, j\} : i, j \in V(K_n), i \neq j\}\). To each hypergraph \(G\), say \(V(G) = n\), we associate an \(n\)-ary monotone Boolean function \(f_G\) whose DNF is given by

\[
f_G = \bigvee_{I \in E(G)} \bigwedge_{i \in I} x_i
\]

Note that every monotone Boolean function is associated with some hypergraph.

Given two hypergraphs \(G\) and \(H\), a homomorphism \(h\) from \(G\) to \(H\) is any mapping \(h : V(G) \to V(H)\) satisfying the condition: if \(I \in E(G)\), then \(h(I) = \{h(i) : i \in I\} \in E(H)\). A homomorphism \(h : V(G) \to V(H)\) is said to be hyperedge-surjective if for each \(J \in E(H)\), there is \(I \in E(G)\) such that \(I = h^{-1}(J)\).

The following lemma provides a characterization of \(\leq V\) restricted to the clone \(M\) of monotone Boolean functions.

**Lemma 4.** Let \(G\) and \(H\) be two hypergraphs, and consider the functions \(f_G\) and \(f_H\) associated with \(G\) and \(H\), respectively. Then there is a hyperedge-surjective homomorphism \(f : V(G) \to V(H)\) if and only if \(f_H \leq V f_G\).

**Proof.** Let \(V(G) = n\) and \(V(H) = m\). Assume first that there is a hyperedge-surjective homomorphism \(h : V(G) \to V(H)\). Define \(m\)-ary projections \(p_1, \ldots, p_n \in I_e\) by \(p_i = x_j\) if and only if \(h(i) = j\). Consider the \(m\)-ary function \(g\) given by \(g = f_G(p_1, \ldots, p_n)\). Note that

\[
g = \bigvee_{I \in E(G) \in I} p_i = \bigvee_{I \in E(G)} \bigwedge_{j \in h(I)} x_j
\]

Now, since \(h\) is a hyperedge-surjective homomorphism, we have that for each \(I \in E(G), h(I) \in E(H)\), and that every \(J \in E(H)\) is of the form \(h(I)\), for some
I \in E(G). Also, both \lor and \land are associative and idempotent operations, and thus
\[ g = \bigvee_{I \in E(G)} \bigwedge_{j \in h(I)} x_j = \bigvee_{J \in E(H)} \bigwedge_{j \in J} x_j = f_H \]
In other words, \( f_H \preceq \lor f_G \).

Now, suppose that \( f_H \preceq \lor f_G \), i.e., there are \( m \)-ary projections \( p_1, \ldots, p_n \in I_c \)
such that \( f_H = f_G(p_1, \ldots, p_n) \). Let \( h \) be the map \( h : V(G) \to V(H) \) satisfying
\( h(i) = j \) if and only if \( p_i = x_j \). We claim that \( h \) is a homomorphism. Indeed, if \( I \in E(G) \), then \( \bigwedge_{i \in I} x_i \) is a conjunct of \( f_G \), and thus \( \bigwedge_{i \in I} x_{h(i)} = \bigwedge_{j \in h(I)} x_j \) is a conjunct of \( f_H \).

By definition of \( f_H \), we have that \( h(I) \in E(H) \). To see that \( h \) is hyperedge-surjective, suppose that \( J \in E(H) \). Then \( \bigwedge_{j \in J} x_j \) is a conjunct of \( f_H \).

By construction, we have that \( I \subseteq V(G) \) such that \( I = h^{-1}(J) \) and \( \bigwedge_{i \in I} x_i \) is a conjunct of \( f_G \). By definition of \( f_G \), it follows that \( I \in E(G) \), and the proof of the lemma is complete. \( \square \)

**Lemma 5.** The family \((H_n)_{n \geq 2}\) of constant-preserving monotone Boolean functions given by
\[ H_n(x_1, \ldots, x_n) = \bigvee_{i \neq j} x_i \land x_j. \]
constitutes an (infinite) antichain of Boolean functions. Furthermore, for each \( n \geq 2 \), the family \((G^n_m)_{m \geq n}\) of composites
\[ G^n_m(x_1, \ldots, x_{m+n-1}) = H_n(x_1, \ldots, x_{n-1}, H_m(x_n, \ldots, x_{m+n-1})) \]
also constitutes an (infinite) antichain of Boolean functions.

**Proof.** To see that the first claim of the lemma holds, observe that for each \( n \geq 2 \), \( H_n \) is the \( n \)-ary function associated with the complete graph \( K_n \). Since there is no hyperedge-surjective homomorphism between \( K_m \) and \( K_n \), whenever \( m \neq n \), by Lemma 4 it follows that \( H_m \) and \( H_n \) are incomparable, whenever \( m \neq n \).

To prove that the second claim of the lemma also holds, we show that if \( m_1 \neq m_2 \), then \( G^n_{m_1} \not\preceq \lor G^n_{m_2} \). Note first that each \( G^n_m \) is associated with a hypergraph whose set of vertices is \( \{1, \ldots, m + n - 1\} \) and whose set of hyperedges is
\[ E(G) = \{ \{i, j\} : 1 \leq i < j \leq n-1\} \cup \{ \{i, k, l\} : 1 \leq i \leq n-1, n \leq k < l \leq m+n-1\}. \]

Now, if \( m_1 > m_2 \), then \( G^n_{m_1} \) and \( G^n_{m_2} \) are associated with graphs \( G_1 \) and \( G_2 \), respectively, such that \( G_1 \) and \( G_2 \) have the same number of 2-element hyperedges, but \( G_1 \) has more 3-element hyperedges than \( G_2 \). From this fact it follows that if \( m_1 > m_2 \), then there is no hyperedge-surjective homomorphism \( h : G_2 \to G_1 \), and by Lemma 4 \( G^n_{m_1} \not\preceq \lor G^n_{m_2} \).

Now, suppose that \( m_1 < m_2 \) and for a contradiction suppose that there is a hyperedge-surjective homomorphism \( h : G_2 \to G_1 \). Clearly, each 2-element hyperedge of \( G_2 \) must be mapped to a 2-element hyperedge of \( G_1 \), and since \( h \) is hyperedge-surjective, we also have that there cannot be two 2-element hyperedges of \( G_2 \) mapped to the same 2-element hyperedge of \( G_1 \). Also, no 3-element hyperedge of \( G_2 \) can be mapped to a 2-element hyperedge \( J \in E(G_1) \), for otherwise \( h^{-1}(J) \) would be of size at least 4 and there is no hyperedge of \( G_2 \) of size greater then
3. Similarly, there cannot be two 3-element hyperedges of $G_2$ mapped to the same 3-element hyperedge of $G_1$. But then there is a 3-element hyperedge $I \in E(G_2)$ such that $h(I) \not\in E(G_1)$, which contradicts our assumption that $h : G_2 \rightarrow G_1$ is a homomorphism.

Thus, if $m_1 < m_2$, then there is no hyperedge-surjective homomorphism $h : G_2 \rightarrow G_1$, and by Lemma 4 it follows that $G^m_{m_1} \not\leq V G^m_{m_2}$, which completes the proof of the lemma. \hfill $\square$

**Lemma 6.** Let $\mathcal{O}$ denote the set of all odd integers $n \geq 7$, and let $\mu_n$ denote the $n$-ary threshold function defined by

$$
\mu_n(x_1, \ldots, x_n) = \begin{cases} 1 & \text{if } \#\{i : x_i = 1\} \geq \frac{n+1}{2} \\
0 & \text{otherwise.}
\end{cases}
$$

The family $(T_n)_{n \in \mathcal{O}}$ given by

$$
T_n(x_1, \ldots, x_n, x_{n+1}, x_{n+2}) = \begin{cases} H_n(x_1, \ldots, x_n) & \text{if } x_{n+1} = x_{n+2} = 1 \\
\mu_n(x_1, \ldots, x_n) & \text{if } x_{n+1} + x_{n+2} = 1 \\
H_n^a(x_1, \ldots, x_n) & \text{if } x_{n+1} = x_{n+2} = 0
\end{cases}
$$

constitutes an (infinite) antichain of Boolean functions. Moreover, the family $(s_n)_{n \in \mathcal{O}}$ defined by

$$
s_n(x_1, \ldots, x_n, x_{n+1}, x_{n+2}) = T_n(x_1, \ldots, x_n, x_{n+1}, x_{n+2})
$$

also constitutes an (infinite) antichain of Boolean functions.

**Proof.** Since each function of $(T_n)_{n \in \mathcal{O}}$, and each function of $(s_n)_{n \in \mathcal{O}}$ has only essential variables, to prove the lemma we only need to show that if $n < m$, then $T_n \not\leq V T_m$ and $s_n \not\leq V s_m$.

So assume that $7 \leq n < m$, and for a contradiction, suppose first that $T_n \leq V T_m$, i.e. there are $n + 2$-ary projections $p_1, \ldots, p_{n+2}$ such that

$$
T_n = T_m(p_1, \ldots, p_{n+2}).
$$

Note that for each $1 \leq i \leq n + 2$, there is at least one $1 \leq i_1 \leq m + 2$ such that $p_i = x_i$ because $T_n$ has no dummy variables.

First we consider the case $p_{n+1} = p_{n+2}$. Let $1 \leq j \leq n$. If both projections are $x_{n+1}$, or $x_{n+2}$, or $x_j$, then for $a_i = 1$ if and only if $i = j, n+1, n+2$, we have

$$
T_n(a_1, \ldots, a_{n+2}) = H_n(a_1, \ldots, a_n) = 0 \\
T_m(p_1(a_1, \ldots, a_{n+2}), \ldots, p_{n+2}(a_1, \ldots, a_{n+2})) = 1
$$

If $p_{n+1} \neq p_{n+2}$, say $p_{n+1} = x_j$ and $p_{n+2} = x_k$, $1 \leq j, k \leq n$, then for $a_i = 1$ if and only if $i \neq i_1, i_2, j, k$, where $1 \leq i_1 < i_2 \leq n$ are indices distinct from $j$ and $k$, we have

$$
T_n(a_1, \ldots, a_{n+2}) = H_n(a_1, \ldots, a_n) = 1 \\
T_m(p_1(a_1, \ldots, a_{n+2}), \ldots, p_{n+2}(a_1, \ldots, a_{n+2})) = 0
$$

Next we consider the case $p_{n+1} \in \{x_{n+1}, x_{n+2}\}$ and $p_{n+2} \in \{x_1, \ldots, x_n\}$, or $p_{n+2} \in \{x_{n+1}, x_{n+2}\}$ and $p_{n+1} \in \{x_1, \ldots, x_n\}$. Without loss of generality, assume
that $p_{m+1} = x_{n+1}$ and $p_{m+2} = x_j$, $1 \leq j \leq n$. If there are at least two $1 \leq i_1 < i_2 \leq m$ such that $p_{i_1} = p_{i_2} = x_{n+2}$, then for $a_i = 1$ if and only if $i = j, n+1, n+2$,

$$T_n(a_1, \ldots, a_{n+2}) = H_n(a_1, \ldots, a_{n}) = 0 \text{ and}$$

$$T_m(p_1(a_1, \ldots, a_{n+2}), \ldots, p_{m+2}(a_1, \ldots, a_{n+2})) =$$

$$H_m(p_1(a_1, \ldots, a_{n+2}), \ldots, p_m(a_1, \ldots, a_{n+2})) = 1$$

If there is a unique $1 \leq k \leq m$ such that $p_k = x_{n+2}$, then let $I = \{i_1, \ldots, i_{m+1}\}$ be a “majority” of indices not containing $j, n+1, n+2$. Thus, for $a_i = 1$ if and only if $i \in I$, we have

$$\mu_n(a_1, \ldots, a_{n}) = 1.$$  

Now, if

$$T_m(p_1(a_1, \ldots, a_{n+2}), \ldots, p_{m+2}(a_1, \ldots, a_{n+2})) = 1$$

with $a_i = 1$ if and only if $i \in I \cup \{n+1\}$, then for $a_i = 1$ if and only if $i \in I \cup \{n+2\}$,

$$T_n(a_1, \ldots, a_{n+2}) = \mu_n(a_1, \ldots, a_{n}) = 1 \text{ and}$$

$$T_m(p_1(a_1, \ldots, a_{n+2}), \ldots, p_{m+2}(a_1, \ldots, a_{n+2})) =$$

$$H_m(p_1(a_1, \ldots, a_{n+2}), \ldots, p_m(a_1, \ldots, a_{n+2})) = 0.$$  

Otherwise, for $a_i = 1$ if and only if $i \in I \cup \{n+1\}$,

$$T_n(a_1, \ldots, a_{n+2}) = \mu_n(a_1, \ldots, a_{n}) = 1 \text{ and}$$

$$T_m(p_1(a_1, \ldots, a_{n+2}), \ldots, p_{m+2}(a_1, \ldots, a_{n+2})) =$$

$$\mu_n(p_1(a_1, \ldots, a_{n+2}), \ldots, p_m(a_1, \ldots, a_{n+2})) = 0.$$  

Finally, we consider the case $p_{m+1} \neq p_{m+2}$ and $p_{m+1}, p_{m+2} \in \{x_{n+1}, x_{n+2}\}$. Without loss of generality, suppose that $p_{m+1} = x_{n+1}$ and $p_{m+2} = x_{n+2}$.

Note that there must be at least one $1 \leq i_1 \leq m$, such that $p_{i_1} = x_{n+1}$ or $p_{i_1} = x_{n+2}$, otherwise, by identifying $x_{n+1} = x_{n+2} = 1$ we would conclude that $H_n \not\leq_V H_m$, which contradicts Lemma 3, or alternatively, by identifying $x_{n+1} = x_{n+2} = 0$ we would conclude that $H_n^d \not\leq_V H_m^d$ which, together with Fact 4, again constitutes a contradiction.

If there are $1 \leq i_1 < i_2 \leq m$, such that $p_{i_1}, p_{i_2} \in \{x_{n+1}, x_{n+2}\}$, then for $a_i = 1$ if and only if $i = n+1, n+2$,

$$T_n(a_1, \ldots, a_{n+2}) = H_n(a_1, \ldots, a_{n}) = 0 \text{ and}$$

$$T_m(p_1(a_1, \ldots, a_{n+2}), \ldots, p_{m+2}(a_1, \ldots, a_{n+2})) =$$

$$H_m(p_1(a_1, \ldots, a_{n+2}), \ldots, p_m(a_1, \ldots, a_{n+2})) = 1.$$  

If there is exactly one $1 \leq i_1 \leq m$, such that $p_{i_1} \in \{x_{n+1}, x_{n+2}\}$, then for $a_i = 1$ if and only if $i = j, n+1, n+2$, for a unique $1 \leq j \leq n$,

$$T_n(a_1, \ldots, a_{n+2}) = H_n(a_1, \ldots, a_{n}) = 0 \text{ and}$$

$$T_m(p_1(a_1, \ldots, a_{n+2}), \ldots, p_{m+2}(a_1, \ldots, a_{n+2})) =$$

$$H_m(p_1(a_1, \ldots, a_{n+2}), \ldots, p_m(a_1, \ldots, a_{n+2})) = 1.$$  

In all possible cases, we derive the same contradiction $T_n \neq T_m(p_1, \ldots, p_{m+2})$, and hence, $T_n \not\leq_V T_m$.

The proof of $s_n \not\leq_V s_m$ can be obtained by minor adjustments in the proof above.  
\qed
4.3. Classification of the closed intervals of $E_B$. In this subsection we provide a complete classification of the closed intervals of $E_B$ in terms of their size. We prove the following theorem:

**Theorem 8.** Let $[C_1, C_2]$ be a non-empty closed interval of $E_B$. Then $[C_1, C_2]$ is countable if and only if one of the following holds:

- $C_2 \subseteq V$,
- $C_2 \subseteq \Lambda$,
- $C_2 \subseteq L$,
- $C \cap M_e \subseteq C_1$ and $C_2 \subseteq C \cap M$ where $C$ is a clone in
  $$\{U_m : m \geq 2\} \cup \{W_m : m \geq 2\} \cup \{\Omega, U_\infty, W_\infty\}.$$  

The proof of Theorem 8 follows from several propositions.

**Proposition 3.** Let $[C_1, C_2]$ be a closed interval of $E_B$. If $C_2 \subseteq \Lambda \cup V \cup L$, then $[C_1, C_2]$ is countable.

**Proof.** Using the description of the clones $\Lambda$, $V$ and $L$, it is easy to verify that every antichain in $\Lambda$, in $V$ or in $L$ is finite. The proof of the proposition follows then from Theorem 5. □

From the fact that $M \setminus M_e = \{0, 1\}I_c$, it follows that:

**Proposition 4.** If $C \in \{U_m : m \geq 2\} \cup \{W_m : m \geq 2\} \cup \{\Omega, U_\infty, W_\infty\}$, then the closed interval $[C \cap M_e, C \cap M]$ of $E_B$ is finite.

Thus if $[C_1, C_2]$ satisfies the conditions of Theorem 8, then it is countable. To prove that these are indeed the only countable closed intervals of $E_B$, we show that the minimal intervals which do not satisfy the conditions of Theorem 8 are uncountable by making use of the antichains provided in Subsection 4.2 and applying Theorem 5. This suffices to complete the proof of Theorem 5 because if a closed interval does not satisfy the conditions of Theorem 8, then it must contain a minimal interval not satisfying the same conditions. In the sequel, we will make use of the following fact.

**Fact 4.** Let $C_1$ and $C_2$ be idempotent classes such that $C_1 \subseteq C_2$. If $(f_i)_{i \in I}$ is an antichain in $C_2 \setminus C_1$, then $(f_{i_i})_{i \in I}$ and $(f_{d_i})_{i \in I}$ are antichains in $C_2 \setminus C_1$ and $C_2 \setminus C_1$, respectively.

**Proposition 5.** Each of the minimal intervals

1. $[C, \Omega]$ where $C \in \{T_0, T_1, L, S, M\}$,
2. $[C, T_0]$ where $C \in \{T_c, L_0, M_0, U_2\}$,
3. $[C, T_1]$ where $C \in \{T_c, L_1, M_1, W_2\}$,
4. $[C, T_c]$ where $C \in \{M_c, S_c, T_cU_2, T_cW_2\}$,

is uncountable.

**Proof.** Note that every member of $(f_n)_{n \geq 4}$ defined in Lemma 1 belongs to $T_0 \setminus T_1 \cup T_c \cup U_2 \cup S \cup M$. Moreover, if $n \geq 5$, then $f_n \notin L$. Thus, using Fact 4 and applying Theorem 5, we conclude that (i),(ii) and (iii) of the proposition hold. The
proof of (iv) follows similarly by observing that every member of \((g_n)_{n \geq 4}\) defined in Lemma 2 belongs to \(T_c \setminus (M_c \cup S_c \cup T_c U_2 \cup T_c W_2)\).

Proposition 6. Each of the minimal intervals
(i) \([C_1 \cap C_2, C_2]\) where \(C_1 \in \{T_c, M\}\) and \(C_2 \in \{U_m : m \geq 2\} \cup \{U_\infty\}\),
(ii) \([C_1 \cap C_2, C_2]\) where \(C_1 \in \{T_c, M\}\) and \(C_2 \in \{W_m : m \geq 2\} \cup \{W_\infty\}\),
(iii) \([M_c \cap C_2, C_2]\) where \(C_2 \in \{T_c U_m : m \geq 2\} \cup \{T_c U_\infty\}\),
(iv) \([M_c \cap C_2, C_2]\) where \(C_2 \in \{T_c W_m : m \geq 2\} \cup \{T_c W_\infty\}\),
is uncountable.

Proof. Observe that, for every \(n \geq 4\), \(u_n \in U_\infty \setminus T_c U_2\), and \(t_n^c \in T_c U_\infty \setminus MU_2\). Thus it follows from Lemma 3 and Theorem 5 that (i) and (iii) hold. The proof of (ii) and (iv) follows similarly by making use of Fact 4.

Proposition 7. Each of the minimal intervals
(i) \([M \cap C, M]\) where \(C \in \{\Lambda, V, U_2, W_2\}\),
(ii) \([MU_2, M_0]\) and \([MW_2, M_1]\),
(iii) \([M_c \cap C, M_c]\) where \(C \in \{U_2, W_2\}\),
(iv) \([SM, C]\) where \(C \in \{M_c U_2, M_c W_2\}\),
is uncountable.

Proof. Observe that for each \(n \geq 4\), we have \(H_n \in M_c W_2 \setminus (U_2 \cup \Lambda \cup V)\), and thus, by Lemma 5 and Fact 4, \((H_n)_{n \geq 4}\) and \((H_n^2)_{n \geq 4}\) constitute infinite antichains in \(M_c W_2 \setminus (U_2 \cup \Lambda \cup V)\) and \(M_c U_2 \setminus (W_2 \cup \Lambda \cup V)\), respectively. Hence, by Theorem 5 the proposition holds.

Proposition 8. For \(n \geq 2\), each of the minimal intervals
(i) \([C_1 \cap U_{n+1}, C_1 \cap U_n]\) where \(C_1 \in \{\Omega, T_c, M, M_c\}\),
(ii) \([C_1 \cap W_{n+1}, C_1 \cap W_n]\) where \(C_1 \in \{\Omega, T_c, M, M_c\}\),
is uncountable.

Proof. It is not difficult to verify that for each \(n \geq 2\), \(H_{n+1} \in M_c W_n \setminus W_{n+1}\) and thus, by Lemma 5, \((H_m)_{m \geq n+1}\) constitute an infinite antichain in \(M_c W_{n+1} \setminus W_\infty\). Furthermore, for each \(n \geq 2\), the family \((G_m^{n+1})_{m \geq n+1}\) is in \(M_c W_n\) but for every \(m \geq n+1\), \(G_m^{n+1} \notin W_{n+1}\); otherwise by identifying the variables \(x_{n+2}, \ldots, x_{m+n+1}\) of \(G_m^{n+1}\), we would conclude that \(H_{n+1} \in W_{n+1}\) which is a contradiction. By Lemma 5, for \(n \geq 2\), \((G_m^{n+1})_{m \geq n+1}\) constitute an infinite antichain in \(M_c W_n \setminus W_{n+1}\). The proof of the proposition follows now by making use of Fact 4 and applying Theorem 5.

Proposition 9. Each of the minimal intervals
(i) \([C \cap \Lambda, C \cap U_\infty]\) where \(C \in \{M_0, M_c\}\),
(ii) \([C \cap V, C \cap W_\infty]\) where \(C \in \{M_1, M_c\}\),
is uncountable.

Proof. Observe that each member of \((G_m^2)_{m \geq 2}\) is in \(M_c W_\infty \setminus V\), and thus, by Lemma 5, \((G_m^2)_{m \geq 2}\) constitute an infinite antichain in \(M_c W_\infty \setminus V\). The proof of the proposition follows now by making use of Fact 4 and applying Theorem 5.
Proposition 10. Each of the minimal intervals

(i) \([L, SM]\),
(ii) \([C \cap S, S]\) where \(C \in \{M, L\}\),
(iii) \([C \cap S, S]\) where \(C \in \{T, L\}\),

is uncountable.

Proof. To prove Proposition 10 we shall make use of the antichains given in Lemma 6. First, we show that the members of \((T_n)_{n \in \mathbb{N}}\) are in \(SM \setminus L\). Observe that if

\[
T_n(x_1, \ldots, x_n, x_{n+1}, x_{n+2}) = H_n(x_1, \ldots, x_n),
\]

then \(x_{n+1} = x_{n+2} = 1\), i.e. \(\bar{x}_{n+1} = \bar{x}_{n+2} = 0\). Hence,

\[
T_n^d(x_1, \ldots, x_n, x_{n+1}, x_{n+2}) = H_n^d(\bar{x}_1, \ldots, \bar{x}_n) = H_n(x_1, \ldots, x_n) = T_n(x_1, \ldots, x_n, x_{n+1}, x_{n+2})
\]

For the case \(T_n(x_1, \ldots, x_n, x_{n+1}, x_{n+2}) = \mu_n(x_1, x_2, \ldots, x_n)\), we note that the identity \(x_{n+1} + x_{n+2} = \bar{x}_{n+1} + \bar{x}_{n+2}\) holds, and that \(\mu_n\) is self-dual, and thus \(T_n^d = T_n\) also holds. Hence, \((T_n)_{n \in \mathbb{N}}\) is a family of self-dual functions. The fact that each function in \((T_n)_{n \in \mathbb{N}}\) is monotone follows immediately from the definition of each \(T_n\).

To see that the members of \((T_n)_{n \in \mathbb{N}}\) are not linear, just note that for each \(n \in \mathbb{N}\),

\[
T_n(1, \ldots, 1, x_{n+1}, x_{n+2}) = T_n(1, \ldots, 1, x_{n+1}, \bar{x}_{n+2}) = T_n(1, \ldots, 1, \bar{x}_{n+1}, \bar{x}_{n+2})
\]

for all \(x_{n+1}, x_{n+2} \in \{0, 1\}\), and \(T_n\) depends essentially on all variables.

Using Fact 4, it follows from Lemma 6 that \((T_n)_{n \in \mathbb{N}}\) and \((\bar{T}_n)_{n \in \mathbb{N}}\) are antichains in \(SM \setminus L\) and \(S \setminus (S_c \cup L)\), respectively. Thus, by Theorem 5 it follows that (i) and (iii) of the proposition hold.

Now we show that the members of \((s_n)_{n \in \mathbb{N}}\) are in \(S_c \setminus (SM \cup L)\). It is easy to verify that indeed, for each \(n \in \mathbb{N}\), \(s_n \in S_c \setminus L\). To see that \(s_n \in S_c \setminus M\), let \(n \in \mathbb{N}\) and consider the \(n+2\)-tuples \(a = (1, 1, 0, \ldots, 0, 0, 0)\) and \(b = (1, 1, 0, \ldots, 0, 1, 1)\). Obviously, \(a \preceq b\) but \(T_n(a) > T_n(b)\). Thus, by Lemma 6, \((s_n)_{n \in \mathbb{N}}\) constitutes an infinite antichain in \(S_c \setminus (SM \cup L)\), and hence, by Theorem 5 it follows that (ii) also holds.

\[\square\]

4.4. Characterization of the closed intervals of \(E_b\). Using the classification of the closed intervals of \(E_b\) given in the previous subsection, we derive the following characterization of the uncountable closed intervals of \(E_b\):

Theorem 9. Let \([C_1, C_2]\) be a closed interval of \(E_b\). Then there are uncountably many equational classes in \([C_1, C_2]\) if and only if \(C_2 \setminus C_1\) contains a non-associative Boolean function.

Proof. Let \([C_1, C_2]\) be a closed interval of \(E_b\). It is not difficult to verify that if \([C_1, C_2]\) satisfies one of the conditions of Theorem 8, then \(C_2 \setminus C_1\) contains only quasi-associative Boolean functions.

To see that the converse holds, it is enough to provide a non-associative function in \(C_2 \setminus C_1\) for each uncountable minimal interval \([C_1, C_2]\). For that it is sufficient to show that the members of the antichains given in Subsection 4.2 are non-associative.
Note that the members of each antichain \((F_n)_{n \in I}\) given in Subsection 4.2 have no inessential variables, and thus by Proposition 1, for each antichain \((F_n)_{n \in I}\), it is enough to show that for some \(n \in I\), and some \(1 \leq i < j \leq m_n\), there is \((a_1, \ldots, a_{2m_n - 1}) \in \mathbb{B}^{2m_n - 1}\), such that

\[
F_n(a_1, \ldots, a_{i-1}, a_{i}, \ldots, a_{i+m_n - 1}, a_{i+m_n}, \ldots, a_{2m_n - 1}) \neq F_n(a_1, \ldots, a_{j-1}, a_{j}, \ldots, a_{j+m_n - 1}, a_{j+m_n}, \ldots, a_{2m_n - 1})
\]

Consider the antichain \((f_n)_{n \geq 4}\) defined in Lemma 1. Let \(n > 4\). To see that \(f_n\) is non-associative, let \(i = 2, j = 3\), and let \((a_1, \ldots, a_{2n-1}) \in \mathbb{B}^{2n-1}\) be defined by \(a_t = 0\) if and only if \(t \in \{1, \ldots, n-1, n+1\}\). Then

\[
f_n(a_1, f_n(a_2, a_3, \ldots, a_{n-1}), a_n, \ldots, a_{2n-1}) = 1 \neq 0 = f_n(a_1, a_2, f_n(a_3, a_4, \ldots, a_{n-2}), a_{n+1}, \ldots, a_{2n-1})
\]

For the antichain \((g_n)_{n \geq 4}\) defined in Lemma 2, let \(n > 4\), \(i = 1, j = 3\), and let \((a_1, \ldots, a_{2n-1}) \in \mathbb{B}^{2n-1}\) be defined by \(a_t = 1\) if and only if \(t \in \{1, n+1, \ldots, 2n+1\}\). Then

\[
g_n(g_n(a_1, a_2, a_3, \ldots, a_{n-1}), a_n, a_{n+1}, \ldots, a_{2n-1}) = 1 \neq 0 = g_n(a_1, a_2, g_n(a_3, a_4, \ldots, a_{n-2}), a_{n+3}, \ldots, a_{2n-1})
\]

For the antichain \((u_n)_{n \geq 4}\) defined in Lemma 3, let \(n > 4\), \(i = 1, j = 2\), and let \((a_1, \ldots, a_{2n+1}) \in \mathbb{B}^{2n+1}\) be defined by \(a_t = 0\) if and only if \(t \in \{2, \ldots, n\}\). Then

\[
u_n(u_n(a_1, a_2, a_3, \ldots, a_{n-1}), a_{n+1}, \ldots, a_{2n+1}) = 1 \land 1 = 0 = 0 \neq 1 = 1 \land 1 = u_n(a_1, u_n(a_2, a_3, \ldots, a_{n-2}), a_{n+3}, \ldots, a_{2n+1})
\]

For the antichain \((t_n)_{n \geq 4}\) also defined in Lemma 3, let \(n > 4\), \(i = 1, j = 2\), and let \((a_1, \ldots, a_{2n+1}) \in \mathbb{B}^{2n+1}\) be defined by \(a_t = 0\) if and only if \(2 \leq t \leq n-1\). Then

\[
t_n(t_n(a_1, a_2, a_3, \ldots, a_{n-1}), a_{n+2}, \ldots, a_{2n+1}) = 1 \land 1 = 0 = 0 \neq 1 = 1 \land 1 = t_n(t_n(a_2, a_3, a_4, \ldots, a_{n+1}), a_{n+3}, \ldots, a_{2n+1})
\]

For the antichain \((H_n)_{n \geq 4}\) defined in Lemma 5, let \(n > 4\), \(i = 1, j = 2\), and let \((a_1, \ldots, a_{2n-1}) \in \mathbb{B}^{2n-1}\) be defined by \(a_t = 1\) if and only if \(1 \leq t \leq n\). Then

\[
H_n(H_n(a_1, a_2, a_3, \ldots, a_{n-1}), a_{n+2}, \ldots, a_{2n-1}) = 0 \neq 1 = H_n(a_1, H_n(a_2, a_3, \ldots, a_{n+1}), a_{n+2}, \ldots, a_{2n-1})
\]

For the antichain \((G_n)_{n \geq 2}\) also defined in Lemma 5, let \(m \geq n \geq 4\), \(i = 2, j = 3\), and let \((a_1, \ldots, a_{2m+2n-3}) \in \mathbb{B}^{2m+2n-3}\) be defined by \(a_t = 1\) if and only if \(1 \leq t \leq 2\). Then

\[
G_n(a_1, a_2, a_3, \ldots, a_{m+2n-3}) = 0 \neq 1 = G_n(a_1, a_2, a_3, \ldots, a_{m+n+1}, a_{m+n+2}, \ldots, a_{2m+2n-3})
\]

For the antichain \((T_n)_{n \in \mathbb{N}}\) defined in Lemma 6, let \(n \in \mathbb{N}\), \(i = 1, j = 2\), and let \((a_1, \ldots, a_{2n+3}) \in \mathbb{B}^{2n+3}\) be defined by \(a_t = 1\) if and only if \(t \in \{1, \ldots, n+2, 2n+2, 2n+3\}\). Then

\[
T_n(T_n(a_1, a_2, a_3, \ldots, a_{n+2}), a_{n+3}, \ldots, a_{2n+3}) = 0 \neq 1 = T_n(a_1, T_n(a_2, a_3, a_{n+4}, \ldots, a_{2n+3}))
\]
For the antichain \((s_n)_{n \in \mathbb{N}}\) also defined in Lemma 6, let \(n \in \mathbb{N}, i = 1, j = 2\), and let \((a_1, \ldots, a_{2n+3}) \in \mathbb{B}^{2n+3}\) be defined by \(a_t = 1\) if and only if \(t \in \{1, \ldots, n + 2\}\). Then
\[
s_n(s_n(a_1, \ldots, a_{n+2}), a_{n+3}, \ldots, a_{2n+3}) = 0 \neq 1 = s_n(a_1, s_n(a_2, \ldots, a_{n+3}), a_{n+4}, \ldots, a_{2n+3})
\]

Note that if a Boolean function \(f\) is non-associative, then its dual is also non-associative. Now, if \([C_1, C_2]\) is a minimal and uncountable closed interval, then \(C_2 \setminus C_1\) contains at least one of the functions above or the dual of one of the functions above, and thus it contains a non-associative function. Since each uncountable closed interval must contain a minimal and uncountable closed interval, we conclude that if \([C_1, C_2]\) is an uncountable closed interval, then \(C_2 \setminus C_1\) contains a non-associative Boolean function, and the proof of the theorem is complete. \(\square\)

References


Department of Mathematics, Statistics and Philosophy, University of Tampere, Kalevantie 4, 33014 Tampere, Finland
E-mail address: Miguel.Couceiro@uta.fi