“Generalized LU-fuzzy Derivative and Numerical Solutions of Fuzzy Differential Equations"

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Generalized LU-fuzzy derivative and numerical solution of Fuzzy Differential Equations

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Abstract—We present a representation of fuzzy numbers and its application to the numerical solution of fuzzy differential (initial value) equations (FDE). The generalized differentiability of a fuzzy-valued function \( g(x) \) of a real variable \( x \in [a, b] \), as recently introduced by Bede and Gal, is expressed in terms of the differentiability of the lower and upper functions defining its level-cuts \( [g(x)]_\alpha = [g_\alpha^-(x), g_\alpha^+(x)] \) where, for any membership level \( \alpha \in [0, 1] \), \( g_\alpha^- \leq g_\alpha^+ \) are ordinary non-fuzzy functions that are monotonic with respect to \( \alpha \). The representation uses a finite decomposition of the membership interval \( 0 = \alpha_0 < \alpha_1 < \ldots \leq \alpha_N = 1 \) and models the level-cuts of the fuzzy numbers and functions to obtain the formulation of a fuzzy differential equation \( y' = f(x, y) \) in terms of a set of ordinary differential equations \( y_\alpha' = \phi_\alpha(x, y), y_\alpha^+ = \phi^+_\alpha(x, y), \quad i = 0, 1, \ldots, N \) \((\phi_\alpha^+ \text{ and } \phi_\alpha^- \text{ depending on the fuzzy-valued function } f(x, y))\) which can be solved by any standard method for ODE’s. Some numerical examples are included.

I. INTRODUCTION

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II. FUZZY NUMBERS AND LU-FUZZY REPRESENTATION

We will consider fuzzy numbers and intervals, i.e. fuzzy sets defined over the field \( \mathbb{R} \) of real numbers having a particular form. A general fuzzy set over \( \mathbb{R} \) is usually defined by its membership function \( \mu : \mathbb{R} \rightarrow T \subseteq [0, 1] \) and a fuzzy (sub)set \( u \) of \( \mathbb{R} \) is uniquely characterized by the pairs \((x, \mu_u(x))\) for each \( x \in \mathbb{R} \); the value \( \mu_u(x) \in [0, 1] \) is the membership grade of \( x \) to the fuzzy set \( u \). Denote by \( \mathcal{F}(\mathbb{R}) \) the collection of the fuzzy sets over \( \mathbb{R} \). Elements of \( \mathcal{F}(\mathbb{R}) \) will be denoted by letters \( u, v, w \) and the corresponding membership functions by \( \mu_u, \mu_v, \mu_w \).

Fundamental concepts in fuzzy theory are the support, the level-sets (or level-cuts) and the core of a fuzzy set:

Definition 1: Let \( \mu_u \) be the membership function of a fuzzy set \( u \) over \( \mathbb{R} \). The support of \( u \) is the (crisp) subset of points of \( \mathbb{R} \) at which the membership grade \( \mu_u(x) \) is positive: \( \text{supp}(u) = \{x|x \in \mathbb{R}, \mu_u(x) > 0\} \). For \( \alpha \in [0, 1] \), the \( \alpha \)-level cut of \( u \) (or simply the \( \alpha \)-cut) is defined by \( [u]_\alpha = \{x|x \in \mathbb{R}, \mu_u(x) \geq \alpha\} \) and for \( \alpha = 0 \) by the closure of the support \( [u]_0 = \text{cl}\{x|x \in \mathbb{R}, \mu_u(x) > 0\} \). The core of \( u \) is the set of elements of \( \mathbb{R} \) having membership grade 1, i.e. \( \text{core}(u) = \{x|x \in \mathbb{R}, \mu_u(x) = 1\} \) and we say that \( u \) is normal if \( \text{core}(u) \neq \emptyset \).

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It is well-known that the level - cuts are "nested", i.e. \( [u]_\alpha \subseteq [u]_\beta \) for \( \alpha > \beta \).

A particular class of fuzzy sets \( u \in \mathcal{F}(\mathbb{R}) \) is when the support is a convex set \((A \text{ is said convex if } (1-t)x' + tx'' \in A \text{ for every } x', x'' \in A \text{ and all } t \in [0, 1])\) and the membership function is quasi-concave, i.e. \( \text{supp}(u) \) is convex and \( \mu_u((1-t)x' + tx'') \geq \min\{\mu_u(x'), \mu_u(x'')\} \) for every \( x', x'' \in \text{supp}(u) \) and \( t \in [0, 1] \). Equivalently, \( \mu_u \) is quasi-concave if the level sets \( [u]_\alpha \) are convex for all \( \alpha \in [0, 1] \).

If the membership function is upper semi-continuous, then the level-cuts are closed.

Definition 2: A fuzzy set \( u \) is a fuzzy quantity if the \( \alpha \)-cuts are nonempty, compact intervals of the form \( [u]_\alpha = [u^-_\alpha, u^+_\alpha] \subseteq \mathbb{R} \).

We denote by \( \mathcal{F} \) the set of fuzzy quantities.

Definition 3: An LU-fuzzy quantity (number or interval) \( u \) is completely determined by any pair \( u = (u^-, u^+ : [0, 1] \rightarrow \mathbb{R}) \) of functions \( u^-, u^+ : [0, 1] \rightarrow \mathbb{R} \), defining the end-points of the \( \alpha \)-cuts, satisfying the three conditions: (i) \( u^- : \alpha \rightarrow u^-_\alpha \subseteq \mathbb{R} \) is a bounded monotonic nondecreasing left-continuous function \( \forall \alpha \in [0, 1] \) and right-continuous for \( \alpha = 0 \); (ii) \( u^+ : \alpha \rightarrow u^+_\alpha \subseteq \mathbb{R} \) is a bounded monotonic nonincreasing left-continuous function for \( \alpha \in [0, 1] \) and right-continuous for \( \alpha = 0 \); (iii) \( u^- \leq u^+_\alpha \forall \alpha \in [0, 1] \).

The support of \( u \) is the interval \([u^-_0, u^+_0]\) and the core is \( [u^-_1, u^+_1] \). We refer to the functions \( u^-_\alpha \) and \( u^+_\alpha \) as the lower and upper branches on \( u \), respectively. If the two branches \( u^-_\alpha \) and \( u^+_\alpha \) are continuous invertible functions then \( \mu_u(.) \) is formed by two continuous branches, the left being the increasing inverse of \( u^-_\alpha \) on \([u^-_\alpha, u^+_\alpha]\) and, the right, the decreasing inverse of \( u^+_\alpha \) on \([u^-_\alpha, u^+_\alpha]\).

Well-known properties of the level - cuts are:

\[
[u]_\alpha \subseteq [u]_\beta \text{ for } \alpha > \beta, \quad (1)\\
[u]_\alpha = \bigcap_{\beta < \alpha} [u]_\beta \text{ for } \alpha \in [0, 1] \quad (2)
\]

and (if \( x \in \text{supp}(u) \), otherwise \( \mu_u(x) = 0 \))

\[
\mu_u(x) = \sup\{\alpha | \alpha \in [0, 1] \text{ for which } x \in [u]_\alpha\}. \quad (3)
\]

A particular class of fuzzy sets \( u \in \mathcal{F}(\mathbb{R}) \) is when the support is a convex set and the membership function is quasi-concave:
Definition 4: Consider $u \in \mathcal{F}(\mathbb{R})$ and assume that $\text{supp}(u)$ is a convex set; we say that the membership function $\mu_u$ is quasi-concave if $\mu_u((1-t)x' + tx'') \geq \min\{\mu_u(x'), \mu_u(x'')\}$ for every $x', x'' \in \text{supp}(u)$ and $t \in [0, 1]$. Equivalently, $\mu_u$ is quasi-concave if the level sets $[u]_\alpha$ are convex sets for all $\alpha \in [0, 1]$.

Definition 5: Consider $u \in \mathcal{F}(\mathbb{R})$; its membership function is said to be upper semicontinuous at every $\widehat{x} \in \text{supp}(u)$

$$
\lim_{x \to \widehat{x}} \sup \mu_u(x) = \mu_u(\widehat{x})
$$
or, equivalently, if the level-cuts $[u]_\alpha$ are closed sets for all $\alpha \in [0, 1]$.

With the definitions above we can define the so called fuzzy quantities.

Definition 6: A fuzzy quantity is a fuzzy set $u \in \mathcal{F}(\mathbb{R})$ with the properties:

(i) $\mu_u$ has bounded support and is normal,

(ii) $\mu_u$ is quasi-concave,

(iii) $\mu_u$ is upper semicontinuous

or, equivalently:

(a.) $[u]_\alpha$ are convex sets for all $\alpha \in [0, 1]$,
(b.) $[u]_\alpha$ are compact sets for all $\alpha \in [0, 1]$,
(c.) $[u]_\alpha$ satisfy conditions (1) and (2).

We will denote by $\mathcal{F}_I$ the set of fuzzy quantities. A fundamental theorem in fuzzy theory characterizes uniquely (up to a scalar multiplication) fuzzy quantities $u \in \mathcal{F}_I$ either in terms of the membership function or in terms of the associated level-cuts: if the membership function $\mu_u$ satisfies (i)–(iii) then its $\alpha$–cuts satisfy (a.)–(c.) and, vice versa, if a family of sets $\{A_\alpha \subset \mathbb{R} | \alpha \in [0, 1]\}$ satisfies (a.)–(c.) then the membership function defined by

$$
\mu(x) = \begin{cases} 
\sup\{\alpha \in [0, 1] \text{ for which } x \in A_\alpha\}, & \text{if } x \in A_0 \\
0, & \text{if } x \notin A_0 
\end{cases}
$$

has properties (i)–(iii) and defines a fuzzy quantity $u \in \mathcal{F}_I$ such that $[u]_\alpha = A_\alpha$, $\forall \alpha \in [0, 1]$.

Using this fact, we can structure $\mathcal{F}_I$ by an addition and a scalar multiplication, defined either by the level sets or, equivalently, by the Zadeh extension principle. Let $u, v \in \mathcal{F}_I$ have membership functions $\mu_u, \mu_v$ and $\alpha$–cuts $[u]_\alpha, [v]_\alpha$, $\alpha \in [0, 1]$ respectively. The addition $u + v \in \mathcal{F}_I$ and the scalar multiplication $ku \in \mathcal{F}_I$ for $k \in \mathbb{R} \setminus \{0\}$ have membership functions (extension principle)

$$
\mu_{u+v}(z) = \sup\{\min\{\mu_u(x), \mu_v(y)\} | z = x + y\} \\
\mu_{ku}(x) = \mu_u(\frac{x}{k})
$$

and level cuts

$$
[u + v]_\alpha = [u]_\alpha + [v]_\alpha = \{x + y | x \in [u]_\alpha, y \in [v]_\alpha\} \\
[ku]_\alpha = k[u]_\alpha = \{kx | x \in [u]_\alpha\}.
$$

The difference of two fuzzy numbers $u - v$ is defined as the addition $u + (-v)$ where $-v = (-1)v$. Also the (Hukuhara) H-difference is used here; it is defined by $u \odot_h v = w \iff u = v + w$, being $+$ the standard fuzzy addition; if $u \odot_h v$ exists, its $\alpha$–cuts are $[u \odot_h v]_\alpha = [u_\alpha - v_\alpha, u_\alpha + v_\alpha]$. Finally, the Hausdorff distance on $\mathcal{F}_I$ is defined by

$$
D(u, v) = \sup_{\alpha \in [0, 1]} \{\max\{[u_\alpha - v_\alpha], |u_\alpha + v_\alpha|\}\}
$$

and $(\mathcal{F}_I, D)$ is a complete metric space.

There are many choices for $\bar{u}(-)$ and $\bar{u}^+(\cdot)$. If we start with two decreasing shape functions $p(\cdot)$ and $q(\cdot)$ and with four numbers $u_0^- \leq u_1^- \leq u_1^+ \leq u_0^+$ defining the support and the core of $u$ then we can model $u_\ominus$ and $u_\oplus$ by $u_\ominus = u_1^- - (u_1^- - u_0^-)q(\alpha)$ and $u_\oplus = u_1^+ - (u_1^+ - u_0^+)q(\alpha)$ for all $\alpha \in [0, 1]$. The simplest fuzzy quantities have linear branches: a trapezoidal fuzzy interval, denoted by $u = (a, b, c, d)$, where $a \leq b \leq c \leq d$, has $\alpha$–cuts $[u]_\alpha = [a + \alpha(b-a), d - \alpha(d-c)]$, $\alpha \in [0, 1]$, obtaining a triangular fuzzy number if $b = c$.

As we have seen in the previous section, the LU representations of fuzzy numbers require to use appropriate (monotonic) shape functions to model the lower and upper branches of the $\alpha$–cuts. In this section we present the basic elements of a parametric representation of the shape functions proposed in [7] and [12] based on monotonic Hermite-type interpolation. The parametric representations can be used both to define the shape functions and to calculate the arithmetic operations by error controlled approximations.

We first introduce some models for "standardized" differentiable monotonic shape functions $p : [0, 1] \to [0, 1]$ such that $p(0) = 0$ and $p(1) = 1$ with $p(t)$ increasing on $[0, 1]$; if interested to decreasing functions, we can start with an increasing function $p(\cdot)$ and simply define corresponding decreasing functions $q : [0, 1] \to [0, 1]$ by $q(t) = 1 - p(t)$ or $q(t) = p(\varphi(t))$ where $\varphi : [0, 1] \to [0, 1]$ is any decreasing bijection (e.g. $\varphi(t) = 1 - t$).

Valid shape functions can be obtained by $p : [0, 1] \to [0, 1]$, satisfying the four Hermite interpolation conditions $p(0) = 0$, $p(1) = 1$ and $p'(0) = \beta_0$, $p'(1) = \beta_1$ for any value of the two nonnegative parameters $\beta_i \geq 0$, $i = 0, 1$.

To explicit the parameters, we denote the interpolating function by $t \mapsto p(t; \beta_0, \beta_1)$ for $t \in [0, 1]$.

We recall here two of the basic forms illustrated in [12]:

- $\circ$ (2,2)-rational spline:
$$
p(t; \beta_0, \beta_1) = \frac{t^2 + \beta_0 t(1-t)}{1 + (\beta_0 + \beta_1 - 2)t(1-t)};
$$

- $\circ$ mixed exponential spline:
$$
p(t; \beta_0, \beta_1) = \frac{1}{a} \left[ e^{2(3-2t)} - \beta_0 \beta_1 - \beta_0 (1-t)^a + \beta_1 t^n \right]
$$

where $a = 1 + \beta_0 + \beta_1$.

Note that in both forms we obtain a linear $p(t) = t$, $\forall t \in [0, 1]$ if $\beta_0 = \beta_1 = 1$ and a quadratic $p(t) = t^2 + \beta_0 t(1-t)$ if $\beta_0 + \beta_1 = 2$.

In order to produce different shapes we can either fix the slopes $\beta_0$ and $\beta_1$ (if we have information on the first derivatives at $t = 0$, $t = 1$) or we can estimate them by knowing values of $p(t)$ in additional points.
The model functions above can be adopted to represent the functions "piecewise", on a decomposition of the interval \([0, 1]\) into \(N\) subintervals \(0 = \alpha_0 < \alpha_1 < \ldots < \alpha_{i+1} < \alpha_i < \ldots < \alpha_N = 1\). It is convenient to use the same subdivision for both the lower \(u_0^\alpha\) and upper \(u_0^+\) branches (we can always reduce this situation by the union of two different subdivisions). In each subinterval \(I_i = [\alpha_{i-1}, \alpha_i]\), the values and the slopes of the two functions are

\[
\begin{align*}
  u^\alpha_{(i+1)} & = u^\alpha_{i}, & u^+_{(i+1)} & = u^+_{i}, \\
  \delta u^\alpha_i & = \frac{u^\alpha_{i+1} - u^\alpha_i}{\delta t}, & \delta u^+_{i} & = \frac{u^+_{i+1} - u^+_{i}}{\delta t},
\end{align*}
\]

(9)

and by the transformation \(t_\alpha = \frac{t - \alpha_{i-1}}{\alpha_i - \alpha_{i-1}}, \quad \alpha \in I_i\), each subinterval \(I_i\) is mapped into the standard \([0, 1]\) interval to determine each piece independently. Globally continuous or more regular \(C^{(1)}\) fuzzy numbers can be obtained directly from the data (for example, \(u_{1,1} = u_{0,i+1}, u^+_{1,1} = u^+_{0,i+1}\) for continuity and \(\delta u^\alpha_{i+1} = \frac{u^\alpha_{i+1} - u^\alpha_i}{\delta t}, \delta u^+_{i+1} = \frac{u^+_{i+1} - u^+_{i}}{\delta t}\))

for differentiability at \(\alpha = \alpha_i\).

Let \(p_j^\alpha (t)\) denote the model function on \(I_i\); we obtain

\[
p_j^\alpha (t) = p(t; \beta^\alpha_{0,i}, \beta^+_{1,i}),
\]

(10)

\[
\begin{align*}
  \alpha & = \frac{-\alpha_{i-1}}{\alpha_i - \alpha_{i-1}}, \\
  \beta^\alpha & = \frac{-\alpha_{i-1}}{\alpha_i - \alpha_{i-1}} d^\alpha_{i,j}, \beta^+ & = \frac{-\alpha_{i-1}}{\alpha_i - \alpha_{i-1}} d^+_{i,j}.
\end{align*}
\]

The illustrated monotonic models suggest a first parametrization of fuzzy numbers on the trivial decomposition of interval \([0, 1]\), with \(N = 1\) (without internal points) and \(\alpha_0 = 0, \alpha_1 = 1\). In this simple case, \(u\) can be represented by a vector of 8 components (the slopes corresponding to \(u^\alpha\) are denoted by \(\delta u^\alpha\), etc)

\[
u = (u_0^\alpha, \delta u_0^\alpha, u_0^+, \delta u_0^+, u_1^-, \delta u_1^-, u_1^+, \delta u_1^+)
\]

(11)

with \(u_0^\alpha, \delta u_0^\alpha, u_0^+, \delta u_0^+\) for the lower branch \(u_0^\alpha\) and \(u_0^-, \delta u_0^-, u_0^+, \delta u_0^+\) for the upper branch \(u_0^+\).

On a decomposition \(0 = \alpha_0 < \alpha_1 < \ldots < \alpha_N = 1\) we can proceed piecewise. For example, a differentiable shape function requires \(4(N + 1)\) parameters

\[
u = (u_i^\alpha \mid \delta u_i^\alpha, u_i^+ \mid \delta u_i^+)\mid_{i=0,1,\ldots,N} \text{ with }
\]

(12)

\[
\begin{align*}
  u_0^- & \leq u_1^- \leq \ldots \leq u_N^- \leq u_N^+ \leq u_{N-1}^- \leq \ldots \leq u_0^+ \text{ (data)} \quad \delta u_i^- \geq 0, \delta u_i^+ \leq 0 \text{ (slopes)}. \quad \alpha \in [0,1]
\end{align*}
\]

and the branches are computed according to (10). In [7] and [12] we have analyzed the advantages of the LU representation in the computation of fuzzy expressions.

III. FUZZY DIFFERENTIAL EQUATIONS

We apply the LU-fuzzy representation of fuzzy numbers to the numerical solution of an ordinary fuzzy differential equation (FODE). For simplicity of notation we consider the unidimensional fuzzy initial value problem on the space \(F\) of fuzzy numbers

\[
y' = f(x, y),
\]

(11)

\[
y(0) = y_0
\]

where \(f : \mathbb{R} \times F \to F\) is a given function of \(x \in \mathbb{R}\) and \(y \in F\), defined in terms of its level cuts \([f(x, y)]_\alpha = [f^-_\alpha(x, y), f^+_\alpha(x, y)]\). Denote also \([y]_\alpha = [y^-_\alpha, y^+_\alpha]\). We adopt here the concept of generalized differentiability introduced by Bede and Gal (see [1] and [1]). The generalized derivative of a fuzzy function \(G \in \mathbb{R}^a \to F\) at a point \(x_0 \in [a, b]\) is defined by considering the four combinations of fuzzy Hukuhara differences \(g(x_0 + h) \circ_h g(x_0), g(x_0 - h) \circ_h g(x_0), g(x_0) \circ_h g(x_0 + h), g(x_0) \circ_h g(x_0 - h)\) and the existence of the limits (in the D metric):

\[
\text{case (i)} \left\{ \begin{array}{ll}
  \lim_{h \to 0^+} \frac{1}{h} [g(x_0 + h) \circ_h g(x_0)] = & \\
  \lim_{h \to 0^-} \frac{1}{h} [g(x_0) \circ_h g(x_0 - h)] = & \\
\end{array} \right.
\]

\[
\text{case (ii)} \left\{ \begin{array}{ll}
  \lim_{h \to 0^+} \frac{1}{h} [g(x_0) \circ_h g(x_0 + h)] = & \\
  \lim_{h \to 0^-} \frac{1}{h} [g(x_0) \circ_h g(x_0)] = & \\
\end{array} \right.
\]

\[
\text{case (iii)} \left\{ \begin{array}{ll}
  \lim_{h \to 0^+} \frac{1}{h} [g(x_0) \circ_h g(x_0 + h)] = & \\
  \lim_{h \to 0^-} \frac{1}{h} [g(x_0) \circ_h g(x_0)] = & \\
\end{array} \right.
\]

\[
\text{case (iv)} \left\{ \begin{array}{ll}
  \lim_{h \to 0^+} \frac{1}{h} [g(x_0) \circ_h g(x_0 + h)] = & \\
  \lim_{h \to 0^-} \frac{1}{h} [g(x_0) \circ_h g(x_0)] = & \\
\end{array} \right.
\]

It is easy to see that, if the level cuts of \(g(x)\) are \([g(x)]_\alpha = [g^-_\alpha(x), g^+_\alpha(x)]\), then the conditions above require that all functions \(g^-_\alpha, g^+_\alpha : [a, b] \to \mathbb{R}\) for \(\alpha \in [0,1]\) be differentiable at \(x_0\) (in the ordinary sense) and that, for each case,

\[
(i) \quad \left\{ \frac{d}{dx} g^-_\alpha(x_0), \frac{d}{dx} g^+_\alpha(x_0) \right\}, \quad \alpha \in [0,1]
\]

(iii) and (iv) \(\frac{d}{dx} g^-_\alpha(x_0) = \frac{d}{dx} g^+_\alpha(x_0), \quad \forall \alpha \in [0,1]\).

IV. CONCLUSIONS

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REFERENCES