Theoretical Analysis of Joint Synchronization Error Effects for OFDMA Systems

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Abstract—Few papers investigate synchronization error effects for the uplink of orthogonal frequency-division multiple access (OFDMA) systems. By contrast, this paper simultaneously analyzes the joint effects of major synchronization errors, including symbol time (ST) offset (STO), carrier frequency offset (CFO) and sampling clock frequency offset (SCFO) for the uplink of OFDMA systems in time-variant multipath fading channels. Such errors degrade the performance of an OFDMA receiver by introducing inter-carrier interference (ICI), inter-symbol interference (ISI) and multiple-access interference (MAI) into the systems. A theoretical signal-to-interference-and-noise ratio (SINR) is formulated to characterize the losses due to synchronization errors in time-variant multipath fading channels. The results provide designers a useful reference in designing suitable synchronization algorithms for the OFDMA applications.

Index Terms—MAI, OFDMA, SINR, Synchronization Errors

I. INTRODUCTION

Orthogonal frequency-division multiple access (OFDMA) is a promising technology for broadband wireless network beyond 3G due to its high spectrum efficiency, and its robustness to the effects of multipath fading channels. However, similar to orthogonal frequency-division multiple access (OFDM), it is sensitive to synchronization errors. OFDMA is a multiple access technique, in which the subchannels (composed of multiple subcarriers) of an OFDM symbol are shared by multiple users to utilize multiuser diversity. Multiple subchannels may be assigned to a user depending on the bandwidth request. Multiuser OFDMA uplink synchronization is more difficult than synchronization in a broadcast or downlink scenarios for a couple of reasons. First, frequency and timing synchronizations cannot be compensated at the base station (BS), as the correction of one user’s offsets would misalign the other users. Second, the estimations become even difficult because the offsets of all users should be estimated and the interferences due to multiple accesses may escalate as the active users increase. To maintain the orthogonality among the subcarriers, the signals from all active users should arrive at the BS synchronously. This is accomplished by an initial uplink synchronization called a ranging process by which ranging subscriber stations (RSSs), a.k.a. asynchronous users, adjust their transmission time instants and transmitted powers.

Figure 1. Three different ST regions for the desired user $p$. This figure depicts a later user $q$ ($\tau^{q} \geq 0$) relative to the desired user $p$, whereas an earlier user ($\tau^{q} < 0$) can be similarly shown. The ST of the desired user is assumed to be in the Good ST region. The corresponding ST for the user $q$ is in the Bad ST1 region. Different users have different STOs.

As shown in Fig. 1, for the desired $p$-th user, the estimated symbol time (ST) is located in one of the three regions of an OFDM symbol: the Bad ST1 region, the Good ST region, and the Bad ST2 region in which the symbol time offsets (STOs) $\tau^{(p)}_{d}$, are confined within the ranges of $-N_{c} \leq \tau^{(p)}_{d} \leq -N_{c} + \tau^{(p)}_{d}$, $-N_{c} + \tau^{(p)}_{d} + 1 \leq \tau^{(p)}_{d} \leq 0$, and $1 \leq \tau^{(p)}_{d} \leq N - 1$, respectively; where $\tau^{(p)}_{d}$ is the maximum delay spread of the channel; and $N$ is the number of subcarriers. In this paper, the superscript $(\cdot)^{(p)}$ denotes the parameter belonging to the $p$-th user. The former two regions are all in the guard interval of length $N_{c}$. When the ST is located in the Good ST region, no inter-symbol interference (ISI) results; however when the ST is located in the Bad ST1 and Bad ST2 regions, the $l$-th symbol has ISI from the $(l-1)$-th symbol and the $(l+1)$-th symbol, respectively. The carrier frequency offset (CFO) and sampling clock frequency offset (SCFO) introduce additional inter-carrier interference (ICI). The interferences incurred by all the other users except the desired one are classified as multiple-access interferences (MAIs).

Many papers treat the synchronization error effects in OFDM systems [1]-[5]. However, few papers analyze the uplink of multiuser OFDMA systems. The work in [6]...
Figure 2. A simplified OFDMA uplink system model.
considers the STO effect in static multipath channels. The work in [7] considers the joint effects of the STO and CFO in static multipath channels. Whereas, [6]-[7] only consider the specific user to which the ST is located in the Good ST region. The work in [8] investigates the CFO effect without the STO. The work in [9] considers the STO, CFO and SCFO separately in static multipath channels. By analyzing the signal model of all the combined synchronization errors assuming time-variant multipath channels, a theoretical signal-to-interference-and-noise ratio (SINR) for the OFDMA systems has been derived in this work.

II. OFDMA UPLINK SYSTEM AND CHANNEL MODEL

A simplified OFDMA uplink system model is shown in Fig. 2. In the figure, \( X_p^{(n)} \) is the transmitted frequency-domain data of the \( p \)-th user on the \( k \)-th subcarrier; \( \tilde{X}_{ik} \) is the aggregately received frequency-domain data from all active users on the \( k \)-th subcarrier; \( U^{(p)} \) is the set of subcarriers that are assigned to user \( p \); \( X_{ik}^{(n)} \) is the \( n \)-th transmitted time-domain data sample of the \( l \)-th symbol; \( 1/T_s^{(n)} \) is the transmitter’s sampling frequency; \( 1/T_s^{(p)} \) is the sample’s frequency, where \( T_{ok}^{(p)} \) is the SCFO normalized by \( 1/T_s^{(p)} \); and \( f_d^{(k)} = (1 + \epsilon_d^{(k)}) f_0^{(k)} = (1 + \epsilon_d^{(k)}) f_c^{(k)} \) is the carrier frequency, where \( \epsilon_d^{(k)} \) is the CFO normalized by subcarrier spacing. On the transmitter side, for \( k \in U^{(p)} \), the subcarriers are modulated by complex data symbols. For \( k \notin U^{(p)} \), the subcarriers are modulated by null (zero) data symbols. The total \( N \) subcarriers are transformed into time-domain by using the inverse fast Fourier transform (IFFT). The last \( N_c \) IFFT samples are copied to form cyclic prefix (CP) and inserted at the beginning of each OFDM symbol. By inserting CP, the guard interval is created so that ISI can be avoided and orthogonality among subcarriers can be sustained. The receiver uses fast Fourier transform (FFT) to demodulate received data.

In this paper, \( h_s^{(p)}(n, \tau) \) denotes the discrete time-variant channel impulse responses (CIR) of the \( l \)-th symbol. Furthermore, three assumptions regarding the channels are made: (a) wide-sense stationary and uncorrelated scattering (WSSUS), and (b) the Doppler spectrum follows the Jakes’ model [10]. Based on these assumptions, the cross-correlation of the CIR can be obtained by

\[
E[h_s^{(p)}(n, \tau) h_s^{(q)}(n, \tau)] = E[h_s^{(p)}(n, \tau) h_s^{(q)}(n, \tau)] \delta(\tau - \tau_d) \delta(t_d - t_d)
\]

where \( \delta(\cdot) \) is the Dirac delta function; \( \sigma_v^2 = E[|h_s^{(p)}(\tau)|^2] \) is the power of the \( \tau \)-th channel tap; \( J_{\rm L}(\cdot) \) is the zeroth-order Bessel function of the first kind; \( \beta^{(p)} = 2\pi f_{\rm d}^{(p)} T_{\rm c}^{(p)}/N \); \( p \neq q; \tau_\delta = \tau_d; \Delta \tau = n - n_d; f_d^{(p)} \) represents the maximum Doppler shift in Hertz; \( T_{\rm c}^{(p)} = N T_{\rm d}^{(p)} \) is the symbol duration; and \( f_d^{(p)} T_{\rm c}^{(p)} \) is the normalized Doppler frequency (NDF).

III. ANALYSIS OF RECEIVED FREQUENCY-DOMAIN DATA AND SINR

In the following derivations, we assume that symbol index \( l \) is the same for both the receiver side and the transmitter side due to ST and/or SCFO compensation. The estimated ST of each user is located in one of those three different regions as depicted in Fig. 1, where the ideal ST of the desired \( p \)-th user is marked by the time index at zero. The STO \( n_1^{(q)} \) of the \( q \)-th user is with reference to its ideal ST of the \( l \)-th symbol. \( \tau_d^{(p)} \) is the signal delay of the \( q \)-th user relative to the \( p \)-th user (with \( \tau_d^{(p)} = 0 \)). Therefore, the STOs of the \( q \)-th user and the \( p \)-th user have the following relationship,

\[
n_1^{(q)} = n_1^{(q)} - \tau_d^{(p)}.\tag{2}
\]

The transmitted signal of the \( l \)-th symbol of the \( p \)-th user is

\[
x_s^{(l)}(t) = \frac{1}{N} \sum_{m \in U^{(p)}} x_{ik}^{(n)} e^{j2\pi (m-\tau_\delta) T_{\rm c}^{(p)}} - N_0 T_{\rm c}^{(p)} \leq t \leq N T_{\rm c}^{(p)}.
\]

where \( m \) is the subcarrier index of the transmitter. The received time-domain signal is

\[
\tilde{x}_s^{(l)}(t) = \sum_{i=1}^{i_{\rm max}} x_{i1}^{(p)}(t - \tau T_{\rm c}^{(p)}) h_s^{(p)}(t-i\tau T_{\rm c}^{(p)}), N_0 T_{\rm c}^{(p)} \leq t \leq (N + \tau T_{\rm c}^{(p)}) T_{\rm c}^{(p)}.
\]

where \( h_s^{(p)}(t, \tau) \) is the continuous-time time-variant CIR of the \( l \)-th symbol. The received signal, under the effect of the CFO, belonging to the \( p \)-th user is

\[
\tilde{x}_s^{(l)}(t) = \sum_{i=1}^{i_{\rm max}} \tilde{x}_s^{(p)}(t - i N T_{\rm c}^{(p)}) + w(t)
\]

where

\[
\tilde{x}_s^{(p)}(t - i N T_{\rm c}^{(p)}) \triangleq x_s^{(p)}(t - i N T_{\rm c}^{(p)}) e^{j2\pi f_d^{(p)} T_{\rm c}^{(p)}};
\]

\[
w(t) \triangleq w(t) e^{j2\pi f_d^{(p)} T_{\rm c}^{(p)}}.
\]
is the additive white Gaussian noise (AWGN), and $N_s = N + N_o$ is the OFDM symbol length including CP. The received discrete-time sample due to the $l$-th transmitted symbol that belonging to the $p$-th user is

$$
\tilde{x}_i^{(p)} = x_i^{(p)}(t - lN_T^{(p)}) \big|_{t \in [0, \tau_p' \gamma \gamma]}.
$$

One can define the discrete-time AWGN as $w_i^{(p)} \triangleq w(t) \big|_{t \in [0, \tau_p' \gamma \gamma]}$.

By treating all the other users except the desired one as the sources of the MAI, the analyses of the self-interference of the desired user and the MAI from all the other users are derived separately in three different regions below.

**Case 1:** The estimated ST is located in the Good ST region $R_1$ of the desired user. The received frequency-domain data belonging to the $p$-th user can be determined as

$$
\tilde{X}^{(p)} = \text{FFT} \{ \tilde{x}_i^{(p)} g_x(n' - n_a) + w_i^{(p)} \} 
$$

where

$$
g_x(n) = \begin{cases} 
1, & 0 \leq n < N \\
0, & \text{otherwise}
\end{cases}
$$

With (3)-(8), and assuming $1/(1 + \epsilon_i^{(p)}(\gamma)) \equiv 1 - \epsilon_i^{(p)}$, $\epsilon_i^{(p)} \epsilon_j^{(p)} \equiv 0$, after some manipulations, it can be shown that

$$
\tilde{X}^{(p)} = \tilde{X}^{(p),i} + N^{(p)}
$$

where

$$
\tilde{X}^{(p),i} \triangleq \sum_{n = 0}^{N - 1} \tilde{X}^{(p),i,\{n\}} + v_i^{(p)},
$$

$$
N^{(p)} \triangleq e^{j2\pi n_a/N}, \quad \text{and} \quad v_i^{(p)} \triangleq \text{FFT} \{ w_i^{(p)} \}. \quad \text{In addition},
$$

$$
\hat{H}^{(p)} \triangleq \frac{1}{N} \sum_{n = 0}^{N - 1} H_i^{(p)}(n', k) W_n \epsilon_i^{(p)}(n' - n_a) W_n',
$$

is the time-averaged time-variant transfer function on the $k$-th subcarrier,

$$
H_i^{(p)}(n', m) \triangleq \sum_{i' = 0}^{M - 1} h_i^{(p)}(n', \tau) W_n m
$$

is the time-variant transfer function, while

$$
\tilde{X}^{(p),i,\{n\}} \triangleq X_i^{(p)} \frac{1}{N} \sum_{n = 0}^{N - 1} H_i^{(p)}(n', m) W_n \epsilon_i^{(p)}(n' - n_a) W_n' \epsilon_i^{(p)}(n' - n_a)
$$

is the ICI. With (11), (14), (13) and utilizing (1), the ICI power incurred from the $p$-th user can be shown to be

$$
E \left[ ||\tilde{X}^{(p)}||^2 \right] = C_i^{(p)} \sigma_i^{(p)} \sum_{m = 0}^{M - 1} (N - |\Delta n|) J_i(\beta_i^{(p)} \Delta n) \times W_n \epsilon_i^{(p)}(n' - n_a) W_n' \epsilon_i^{(p)}(n' - n_a)
$$

where $\sigma_i^{(p)}$ is the signal power, $\sigma_{i,\gamma}^{(p)}$ is the AWGN power and

$$
C_i^{(p)} = \sum_{m = 0}^{M - 1} \sigma_i^{(p)} / N = \sigma_i^{(p)} / N \gamma. \quad \text{By (10), (12), (13) and (1), the desired signal power can be shown to be}
$$

$$
E \left[ ||\tilde{X}^{(p)}||^2 \right] = C_i^{(p)} \sigma_i^{(p)} \sum_{m = 0}^{M - 1} (N - |\Delta n|) J_i(\beta_i^{(p)} \Delta n) W_n \epsilon_i^{(p)}(n' - n_a) W_n' \epsilon_i^{(p)}(n' - n_a).
$$

Similarly, the MAI power from all the other users with their STs located in this region can be shown to be

$$
E \left[ ||\tilde{X}^{(p),i,\{n\}}||^2 \right] = E \left[ \sum_{k = 1}^{K} \sum_{\{n\}} \tilde{X}^{(p),i,\{n\}} \right] = \sum_{k = 1}^{K} \sum_{\{n\}} (N - |\Delta n|) J_i(\beta_i^{(p)} \Delta n) W_n \epsilon_i^{(p)}(n' - n_a) W_n' \epsilon_i^{(p)}(n' - n_a).
$$

**Case 2:** The estimated ST is located in the Bad ST1 region $R_2$ of the desired user. From (5), by considering the ISI from the $(l-1)$-th symbol, the received frequency-domain data belonging to the $p$-th user can be determined as

$$
\tilde{X}^{(p)} = \text{FFT} \{ \tilde{x}_i^{(p)} g_x(n' - n_a) - \tilde{x}_i^{(p)} - \tilde{x}_i^{(p)} g_x(n' - n_a) + w_i^{(p)} \}
$$

where $N_i^{(p)} = -N_o + \tau_p^{(p)} - n_a^{(p)} + 1$. As the derivation is similar to Case 1, the desired signal power and MAI power can be shown to be

$$
E \left[ ||\tilde{X}^{(p),i,\{n\}}||^2 \right] = C_i^{(p)} \sigma_i^{(p)} \times \sum_{\{n\}} (N - N_i^{(p)}) |\Delta n| J_i(\beta_i^{(p)} \Delta n) W_n \epsilon_i^{(p)}(n' - n_a) W_n' \epsilon_i^{(p)}(n' - n_a)
$$

and

$$
E \left[ ||\tilde{X}^{(p),i,\{n\}}||^2 \right] = \sum_{\{n\}} C_i^{(p)} \sigma_i^{(p)} \sum_{k = 1}^{K} \sum_{\{n\}} (N - N_i^{(p)}) |\Delta n| J_i(\beta_i^{(p)} \Delta n) W_n \epsilon_i^{(p)}(n' - n_a) W_n' \epsilon_i^{(p)}(n' - n_a)
$$

$$
+ \sum_{k = 1}^{K} \sum_{\{n\}} (N - N_i^{(p)}) |\Delta n| J_i(\beta_i^{(p)} \Delta n) W_n \epsilon_i^{(p)}(n' - n_a) W_n' \epsilon_i^{(p)}(n' - n_a) \sum_{k = 1}^{K} \sigma_i^{(p)}.
$$
Case 3: The estimated ST is located in the Bad ST2 region $R_3$ of the desired user. Since the derivations are similar to Case 2, they are omitted here.

Finally, with (15) and (16), the SINR on the $k$-th subcarrier of the desired user $p$ in the Good ST region can be determined as

$$
\eta^{(p)}_k = E \left[ R^{(p)}_k / \left( \eta^{(p)}_k \right) \right] + E \left[ N^{(p)}_{k,i} \right],
$$

(21)

Similarly, the SINR on the $k$-th subcarrier for the Bad ST1 and Bad ST2 regions can be easily derived and omitted here.

IV. NUMERICAL RESULTS

We consider an OFDMA system of $N = 512$ subcarriers and GI of $N_G = N/8 = 64$ samples. The numerical results are based on distributed carrier assignment scheme. Each subchannel has 32 interleaved subcarriers from the available subcarrier set $\{0, \ldots, N-1\}$. Each user is assigned a subchannel. The eighth subchannel is assigned to the desired user. The maximum number of active users is 16. To simplify the analysis, we assume that the channels, CFO, and SCFO are the same for all users. In addition, the scenario after the initial ranging process is considered. After the initial ranging process, the deviations of the STs of the active users are less than $\frac{N_G}{4}$ samples [1].

The STs of other active users are assumed to be randomly and uniformly distributed in this interval, and the MAI power is averaged over this interval. The SIRs of all the subcarriers assigned to the desired user are also averaged. The maximum delay spread $\tau_c$ of the channel is 24 samples. The channel taps are randomly generated by independent complex Gaussian random variables.

To exemplify those arguments that affect the SIR, the SIRs under various STOs, CFOs, and NDFs are shown in Fig. 3; the SIRs under various STOs and SCFOs are shown in Fig. 4; the SIRs under various STOs and active users are shown in Fig. 5; the SIRs under various NDFs and SCFOs are shown in Fig. 6; and the SIRs under various NDFs and CFOs are shown in Fig. 7. These figures show the SIRs of the desired user against various synchronization errors. The number of active users includes the desired user and all the other users who cause MAIs. The case of one active user means that only the desired user is incorporated in the system and there is no other user.

First, the impact of single synchronization error is investigated. The following examples are demonstrated to achieve the typical condition of SIR $> 20$ dB. For the condition of SIR $> 20$ dB to be satisfied, under the case of 16 active users, the NDF should be less than 8% and the CFO should be less than 5.66% as shown in Fig. 3. The SIRs are the same when $\epsilon_T = \epsilon_f = 0$. As can be seen in Fig. 3, the SIRs have plateau in the Good ST region over the interval of $[-40, 0]$. As can be seen in Fig. 4, the SCFO has a very minor effect on the SIR. In comparison, the effect of the STO weakens considerably the effect of the SCFO. As shown in Fig. 5, for SIR $> 20$ dB, under the case of 16 active users, the STO should be less than 8 samples. Besides, the MAI power is negligible in the Good ST region, while it is significant in the Bad ST regions. The SIR worsens when the number of active users increases due to the increasingly incurred MAI.

Second, we illustrate the impacts of the combined synchronization errors

1) The impact due to the combined synchronization errors of the NDF and STO: As can be seen in Figs. 3 and 5,

1 This observation may be explained by the fact that mobility results in frequency shift of the received signal similar to the CFO.
under the case of 16 active users, the SIRs are 22.2 dB and 20.9 dB due to the single error of NDF = 0.06 and STO = 6 (samples), respectively. However, as can be seen in Fig. 3, when both errors of NDF = 0.06 and STO = 6 co-exist, the SIR drops to 18.5 dB. The degradation due to the combined synchronization errors is 3.7 dB more than the single error of NDF, while 2.4 dB more than the single error of STO. The impact due to the combined synchronization errors of the CFO and STO is similar to the combined NDF and STO effect.

2) The impact due to the combined synchronization errors of the STO and SCFO: As can be seen in Fig. 4, under the case of 16 active users, the SIR is mainly influenced by the STO.

3) The impact due to the combined synchronization errors of the NDF and SCFO: As can be seen in Fig. 6, under the case of 16 active users, the SIR is also mainly influenced by the NDF. The effect of the combined CFO and SCFO errors is similar to the combined NDF and STO effect.

4) The impact due to the combined synchronization errors of the NDF and CFO: As can be seen in Fig. 7, the SIRs due to the single error of NDF = 0.06 and CFO = 0.0424 are both 22.25 dB. However, when the NDF = 0.06 and CFO = 0.0424 coexist, the SIR drops to 19.24 dB. The degradation due to the combined synchronization errors is about 3 dB more than both of the single errors of CFO and SCFO.

5) In short, the impact due to the combined synchronization errors on the SINR can be also easily derived. Besides, the degradation of the SINR due to the combined synchronization errors may be much more severe than a single synchronization error.

V. CONCLUSION

Based on the analyses presented in this paper, it can be seen that the NDF and CFO have similar impacts on the SIR. The SIRs remain the same when $f_{c}^{c+n} = \sqrt{5f_{c}^{c+n}}$. To achieve the typical condition of SIR > 20 dB, the synchronization error constraints have been derived. For the case of 16 active users, the single error of NDF, CFO, and STO should be less than 8%, 5.66%, and 8 samples, respectively. The requirements (to achieve SIR > 20 dB) for the combined synchronization errors can be also easily derived. In summary, the analysis result of the combined effects of the synchronization errors can be used as a reference for designing synchronization algorithms of the OFDMA applications in double-selective fading channels.

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