ON TURÁN'S INEQUALITY FOR ULTRASPHERICAL POLYNOMIALS

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0. Introduction. Some years ago P. Turán noticed the interesting inequality

$$(0.1)\qquad \Delta_n(x) \equiv [P_n(x)]^2 - P_{n+1}(x)P_{n-1}(x) \geqq 0, \qquad n \geqq 1, \ |x| \leqq 1,$$

where $P_n(x)$ is the Legendre polynomial of order n and the above inequality has been extended in recent years to the case of some other orthogonal polynomials as well as the ordinary and modified Bessel functions [1; 2]. B. S. Madhava Rao and V. R. Thiruvenkatachar [3] deepened the inequality (0.1) by showing that

$$(0.2)\qquad \frac{d^2}{dx^2}\Delta_n(x) = -\frac{2}{n(n+1)}\left[\frac{d}{dx}P_n(x)\right]^2 \leqq 0.$$

The inequality (0.1) has also been generalized to the case of ultraspherical polynomials $P_n^{(\lambda)}(x)$ in the form

$$(0.3)\qquad \Delta_{n,\lambda}(x) \equiv [F_{n,\lambda}(x)]^2 - F_{n+1,\lambda}(x)F_{n-1,\lambda}(x) \geqq \text{ or } < 0,$$

according as $|x| \leqq$ or >1, where $F_{n,\lambda}(x) = P_n^{(\lambda)}(x)/P_n^{(\lambda)}(1)$. V. R. Thiruvenkatachar and T. S. Nanjundiah [4] and A. E. Danese [5] have obtained series of positive functions for $\Delta_n(x)$ and its analogue in the case of the ultraspherical, Laguerre and Hermite polynomials.

In the present paper we deepen the above results on ultraspherical polynomials by determining the signs of the first and second derivatives of $\Delta_n^{(\lambda)}(x) \equiv [P_n^{(\lambda)}(x)]^2 - P_{n+1}^{(\lambda)}(x)P_{n-1}^{(\lambda)}(x)$ for various values of λ and x. We notice first of all that $d\Delta_n^{(\lambda)}(x)/dx$ can be represented as a numerical multiple of the Wronskian $P_{n+1}^{(\lambda)}(x)dP_{n-1}^{(\lambda)}(x)/dx - P_{n-1}^{(\lambda)}(x)(dP_{n+1}^{(\lambda)}(x)/dx)$ and express it also via the Christoffel-Darboux formula as a series of functions of constant sign. The sign of $d\Delta_n^{(\lambda)}(x)/dx$ for various values of λ and x follows easily from this. We determine the sign of $(\lambda-1)x(d\Delta_n^{(\lambda)}(x)/dx)$ for a general value of n and λ and show that $(d^2/dx^2)\Delta_n^{(\lambda)}(x) \gtrless 0$ according as $\lambda > 1$ or $1/2 \leqq \lambda < 1$, thus extending the result of B. S. Madhava Rao and V. R. Thiruvenkatachar to the case of ultraspherical polynomials. For integer values of n we obtain a series of positive functions for $(d^2/dx^2)\Delta_n^{(\lambda)}(x)$. Next we show that $(1-x^2)\Delta_n^{(\lambda)}(x)$ decreases steadily in (0, 1) when $1 < \lambda \leqq 3/2$. From the signs of $d\Delta_n^{(\lambda)}(x)/dx$,

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$d^2\Delta_n^{(\lambda)}(x)/dx^2$ we prove the concavity of $\Delta_{n,\lambda}(x)$ for $0<\lambda\leqq 1/2$ and also arrive at inequalities for $\Delta_{n,\lambda}(x)$ in the range $|x|\leqq 1$, which are not only simpler but also cover a wider range of λ than the corresponding inequalities given by O. Śzasz [2]. Finally we determine an integral representation for $\Delta_n(x)$.

1. **Expressing $d\Delta_n^{(\lambda)}(x)/dx$ as a Wronskian.** We append below the well known relations satisfied by the ultraspherical polynomials [6] for frequent use in the following.

$$(1.1)\quad (1-x^2)\frac{d^2y}{dx^2}-(2\lambda+1)x\frac{dy}{dx}+n(n+2\lambda)y=0, \qquad y=P_n^{(\lambda)}(x)$$

$$(1.2)\quad nP_n^{(\lambda)}(x)-2(n+\lambda-1)xP_{n-1}^{(\lambda)}(x)+(n+2\lambda-2)P_{n-2}^{(\lambda)}(x)=0, \qquad n\geqq 2$$

$$(1.3)\qquad \frac{d^r}{dx^r}P_n^{(\lambda)}(x)=2^r(\Gamma(\lambda+r)/\Gamma(\lambda))P_{n-r}^{(\lambda+r)}(x), \qquad r\leqq n$$

$$(1.4)\qquad nP_n^{(\lambda)}(x)=x\frac{d}{dx}P_n^{(\lambda)}(x)-\frac{d}{dx}P_{n-1}^{(\lambda)}(x),$$

$$(1.5)\qquad (n+2\lambda)P_n^{(\lambda)}(x)=\frac{d}{dx}P_{n+1}^{(\lambda)}(x)-x\frac{d}{dx}P_n^{(\lambda)}(x),$$

$$(1.6)\qquad 2(n+\lambda)P_n^{(\lambda)}(x)=\frac{d}{dx}P_{n+1}^{(\lambda)}(x)-\frac{d}{dx}P_{n-1}^{(\lambda)}(x),$$

$$(1.7)\quad (1-x^2)\frac{d}{dx}P_n^{(\lambda)}(x)=-nxP_n^{(\lambda)}(x)+(n+2\lambda-1)P_{n-1}^{(\lambda)}(x),$$

$$(1.8)\quad (1-x^2)\frac{d}{dx}P_n^{(\lambda)}(x)=(n+2\lambda)xP_n^{(\lambda)}(x)-(n+1)P_{n+1}^{(\lambda)}(x).$$

Differentiation of

$$(1.9)\qquad \Delta_n^{(\lambda)}(x)\equiv\left[P_n^{(\lambda)}(x)\right]^2-P_{n+1}^{(\lambda)}(x)P_{n-1}^{(\lambda)}(x)$$

yields

$$(1.10)\qquad \frac{d}{dx}\Delta_n^{(\lambda)}(x)=\alpha_n^{(\lambda)}(x)-\beta_n^{(\lambda)}(x)$$

where

$$\alpha_n^{(\lambda)}(x)=P_n^{(\lambda)}(x)\frac{d}{dx}P_n^{(\lambda)}(x)-P_{n-1}^{(\lambda)}(x)\frac{d}{dx}P_{n+1}^{(\lambda)}(x)$$

and

$$\beta_n^{(\lambda)}(x) = P_{n+1}^{(\lambda)}(x)\frac{d}{dx}P_{n-1}^{(\lambda)}(x) - P_n^{(\lambda)}(x)\frac{d}{dx}P_n^{(\lambda)}(x).$$

We have then

$$(1.11)\qquad \alpha_n^{(\lambda)}(x) + \beta_n^{(\lambda)}(x) = \begin{vmatrix} P_{n+1}^{(\lambda)}(x) & P_{n-1}^{(\lambda)}(x) \\ \frac{d}{dx}P_{n+1}^{(\lambda)}(x) & \frac{d}{dx}P_{n-1}^{(\lambda)}(x) \end{vmatrix}.$$

Also

$$\begin{aligned} &n\alpha_n^{(\lambda)}(x) - (n+2\lambda)\beta_n^{(\lambda)}(x) \\ &\quad = 2(n+\lambda)P_n^{(\lambda)}(x)\frac{d}{dx}P_n^{(\lambda)}(x) - nP_{n-1}^{(\lambda)}(x)\frac{d}{dx}P_{n+1}^{(\lambda)}(x) \\ &\quad\quad - (n+2\lambda)P_{n+1}^{(\lambda)}(x)\frac{d}{dx}P_{n-1}^{(\lambda)}(x) \\ &\quad = 2(n+\lambda)P_n^{(\lambda)}(x)\frac{d}{dx}P_n^{(\lambda)}(x) - (n+2\lambda-1)P_{n-1}^{(\lambda)}(x)\frac{d}{dx}P_{n+1}^{(\lambda)}(x) \\ &\quad\quad - (n+1)P_{n+1}^{(\lambda)}(x)\frac{d}{dx}P_{n-1}^{(\lambda)}(x) \\ &\quad\quad - (2\lambda-1)\left\{P_{n+1}^{(\lambda)}(x)\frac{d}{dx}P_{n-1}^{(\lambda)}(x) - P_{n-1}^{(\lambda)}(x)\frac{d}{dx}P_{n+1}^{(\lambda)}(x)\right\}. \end{aligned}$$

The first three terms together are found to vanish on using (1.4), (1.5), (1.6) and we get

$$n\alpha_n^{(\lambda)}(x) - (n+2\lambda)\beta_n^{(\lambda)}(x) = -(2\lambda-1)\left[\alpha_n^{(\lambda)}(x) + \beta_n^{(\lambda)}(x)\right],$$

so that there follows the relation

$$(1.12)\qquad (n+2\lambda-1)\alpha_n^{(\lambda)}(x) = (n+1)\beta_n^{(\lambda)}(x).$$

Solving (1.11) and (1.12) for $\alpha_n^{(\lambda)}(x)$, $\beta_n^{(\lambda)}(x)$ and using (1.10), we have

$$(1.13)\qquad \frac{d}{dx}\Delta_n^{(\lambda)}(x) = \frac{1-\lambda}{n+\lambda}\begin{vmatrix} P_{n+1}^{(\lambda)}(x) & P_{n-1}^{(\lambda)}(x) \\ \frac{d}{dx}P_{n+1}^{(\lambda)}(x) & \frac{d}{dx}P_{n-1}^{(\lambda)}(x) \end{vmatrix}.$$

From (1.3) and (1.13) we can also express the first derivative of

$$\Delta_{n,r}^{(\lambda)}(x) \equiv \left[\frac{d^r}{dx^r} P_n^{(\lambda)}(x)\right]^2 - \frac{d^r}{dx^r} P_{n+1}^{(\lambda)}(x) \frac{d^r}{dx^r} P_{n-1}^{(\lambda)}(x)$$

as a constant times the Wronskian of $P_{n-r+1}^{(\lambda+r)}(x)$ and $P_{n-r-1}^{(\lambda+r)}(x)$.

2. **Series of functions of constant sign for** $d\Delta_n^{(\lambda)}(x)/dx$. Using (1.2) in succession we can write

$$P_{n+1}^{(\lambda)}(x) = a_n x^2 P_{n-1}^{(\lambda)}(x) + b_n P_{n-1}^{(\lambda)}(x) + c_n P_{n-3}^{(\lambda)}(x), \tag{2.1}$$

where

$$a_n = \frac{4(n+\lambda)(n+\lambda-1)}{n(n+1)}, \quad c_n = -\frac{(n+\lambda)(n+2\lambda-2)(n+2\lambda-3)}{n(n+1)(n+\lambda-2)}. \tag{2.2}$$

Hence the determinant

$$\delta_n(x, y) = \begin{vmatrix} P_{n+1}^{(\lambda)}(x) & P_{n-1}^{(\lambda)}(x) \\ P_{n+1}^{(\lambda)}(y) & P_{n-1}^{(\lambda)}(y) \end{vmatrix}$$

can be written in the form

$$\begin{vmatrix} a_n x^2 P_{n-1}^{(\lambda)}(x) + b_n P_{n-1}^{(\lambda)}(x) + c_n P_{n-3}^{(\lambda)}(x) & P_{n-1}^{(\lambda)}(x) \\ a_n y^2 P_{n-1}^{(\lambda)}(y) + b_n P_{n-1}^{(\lambda)}(y) + c_n P_{n-3}^{(\lambda)}(y) & P_{n-1}^{(\lambda)}(y) \end{vmatrix}$$
$$= a_n(x^2 - y^2) P_{n-1}^{(\lambda)}(x) P_{n-1}^{(\lambda)}(y) - c_n \delta_{n-2}(x, y)$$

and so we get

$$\delta_n(x, y) + c_n \delta_{n-2}(x, y) = a_n(x^2 - y^2) P_{n-1}^{(\lambda)}(x) P_{n-1}^{(\lambda)}(y). \tag{2.3}$$

On observing that

$$\operatorname*{Lt}_{y\to x} \frac{\delta_n(x, y)}{y - x} = \begin{vmatrix} P_{n+1}^{(\lambda)}(x) & P_{n-1}^{(\lambda)}(x) \\ \dfrac{d}{dx} P_{n+1}^{(\lambda)}(x) & \dfrac{d}{dx} P_{n-1}^{(\lambda)}(x) \end{vmatrix},$$

we obtain from (2.3), the relation

$$\frac{(n+\lambda)}{1-\lambda} \frac{d}{dx} \Delta_n^{(\lambda)}(x) + c_n \frac{n-2+\lambda}{1-\lambda} \frac{d}{dx} \Delta_{n-2}^{(\lambda)}(x) = -2a_n x [P_{n-1}^{(\lambda)}(x)]^2,$$

which can be rewritten in the form

$$(2.4)\quad \frac{d}{dx}\Delta_n^{(\lambda)}(x) - \frac{(n+2\lambda-2)(n+2\lambda-3)}{n(n+1)}\frac{d}{dx}\Delta_{n-2}^{(\lambda)}(x) = -\frac{8(n+\lambda-1)}{n(n+1)}(1-\lambda)x\left[P_{n-1}^{(\lambda)}(x)\right]^2.$$

Solving this difference equation for $d\Delta^{(n)}(x)/dx$ we obtain

$$(2.5)\quad \frac{d}{dx}\Delta_n^{(\lambda)}(x) = \frac{8(\lambda-1)\Gamma(n+2\lambda-1)}{(n+1)!}x \cdot \sum_{k=1}^{[(n+1)/2]} \frac{(n-2k+1)!(n-2k+\lambda+1)}{\Gamma(n-2k+2\lambda+1)}\left[P_{n-2k+1}^{(\lambda)}(x)\right]^2,$$

which is the desired series expansion. Differentiation of (2.5) gives a series for $(d^2/dx^2)\Delta_n^{(\lambda)}(x)$ which has however no particularly elegant form. In the case of Legendre polynomials ($\lambda=1/2$), (2.5) becomes

$$\frac{d}{dx}\Delta_n(x) = \frac{-2x}{n(n+1)}\sum_{k=1}^{[(n+1)/2]}(2n-4k+3)[P_{n-2k+1}(x)]^2.$$

Differentiating this we see that

$$\frac{d^2}{dx^2}\Delta_n(x) = \frac{-2}{n(n+1)}\sum_{k=1}^{[(n+1)/2]}(2n-4k+3)P_{n-2k+1}(x)\left[P_{n-2k+1}(x)+2xP'_{n-2k+1}(x)\right]$$

which is the same as identity (0.2). From (2.5) it follows that

$$(2.6)\quad \operatorname{sgn}\left[\frac{d}{dx}\Delta_n^{(\lambda)}(x)\right] = \operatorname{sgn}(\lambda-1)x \qquad (\lambda>0)$$

which amounts to the following property of $\Delta_n^{(\lambda)}(x)$.

The Turán expression $\Delta_n^{(\lambda)}(x)$ is an increasing (a decreasing) function for positive (negative) values of x when $\lambda>1$. This is to be reversed when $0<\lambda<1$.

From this we see that

when $\lambda>1$, $\quad \Delta_n^{(\lambda)}(x) \geqq \Delta_n^{(\lambda)}(0) > 0$ $\quad$ for all values of x;

when $1/2 \leqq \lambda < 1$, $\quad \Delta_n^{(\lambda)}(x) \geqq \Delta_n^{(\lambda)}(1) \geqq 0$ $\quad$ for $|x| \leqq 1$;

when $0<\lambda\leqq 1/2$, $\quad \Delta_n^{(\lambda)}(x) \leqq \Delta_n^{(\lambda)}(1) \leqq 0$ $\quad$ for $|x| \geqq 1$.

3. **Sign of** $(\lambda-1)xd\Delta_n^{(\lambda)}(x)/dx$ **for a general index** n. For a general index n (which will be restricted in a way later on) we take $P_n^{(\lambda)}(x)$ as a solution of the differential equation (1.1) which remains regular at $x=+1$ (or at $x=-1$) either of which is a regular singular point of the differential equation. We stipulate that the value of the function as well as its slope at $x=+1$ are positive. Defining $\Delta_n^{(\lambda)}(x)$ as in (1.9) we can again derive the relation (1.13) as before. Multiplying this last relation by $(1-x^2)^{\lambda+1/2}$ and differentiating the result, we get

$$\frac{d}{dx}\left[(1-x^2)^{\lambda+1/2}\frac{d}{dx}\Delta_n^{(\lambda)}(x)\right]$$

$$=\frac{1-\lambda}{n+\lambda}\begin{vmatrix} P_{n+1}^{(\lambda)}(x) & P_{n-1}^{(\lambda)}(x) \\ \frac{d}{dx}\left[(1-x^2)^{\lambda+1/2}\frac{d}{dx}P_{n+1}^{(\lambda)}(x)\right] & \frac{d}{dx}\left[(1-x^2)^{\lambda+1/2}\frac{d}{dx}P_{n-1}^{(\lambda)}(x)\right]\end{vmatrix}.$$

Simplifying the second row of the determinant by means of the differential equation we are led to

$$\frac{d}{dx}\left[(1-x^2)^{\lambda+1/2}\frac{d}{dx}\Delta_n^{(\lambda)}(x)\right] = 4(1-\lambda)(1-x^2)^{\lambda-1/2}P_{n+1}^{(\lambda)}(x)P_{n-1}^{(\lambda)}(x). \tag{3.1}$$

Hence the extrema of $(1-x^2)^{\lambda+1/2}d\Delta_n^{(\lambda)}(x)/dx$ occur only at the zeros of $P_{n+1}^{(\lambda)}(x)$, $P_{n-1}^{(\lambda)}(x)$. The solution $P_n^{(\lambda)}(x)$ as defined above cannot have zeros on the real segment $(1, \infty)$. It increases from a positive value at $x=1$. If it attains a maximum value at a point x_1 (>1), we have the conditions $(dy/dx)_{x_1}=0$, $(d^2y/dx^2)_{x_1}<0$ and from the differential equation we notice a contradiction if $n(n+2\lambda)$ is positive. Hence $P_n^{(\lambda)}(x)$ is increasing in $x>1$ and consequently does not vanish for any x on $(1, \infty)$. Let us denote by

$$(\alpha):\ \alpha_1, \alpha_2, \alpha_3, \cdots$$

$$(\beta):\ \beta_1, \beta_2, \beta_3, \cdots$$

the zeros in $(-1, +1)$ of $P_{n+1}^{(\lambda)}(x)$ and $P_{n-1}^{(\lambda)}(x)$ respectively. From (1.13) we have

$$\operatorname{sgn}\left[\frac{d}{dx}\Delta_n^{(\lambda)}(x)\right]_{x=\alpha} = \operatorname{sgn}\left[\frac{\lambda-1}{n+\lambda}P_{n-1}^{(\lambda)}(x)\frac{d}{dx}P_{n+1}^{(\lambda)}(x)\right]_{x=\alpha}.$$

From (1.7) we may write

$$(1-\alpha^2)\left[\frac{d}{dx}P_{n+1}^{(\lambda)}(x)\right]_{x=\alpha} = (n+2\lambda)P_n^{(\lambda)}(\alpha)$$

and hence

$$\operatorname{sgn}\left[\frac{d}{dx}\Delta_n^{(\lambda)}(x)\right]_{x=\alpha} = \operatorname{sgn}\left[\frac{(\lambda-1)(n+2\lambda)P_{n-1}^{(\lambda)}(\alpha)P_n^{(\lambda)}(\alpha)}{(n+\lambda)(1-\alpha^2)}\right].$$

Using the relation $2(n+\lambda)\alpha P_n^{(\lambda)}(\alpha)=(n+2\lambda-1)P_{n-1}^{(\lambda)}(\alpha)$ which follows from (1.2), we get

$$\operatorname{sgn}\left[\frac{d}{dx}\Delta_n^{(\lambda)}(x)\right]_{x=\alpha} = \operatorname{sgn}\left[(\lambda-1)(n+2\lambda)(n+2\lambda-1)/\alpha\right].$$

In a similar way we observe that

$$\operatorname{sgn}\left[\frac{d}{dx}\Delta_n^{(\lambda)}(x)\right]_{x=\beta} = \operatorname{sgn}\left[(\lambda-1)n\beta/(n+1)\right].$$

Combining the above two facts we see that whenever $n \geqq 1$ and $\lambda > 0$

$$\operatorname{sgn}\left[x\frac{d}{dx}\Delta_n^{(\lambda)}(x)\right]_{x=\alpha,\beta} = \operatorname{sgn}(\lambda-1).$$

Thus at all possible extrema of $(1-x^2)^{\lambda+1/2}(d\Delta_n^{(\lambda)}(x)/dx)$, $(\lambda-1)x(d\Delta_n^{(\lambda)}(x)/dx)$ is positive and hence

$$\operatorname{sgn}\left[(\lambda-1)x\frac{d}{dx}\Delta_n^{(\lambda)}(x)\right] = +1, \quad \text{for } n \geqq 1, \lambda > 0. \tag{3.2}$$

For positive integer values of n this is the same as (2.6).

4. **Sign of** $(d^2/dx^2)\Delta_n^{(\lambda)}(x)$. Differentiating (1.13) which also holds for general n we get

$$\frac{d^2}{dx^2}\Delta_n^{(\lambda)}(x) = \frac{1-\lambda}{n+\lambda}\begin{vmatrix} P_{n+1}^{(\lambda)}(x) & P_{n-1}^{(\lambda)}(x) \\ \dfrac{d^2}{dx^2}P_{n+1}^{(\lambda)}(x) & \dfrac{d^2}{dx^2}P_{n-1}^{(\lambda)}(x) \end{vmatrix}.$$

Replacing the second row elements by means of the equations

$$\frac{d^2}{dx^2}P_{n+1}^{(\lambda)}(x) = x\frac{d^2}{dx^2}P_n^{(\lambda)}(x) + (n+2\lambda+1)\frac{d}{dx}P_n^{(\lambda)}(x),$$

$$\frac{d^2}{dx^2}P_{n-1}^{(\lambda)}(x) = x\frac{d^2}{dx^2}P_n^{(\lambda)}(x) - (n-1)\frac{d}{dx}P_n^{(\lambda)}(x)$$

(these follow on differentiating (1.5) and (1.4)) and replacing the first row elements by means of (1.7) and (1.8) we get

$$\frac{d^2}{dx^2}\Delta_n^{(\lambda)}(x) = \frac{1-\lambda}{n+\lambda}$$

$$\cdot\begin{vmatrix} \dfrac{(n+2\lambda)xP_n^{(\lambda)}(x)-(1-x^2)\dfrac{d}{dx}P_n^{(\lambda)}(x)}{n+1} & \dfrac{nxP_n^{(\lambda)}(x)+(1-x^2)\dfrac{d}{dx}P_n^{(\lambda)}(x)}{n+2\lambda-1} \\ x\dfrac{d^2}{dx^2}P_n^{(\lambda)}(x)+(n+2\lambda+1)\dfrac{d}{dx}P_n^{(\lambda)}(x) & x\dfrac{d^2}{dx^2}P_n^{(\lambda)}(x)-(n-1)\dfrac{d}{dx}P_n^{(\lambda)}(x) \end{vmatrix}$$

a form in which the right hand side involves only algebraic combinations of $P_n^{(\lambda)}(x)$ and its first and second derivatives. After simplifying this determinant we arrive at the equation

$$\frac{(n+1)(n+2\lambda-1)}{2(\lambda-1)}\frac{d^2}{dx^2}\Delta_n^{(\lambda)}(x) = 2\left[\frac{d}{dx}P_n^{(\lambda)}(x)\right]^2$$

$$(4.1)\qquad + (2\lambda-1)x\left\{x\left[\left(\frac{d}{dx}P_n^{(\lambda)}(x)\right)^2 - P_n^{(\lambda)}(x)\frac{d^2}{dx^2}P_n^{(\lambda)}(x)\right]\right.$$

$$\left. - P_n^{(\lambda)}(x)\frac{d}{dx}P_n^{(\lambda)}(x)\right\}.$$

If

$$(4.2)\qquad D_n^{(\lambda)}(x) \equiv \Delta_{n,1}^{(\lambda)}(x) \equiv \left[\frac{d}{dx}P_n^{(\lambda)}(x)\right]^2 - \frac{d}{dx}P_{n+1}^{(\lambda)}(x)\frac{d}{dx}P_{n-1}^{(\lambda)}(x),$$

we can easily see that $D_n^{(\lambda)}(x) = 4\lambda^2\Delta_{n-1}^{(\lambda+1)}(x)$ and

$$\frac{d}{dx}D_n^{(\lambda)}(x)$$

$$= 2\lambda\left\{x\left[\left(\frac{d}{dx}P_n^{(\lambda)}(x)\right)^2 - P_n^{(\lambda)}(x)\frac{d^2}{dx^2}P_n^{(\lambda)}(x)\right] - P_n^{(\lambda)}(x)\cdot\frac{d}{dx}P_n^{(\lambda)}(x)\right\}.$$

We may therefore write (4.1) in the form

$$(4.3)\qquad \begin{aligned} &\frac{(n+1)(n+2\lambda-1)}{2(\lambda-1)}\frac{d^2}{dx^2}\Delta_n^{(\lambda)}(x) \\ &\qquad = 2\left[\frac{d}{dx}P_n^{(\lambda)}(x)\right]^2 + 2\lambda(2\lambda-1)x\frac{d}{dx}\Delta_{n-1}^{(\lambda+1)}(x). \end{aligned}$$

Either of (4.1) and (4.3) is an extension of (0.2) to the case of ultraspherical polynomials. When n is a positive integer we may replace the last term on the right side of (4.3) by means of the series (2.5) and so arrive at the following series for $d^2\Delta_n^{(\lambda)}(x)/dx^2$

$$(4.4)\quad \begin{aligned} &\frac{(n+1)(n+2\lambda-1)}{2(\lambda-1)}\frac{d^2}{dx^2}\Delta_n^{(\lambda)}(x) \\ &\qquad = 2\left[\frac{d}{dx}P_n^{(\lambda)}(x)\right]^2 + (2\lambda-1)\frac{\Gamma(n+2\lambda)}{n!}16\lambda^2x^2 \\ &\qquad \cdot \sum_{k=1}^{[(n/2)]}\frac{(n-2k)!(n-2k+\lambda+1)}{\Gamma(n-2k+2\lambda+2)}[P_{n-2k}^{(\lambda+1)}(x)]^2. \end{aligned}$$

From (3.2) it follows that $\lambda x d\Delta_{n-1}^{(\lambda+1)}(x)/dx>0$ for $n\geqq 2$, $\lambda>-1/2$. Using this in (4.3) we deduce that

$$(4.5)\quad \operatorname{sgn}\frac{d^2}{dx^2}\Delta_n^{(\lambda)}(x) = \begin{cases} +1 & (\lambda>1), \\ -1 & (1/2\leqq\lambda<1). \end{cases}$$

Hence $\Delta_n^{(\lambda)}(x)$ is a convex function of x for $\lambda>1$ and a concave function of x for $1/2\leqq\lambda<1$.

5. **Decreasing nature of** $(1-x^2)\Delta_n^{(\lambda)}(x)$. From (3.1) we know that $\Delta_n^{(\lambda)}(x)$ satisfies the differential equation

$$(5.1)\quad \begin{aligned} &\left[(1-x^2)\frac{d^2}{dx^2}-(2\lambda+1)x\frac{d}{dx}+4(1-\lambda)\right]\Delta_n^{(\lambda)}(x) \\ &\qquad = 4(1-\lambda)[P_n^{(\lambda)}(x)]^2. \end{aligned}$$

If $(1-x^2)\Delta_n^{(\lambda)}(x)$ is denoted by y, we get

$$(5.2)\quad \begin{cases} \dfrac{dy}{dx} = (1-x^2)\dfrac{d}{dx}\Delta_n^{(\lambda)}(x) - 2x\Delta_n^{(\lambda)}(x), \\ \dfrac{d^2y}{dx^2} = (1-x^2)\dfrac{d^2}{dx^2}\Delta_n^{(\lambda)}(x) - 4x\dfrac{d}{dx}\Delta_n^{(\lambda)}(x) - 2\Delta_n^{(\lambda)}(x). \end{cases}$$

We find from (5.1) and (5.2) that y satisfies the differential equation

$$(5.3)\quad \begin{aligned} &\left[(1-x^2)\frac{d^2}{dx^2}-(2\lambda-3)x\frac{d}{dx}-(4\lambda-6)\right]y \\ &\qquad = (4\lambda-6)x^2\Delta_n^{(\lambda)}(x)+4(1-\lambda)(1-x^2)[P_n^{(\lambda)}(x)]^2. \end{aligned}$$

Now $y(0)=\Delta_n^{(\lambda)}(0)$ is positive when $\lambda>0$ and $y(1)=0$. When $\lambda>1$, dy/dx is negative for all $x>1$ and hence y decreases for $x>1$ when $\lambda>1$. We shall now see that y steadily decreases even in (0, 1) when $1<\lambda\leqq 3/2$. On using (2.6) we have $(dy/dx)_{x=0}=0$. From (5.3) we have $(d^2y/dx^2)_{x=0}=(4\lambda-6)\Delta_n^{(\lambda)}(0)+4(1-\lambda)[P_n^{(\lambda)}(0)]^2$ and so when

$1<\lambda\leqq 3/2$, $(d^2y/dx^2)_{x=0}$ is negative. Hence when $1<\lambda\leqq 3/2$, y reaches a maximum at $x=0$ with the maximum value $=\Delta_n^{(\lambda)}(0)$ and decreases in the neighbourhood of $x=0$. Also $(dy/dx)_{x=1}=-2\Delta_n^{(\lambda)}(1)$ is negative when $\lambda>1/2$ and so $y(x)$ is decreasing in the neighbourhood of $x=1$ also when $\lambda>1$. If $y(x)$ is not steadily decreasing in (0, 1) it must reach a minimum value at some point $x=\alpha<1$, then increase to a maximum at some point $x=\beta$ between α and 1 and then decrease again. We show that this possibility cannot occur when $1<\lambda\leqq 3/2$. From the conditions for minimum, we have at $x=\alpha$, $(dy/dx)=0$ and $(d^2y/dx^2)>0$ and from the differential equation (5.3) we get

$$(1-\alpha^2)\left(\frac{d^2y}{dx^2}\right)_{x=\alpha} = (4\lambda-6)y(\alpha)+(4\lambda-6)\alpha^2\Delta_n^{(\lambda)}(\alpha) + 4(1-\lambda)(1-\alpha^2)[P_n^{(\lambda)}(\alpha)]^2.$$

The left hand side is positive while the right hand side is negative. We conclude that y cannot have a minimum anywhere in (0, 1). *Therefore* $y(x)$ *must steadily decrease in* (0, 1) *when* $1<\lambda\leqq 3/2$. It is to be expected that for large values of λ, $y(x)$ decreases first to a minimum and then attains a maximum in (0, 1) before decreasing again at and near $x=1$.

Eliminating $(1-x^2)d^2\Delta_n^{(\lambda)}(x)/dx^2$ in d^2y/dx^2 by means of (5.1) we obtain

$$\frac{d^2y}{dx^2} = (2\lambda-3)x\frac{d}{dx}\Delta_n^{(\lambda)}(x)+(4\lambda-6)\Delta_n^{(\lambda)}(x)+4(1-\lambda)[P_n^{(\lambda)}(x)]^2.$$

Using (3.2) we notice that sgn $d^2y/dx^2=-1$ when $1<\lambda\leqq 3/2$.

6. **Inequalities for $\Delta_{n,\lambda}(x)$.** Let

$$\Delta_{n,\lambda}(x) = [P_n^{(\lambda)}(x)/P_n^{(\lambda)}(1)]^2 - [P_{n+1}^{(\lambda)}(x)/P_{n+1}^{(\lambda)}(1)]\cdot[P_{n-1}^{(\lambda)}(x)/P_{n-1}^{(\lambda)}(1)].$$

From the following well-known relations (see (4), (5))

$$(6.1)\qquad \Delta_{n,\lambda}(x) = \frac{(1-x^2)D_n^{(\lambda)}(x)}{k_{n,\lambda}} = \frac{4\lambda^2}{k_{n,\lambda}}(1-x^2)\Delta_{n-1}^{(\lambda+1)}(x)$$

where $k_{n,\lambda}=n(n+2\lambda)[P_n^{(\lambda)}(1)]^2$, and the conclusions of the previous section, we have

(a) $\Delta_{n,\lambda}(x)$ is steadily decreasing in (0, 1) for $0<\lambda\leqq 1/2$;

(b) $\operatorname{sgn}\dfrac{d^2}{dx^2}\Delta_{n,\lambda}(x) = -1$ when $0<\lambda\leqq 1/2$, showing the concavity property of $\Delta_{n,\lambda}(x)$. From (a) we can immediately deduce the well known inequality viz., the Turán expression $\Delta_{n,\lambda}(x)>0$ in $|x|<1$ when $0<\lambda\leqq 1/2$. From (6.1) and the concluding results of §(2) we can prove Turán's inequality in full for $\lambda>0$.

We have seen in (3.2) that

$$\text{sgn}\left[(\lambda-1)x\frac{d}{dx}\Delta_n^{(\lambda)}(x)\right]=+1 \quad \text{for } n\geqq 1, \lambda>0 \tag{3.2}$$

and in (4.5) that

$$\text{sgn}\frac{d^2}{dx^2}\Delta_n^{(\lambda)}(x)=\begin{cases}+1 & \lambda>1\\ -1 & 1/2\leqq\lambda<1\end{cases} \qquad n\geqq 2. \tag{4.5}$$

From the identity $D_n^{(\lambda)}(x)=4\lambda^2\Delta_{n-1}^{(\lambda+1)}(x)$ it therefore follows that

$$\text{sgn}\,\lambda x\frac{d}{dx}D_n^{(\lambda)}(x)=+1 \quad (n\geqq 2, \lambda>-1/2), \tag{6.2}$$

$$\text{sgn}\frac{d^2}{dx^2}D_n^{(\lambda)}(x)=\begin{cases}+1 & (\lambda>0),\\ -1 & (-1/2<\lambda<0).\end{cases} \tag{6.3}$$

These show the monotonic increasing-decreasing behaviour and the convexity-concavity property of $D_n^{(\lambda)}(x)$ for various values of λ, and enable us to write the following chain of inequalities: When

$$\lambda>0,\quad D_n^{(\lambda)}(0)\leqq D_n^{(\lambda)}(x)\leqq D_n^{(\lambda)}(0)(1-x/a)+D_n^{(\lambda)}(a)\,x/a$$
$$\leqq D_n^{(\lambda)}(a),$$

when

$$-1/2<\lambda<0,\quad D_n^{(\lambda)}(0)\geqq D_n^{(\lambda)}(x)\geqq D_n^{(\lambda)}(0)(1-x/a)+D_n^{(\lambda)}(a)\,x/a$$
$$\geqq D_n^{(\lambda)}(a) \qquad 0\leqq x\leqq a.$$

Taking $a=1$ and multiplying the above inequalities by $(1-x^2)$ we get the following inequalities for $\Delta_{n,\lambda}(x)$ in the range $|x|\leqq 1$, for $n\geqq 2$.

$$\Delta_{n,\lambda}(0)(1-x^2)\leqq\Delta_{n,\lambda}(x)\leqq\left[\Delta_{n,\lambda}(0)(1-x)+\frac{x}{2\lambda+1}\right](1-x^2)$$
$$\leqq(1-x^2)/(2\lambda+1) \qquad (\lambda>0),$$
$$\frac{1-x^2}{2\lambda+1}\leqq\left\{\frac{x}{2\lambda+1}+\Delta_{n,\lambda}(0)(1-x)\right\}(1-x^2)\leqq\Delta_{n,\lambda}(x)$$
$$\leqq\Delta_{n,\lambda}(0)(1-x^2) \qquad (-1/2<\lambda<0). \tag{6.4}$$

These are simpler than the inequalities for $\Delta_{n,\lambda}(x)$ in O. Sźasz's paper [2] and are at the same time an improvement over the corresponding ones in [4].

7. **Integral representation for** $\Delta_n(x)$. From (3.1) we have

$$\frac{d}{dx}\left[(1-x^2)^{\lambda+1/2}\frac{d}{dx}\Delta_n^{(\lambda)}(x)\right]+4(1-\lambda)(1-x^2)^{\lambda-1/2}\Delta_n^{(\lambda)}(x) = 4(1-\lambda)(1-x^2)^{\lambda-1/2}\left[P_n^{(\lambda)}(x)\right]^2. \tag{7.1}$$

If we denote $[(1-x^2)^{\lambda-1/2}\Delta_n^{(\lambda)}(x)]/x$ by $f_n^{(\lambda)}(x)$, then

$$(1-x^2)^{\lambda+1/2}\frac{d}{dx}\Delta_n^{(\lambda)}(x)=\frac{d}{dx}\left[x(1-x^2)f_n^{(\lambda)}(x)\right]+(2\lambda+1)x^2 f_n^{(\lambda)}(x).$$

Hence from (7.1) we get

$$\frac{d^2}{dx^2}\left[x(1-x^2)f_n^{(\lambda)}(x)\right]+(2\lambda+1)x^2\frac{d}{dx}f_n^{(\lambda)}(x)+6xf_n^{(\lambda)}(x) = 4(1-\lambda)(1-x^2)^{\lambda-1/2}\left[P_n^{(\lambda)}(x)\right]^2,$$

which may also be written in the form

$$\frac{d}{dx}\left[x^2(1-x^2)^{-\lambda+3/2}\frac{d}{dx}f_n^{(\lambda)}(x)\right]=4(1-\lambda)x\left[P_n^{(\lambda)}(x)\right]^2.$$

For Legendre functions $(\lambda=1/2)$ this becomes

$$\frac{d}{dx}\left(x^2(1-x^2)\frac{d}{dx}\left[\Delta_n(x)/x\right]\right)=2x[P_n(x)]^2. \tag{7.2}$$

Hence $x^2(1-x^2)(d[\Delta_n(x)/x]/dx)$ is an increasing (a decreasing) function for positive (negative) values of x and vanishes at $x=\pm 1$. From (7.2) we have in succession

$$\frac{d}{dx}\left(\Delta_n(x)/x\right)=\frac{2}{x^2(1-x^2)}\int_1^x t[P_n(t)]^2dt,$$

and

$$\Delta_n(x)=2x\int_1^x\frac{du}{u^2(1-u^2)}\int_1^u t[P_n(t)]^2dt, \tag{7.3}$$

the latter giving a positive integral representation for $\Delta_n(x)$. Using **the** relation

$$\frac{d}{dx}\left[(1-x^2)[P_n(x)]^2+\frac{(1-x^2)\left[\frac{d}{dx}P_n(x)\right]^2}{n(n+1)}\right]=-2x[P_n(x)]^2,$$

we may also write $\Delta_n(x)$ in the form

$$\Delta_n(x)=x\int_x^1\left\{[P_n(t)]^2+\frac{(1-t^2)\left[\frac{d}{dt}P_n(t)\right]^2}{n(n+1)}\right\}t^{-2}dt. \tag{7.4}$$

REFERENCES

1. G. Szegö, *On an inequality of P. Turán concerning Legendre polynomials*, Bull. Amer. Math. Soc. vol. 54 (1948) pp. 401–405.

2. O. Száasz, *Inequalities concerning orthogonal polynomials and Bessel functions*, Proc. Amer. Math. Soc. vol. 1 (1950) pp. 256–267.

3. B. S. Madhava Rao and V. R. Thiruvenkatachar, *On an inequality concerning orthogonal polynomials*, Proceedings of the Indian Academy of Sciences, Sect. A vol. 29 (1949) pp. 391–393.

4. V. R. Thiruvenkatachar and T. S. Nanjundiah, *Inequalities concerning Bessel functions and orthogonal polynomials*, ibid. Sect. A vol. 33 (1951) pp. 373–384.

5. A. E. Danese, *Explicit evaluations of Turán expressions*, Annali di Matematica Pura ed Applicata, Serie IV vol. 38 (1955) pp. 339–348.

6. G. Szegö, *Orthogonal polynomials*, (1939) pp. 80–84.

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