On Self-duality of Branchwidth in Graphs of Bounded Genus

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Abstract

A graph parameter is self-dual in some class of graphs embeddable in some surface if its value does not change in the dual graph more than a constant factor. Self-duality has been examined for several width-parameters, such as branchwidth, pathwidth, and treewidth. In this paper, we give a direct proof of the self-duality of branchwidth in graphs embedded in some surface. In this direction, we prove that $bw(G^*) \leq 6 \cdot bw(G) + 2g - 4$ for any graph $G$ embedded in a surface of Euler genus $g$.

Key words: graphs on surfaces, branchwidth, duality, polyhedral embedding.

1 Preliminaries

Our main reference for graphs on surfaces is the monograph by Mohar and Thomassen [10]. A surface $\Sigma$ can be obtained, up to homeomorphism, by adding $eg(\Sigma)$ crosscaps to the sphere, and $eg(\Sigma)$ is called the Euler genus of $\Sigma$. We denote by $(G, \Sigma)$ a graph $G$ embedded in a surface $\Sigma$, that is, drawn in $\Sigma$ without edge crossings. A subset of $\Sigma$ meeting the drawing only at vertices of $G$ is called $G$-normal. An $O$-arc on $\Sigma$ is a subset that is homeomorphic to a cycle. If an $O$-arc is $G$-normal, then we call it a noose. A noose $N$ is contractible if it is the boundary of some disk on $\Sigma$ and is surface separating if $\Sigma \setminus N$ is

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disconnected. The length of a noose is the number of the vertices it meets. Representativity, or face-width, is a parameter that quantifies local planarity and density of embeddings. The representativity \( \text{rep}(G, \Sigma) \) of a graph embedding \((G, \Sigma)\) is the smallest length of a non-contractible noose in \(\Sigma\). We call an embedding \((G, \Sigma)\) polyhedral if \(G\) is 3-connected and \(\text{rep}(G, \Sigma) \geq 3\).

For a given embedding \((G, \Sigma)\), we denote by \((G^*, \Sigma)\) its dual embedding. Thus \(G^*\) is the geometric dual of \(G\). Each vertex \(v\) (resp. face \(r\)) in \((G, \Sigma)\) corresponds to some face \(v^*\) (resp. vertex \(r^*\)) in \((G^*, \Sigma)\). Also, given a set \(X \subseteq E(G)\), we denote as \(X^*\) the set of the duals of the edges in \(X\).

Let \(\mathcal{G}\) be a class of graphs embeddable in a surface \(\Sigma\). We say that a graph parameter \(p\) is \((c,d)-self-dual\) on \(\mathcal{G}\) if for every graph \(G \in \mathcal{G}\) and for its geometric dual \(G^*\), \(p(G^*) \leq c \cdot p(G) + d\). Self-duality of treewidth, pathwidth, or branchwidth (defined in Section 2) has played a fundamental role in the proof of the celebrated Graph Minors Theorem [13], as well as being useful for finding polynomial-time approximation algorithms for these parameters [2].

Most of the research concerning self-duality of graph parameters has been devoted to treewidth. Lapoire proved [7], using algebraic methods, that treewidth is \((1,1)-self-dual\) in planar graphs, settling a conjecture stated by Robertson and Seymour [11]. Bouchitté et al. [3] gave a much shorter proof of this result, exploiting the properties of minimal separators in planar graphs.

Fomin and Thilikos [5] proved that pathwidth is \((6,6g-2)-self-dual\) in graphs polyhedrically embedded in surfaces of Euler genus at most \(g\). This result was improved for planar graphs by Amini et al. [1], who proved that pathwidth is \((3,2)-self-dual\) in 3-connected planar graphs and \((2,1)-self-dual\) in planar graphs with a Hamiltonian path.

Concerning branchwidth, Seymour and Thomas [14] proved that it is \((1,0)-self-dual\) in planar graphs that are not forests (for more direct proofs, see also [9] and [6]). In this note, we give a short proof that branchwidth is \((6,2g-4)-self-dual\) in graphs of Euler genus at most \(g\). We also believe that our result can be considerably improved. In particular, we conjecture that branchwidth is \((1,g)-self-dual\).

2 Self-duality of branchwidth

Given a graph \(G\) and a set \(X \subseteq E(G)\), we define \(\partial X = (\bigcup_{e \in X} e) \cap (\bigcup_{e \in E(G) \setminus X} e)\), where edges are naturally taken as pairs of vertices (notice that \(\partial X = \partial (E(G) \setminus X)\)). A branch decomposition \((T, \mu)\) of a graph \(G\) consists of an unrooted ternary tree \(T\) (i.e., all internal vertices are of degree three) and a bijection
\[ \mu : L \to E(G) \] from the set \( L \) of leaves of \( T \) to the edge set of \( G \). For every edge \( f = \{t_1, t_2\} \) of \( T \) we define the \textit{middle set} \( \text{mid}(e) \subseteq V(G) \) as follows: Let \( L_1 \) be the leaves of the connected component of \( T \setminus \{e\} \) that contain \( t_1 \). Then \( \text{mid}(e) = \partial \mu(L_1) \). The \textit{width} of \( (T, \mu) \) is defined as \( \max \{ |\text{mid}(e)| : e \in T \} \).

An optimal branch decomposition of \( G \) is defined by a tree \( T \) and a bijection \( \mu \) which give the minimum width, called the \textit{branchwidth} of \( G \), and denoted by \( \text{bw}(G) \).

If \( (G, \Sigma) \) is a polyhedral embedding, then the following proposition follows by an easy modification of the proof of [5, Theorem 1].

**Proposition 1** Let \( (G, \Sigma) \) and \( (G^*, \Sigma) \) be dual polyhedral embeddings in a surface of Euler genus \( g \). Then \( \text{bw}(G^*) \leq 6 \cdot \text{bw}(G) + 2g - 4 \).

In the sequel, we focus on generalizing Proposition 1 to arbitrary embeddings. For this, we first need some technical lemmata, whose proofs are easy or well known, and omitted in this short note. Note that the removal of a vertex in \( G \) corresponds to the contraction of a face in \( G^* \), and viceversa (the contraction of a face is the contraction of all the edges incident to it to a single vertex).

**Lemma 1** Branchwidth is closed under taking of minors, i.e., the branchwidth of a graph is no less than the branchwidth of any of its minors.

**Lemma 2** The removal of a vertex or the contraction of a face from an embedded graph decreases its branchwidth by at most 1.

**Lemma 3** (Fomin and Thilikos [4]) Let \( G_1 \) and \( G_2 \) be graphs with one edge or one vertex in common. Then \( \text{bw}(G_1 \cup G_2) \leq \max\{\text{bw}(G_1), \text{bw}(G_2), 2\} \).

We need a technical definition before stating our main result. Suppose that \( G_1 \) and \( G_2 \) are graphs with disjoint vertex-sets and \( k \geq 0 \) is an integer. For \( i = 1, 2 \), let \( W_i \subseteq V(G_i) \) form a clique of size \( k \) and let \( G_i' \) \((i = 1, 2)\) be obtained from \( G_i \) by deleting some (possibly none) of the edges from \( G_i[W_i] \) with both endpoints in \( W_i \). Consider a bijection \( h : W_1 \to W_2 \). We define a \textit{clique-sum} \( G_1 \oplus G_2 \) of \( G_1 \) and \( G_2 \) to be the graph obtained from the union of \( G_1' \) and \( G_2' \) by identifying \( w \) with \( h(w) \) for all \( w \in W_1 \).

**Theorem 1** Let \( (G, \Sigma) \) be an embedding with \( g = \text{eg}(\Sigma) \). Then \( \text{bw}(G^*) \leq 6 \cdot \text{bw}(G) + 2g - 4 \).

**Proof.** The proof uses the following procedure that applies a series of cutting operations to decompose \( G \) into polyhedral pieces plus a set of vertices whose size is linearly bounded by \( \text{eg}(\Sigma) \). The input is the graph \( G \) and its dual \( G^* \) embedded in \( \Sigma \).

1. Set \( \mathcal{B} = \{G\} \), and \( \mathcal{B}^* = \{G^*\} \) (we call the members of \( \mathcal{B} \) and \( \mathcal{B}^* \) \textit{blocks}).
2. If \((G, \Sigma)\) has a minimal separator \(S\) with \(|S| \leq 2\), let \(C_1, \ldots, C_\rho\) be the connected components of \(G[V(G) \setminus S]\) and, for \(i = 1, \ldots, \rho\), let \(G_i\) be the graph obtained by \(G[V(C_i) \cup S]\) by adding an edge with both endpoints in \(S\) in the case where \(|S| = 2\) and such an edge does not already exist (we refer to this operation as cutting \(G\) along the separator \(S\)). Notice that a separator \(S\) of \(G\) with \(|S| = 1\) corresponds to a separator \(S^*\) of \(G^*\) with \(|S^*| = 1\), given by the vertex of \(G^*\) corresponding to the external face of \(G\).

Also, to a separator \(S\) of \(G\) with \(|S| = 2\) we can associate a separator \(S^*\) of \(G^*\) with \(|S^*| = 2\), given by the vertex of \(G^*\) corresponding to the external face of \(G\) and a vertex of \(G^*\) corresponding to a face of \(G\) containing both vertices in \(S\). Let \(G_i^*, i = 1, \ldots, \rho\) be the graphs obtained by cutting \(G^*\) along the corresponding separator \(S^*\). We say that each \(G_i\) (resp. \(G_i^*\)) is a block of \(G\) (resp. \(G^*\)) and notice that each \(G_i\) and \(G_i^*\) is the clique sum of its blocks. Therefore, from Lemma 3,

\[
\text{bw}(G^*) \leq \max\{2, \max\{\text{bw}(G_i^*) \mid i = 1, \ldots, \rho\}\}. \tag{1}
\]

Observe now that for each \(i = 1, \ldots, \rho\), \(G_i\) and \(G_i^*\) are embedded in a surface \(\Sigma_i\) such that \(G_i\) is the dual of \(G_i^*\) and \(\text{eg}(\Sigma) = \sum_{i=1}^\rho \text{eg}(\Sigma_i)\). Notice also that

\[
\text{bw}(G_i) \leq \text{bw}(G), i = 1, \ldots, \rho, \tag{2}
\]

as the possible edge addition does not increase the branchwidth, since each block of \(G\) is a minor of \(G\) and Lemma 1 applies. We set \(B \leftarrow B \setminus \{G\} \cup \{G_1, \ldots, G_\rho\}\) and \(B^* \leftarrow B^* \setminus \{G^*\} \cup \{G_1^*, \ldots, G_\rho^*\}\).

3. If \((G, \Sigma)\) has a non-contractible and non-surface-separating noose meeting a set \(S \subseteq V(G)\) with \(|S| \leq 2\), let \(G' = G[V(G) \setminus S]\) and let \(F\) be the set of faces in \(G^*\) corresponding to the vertices in \(S\). Observe that the obtained graph \(G'\) has an embedding to some surface \(\Sigma'\) of Euler genus strictly smaller than \(\Sigma\) that, in turn, has some dual \(G'^*\) in \(\Sigma'\). Therefore \(\text{eg}(\Sigma') < \text{eg}(\Sigma)\).

Moreover, \(G'^*\) is the result of the contraction in \(G^*\) of the \(|S|\) faces in \(F\). From Lemma 2,

\[
\text{bw}(G^*) \leq \text{bw}(G') + |S|. \tag{3}
\]

Set \(B \leftarrow B \setminus \{G\} \cup \{G'\}\) and \(B^* \leftarrow B^* \setminus \{G^*\} \cup \{G'^*\}\).

4. As long as this is possible, apply (recursively) Steps 2–4 for each block \(G \in B\) and its dual.

We now claim that before each recursive call of Steps 2 and 3, it holds that \(\text{bw}(G^*) \leq 6 \cdot \text{bw}(G) + 2\text{eg}(\Sigma) - 4\). The proof uses descending induction on the the distance from the root of the recursion tree of the above procedure.
Notice that all embeddings of graphs in the collections $\mathcal{B}$ and $\mathcal{B}^*$ constructed by the above algorithm are polyhedral (except from the trivial cases that they have size at most 3). Then the theorem follows directly from Proposition 1.

Suppose that $G$ (resp. $G^*$) is the clique sum of its blocks $G_1, \ldots, G_\rho$ (resp. $G^*_1, \ldots, G^*_\rho$) embedded in the surfaces $\Sigma_1, \ldots, \Sigma_\rho$ (Step 2). By induction, we have that $bw(G_i^*) \leq 6 \cdot bw(G_i) + 2eg(\Sigma_i) - 4, i = 1, \ldots, \rho$ and the claim follows from Relations (1) and (2) and the fact that $eg(\Sigma) = \sum_{i=1}^{\rho} eg(\Sigma)$.

Suppose now (Step 3) that $G$ (resp. $G^*$) occurs from some graph $G'$ (resp. $G'^*$) embedded in a surface $\Sigma'$ where $eg(\Sigma') < eg(\Sigma)$ after adding the vertices in $S$ (resp. $S^*$). From the induction hypothesis, $bw(G'^*) \leq 6 \cdot bw(G') + 2eg(\Sigma') - 4 \leq 6 \cdot bw(G') + 2eg(\Sigma) - 2 - 4$ and the claim follows directly from Relation (3) as $|S| \leq 2$ and $bw(G') \leq bw(G)$.

3 Recent results and a conjecture

Recently, Mazoit [8] proved that treewidth is a $(1, \rho + 1)$-self-dual parameter in graphs embeddable in surfaces of Euler genus $\rho$, using completely different techniques. Since the branchwidth and the treewidth of a graph $G$, with $|E(G)| \geq 3$, satisfy $bw(G) \leq tw(G) + 1 \leq \frac{3}{2}bw(G)$ [12], this implies that $bw(G^*) \leq \frac{3}{2}bw(G) + g + 2$, improving the constants of Theorem 1. We believe that an even tighter self-duality relation holds for branchwidth and hope that the approach of this paper will be helpful to settle the following conjecture.

**Conjecture 1** If $G$ is a graph embedded in some surface $\Sigma$, then $bw(G^*) \leq bw(G) + eg(\Sigma)$.

References


