Nonexistence Results for the Korteweg-deVries and Kadomtsev-Petviashvili Equations

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Abstract

We study characteristic Cauchy problems for the Korteweg-deVries (KdV) equation $u_t = uu_x + u_{xxx}$, and the Kadomtsev-Petviashvili (KP) equation $u_{yy} = \big(u_{xxx}+uu_x+u_t\big)_x$ with holomorphic initial data possessing nonnegative Taylor coefficients around the origin. For the KdV equation with initial value $u(0,x) = u_0(x)$, we show that there is no solution holomorphic in any neighbourhood of $(t,x) = (0,0)$ in $\mathbb{C}^2$ unless $u_0(x) = a_0 + a_1 x$. This also furnishes a nonexistence result for a class of y-independent solutions of the KP equation. We extend this to y-dependent cases by considering initial values given at $y = 0$, $u(t,x,0) = u_0(x,t)$, $u_y(t,x,0) = u_1(x,t)$, where the Taylor coefficients of u_0 and u_1 around $t = 0$, $x = 0$ are assumed nonnegative. We prove that there is no holomorphic solution around the origin in $\mathbb{C}^3$ unless u_0 and u_1 are polynomials of degree 2 or lower.

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1 Introduction

Completely integrable partial differential equations such as the Korteweg-deVries (KdV) equation

$$u_t = uu_x + u_{xxx}$$

and the Kadomtsev-Petviashvili (KP) equation

$$u_{yy} = \big(u_{xxx} + uu_x + u_t\big)_x$$

are widely believed to have the Painlevé property [1, 2, 3], i.e. all solutions are suspected to be single-valued around all movable noncharacteristic analytic singularity manifolds. Although this is a property described in the complex space of independent variables, very few studies of the initial value problem in complex space have been carried out. None to our knowledge have considered analyticity in $(t, x) \in \mathbb{C}^2$.

We carry out such a study for a restricted class of holomorphic initial data, as a first step towards illuminating the Painlevé property of such equations. In this first step, we consider real initial data that grow as $\Re(x) \to +\infty$. As we explain below, these are equivalent to a one-complex-parameter family of complex initial data.

The initial value problems we study are

$$\begin{cases} u_t = uu_x + u_{xxx} \\ u(0, x) = u_0(x) \end{cases} \tag{1.1}$$

and

$$\begin{cases} u_{yy} = \big(u_{xxx} + uu_x + u_t\big)_x \\ u(t, x, 0) = u_0(t, x) \\ u_y(t, x, 0) = u_1(t, x) \end{cases} \tag{1.2}$$

where u_0 and u_1 are assumed holomorphic with Taylor coefficients real and nonnegative. Note that this includes other cases equivalent to it by a complex changes of variables. E.g. for the KdV equation, if we take $u(x, t) \mapsto -U(\xi, \tau)$ with $x = i\xi$, $t = -i\tau$, we get $-iU_\tau = -iUU_\xi - iU_{\xi\xi\xi}$. The latter is the same equation, but now the condition on the initial value has changed. More generally, we can exploit the scaling symmetries of the KdV and KP equations to transform our hypothesis to allow complex Taylor coefficients dependent on one parameter. Again for the KdV equation, under the scaling $u(x,t) = \lambda U(\xi,\tau)$, $\xi = \alpha x$, $\tau = \beta t$, where $\lambda = \alpha^2$, $\beta = \alpha^3$, our hypothesis allows initial value $U_0(\xi)$ with n-th Taylor coefficients of the form $\alpha^{-2-n}a_n$ where a_n are real nonnegative, for arbitrary, complex, nonzero α. Note also that by translation invariance, the above initial value problem can be shifted to the neighbourhood of any complex point in x or t. A similar equivalent family of initial data is valid for the KP equation.

We show that unless these data are polynomials in x (of first degree for the KdV equation and of second degree for the KP equation), no holomorphic solution exists in any neighbourhood of the origin in $\mathbb{C}^N$, where $N = 2$ for the KdV and $N = 3$ for the KP equation. Our main results are

Theorem 1.1 *The initial value problem (1.1) with initial data*

$$u_0(x) = \sum_{n=0}^{\infty} a_n x^n \tag{1.3}$$

where $a_n \geq 0$ for all n, has no solution holomorphic in any neighbourhood of the origin in $\mathbb{C}^2$, unless $u_0(x) = a_0 + a_1 x$.

and

Theorem 1.2 *The initial value problem (1.2) with initial data*

$$u_0(t,x) = \sum_{n=0}^{\infty} c_{0,n}(t)x^n, \quad u_1(t,x) = \sum_{n=0}^{\infty} c_{1,n}(t)x^n \tag{1.4}$$

where $c_{j,n}(t)$ are analytic and have real, nonnegative Taylor coefficients in t for all n, $j = 0,1$, has no solution holomorphic in any neighbourhood of the origin in $\mathbb{C}^3$, unless u_0 and u_1 are polynomials in x of degree less than or equal to two.

An illustrative example is given by the initial value

$$u_0(x) = \frac{c}{(a-x)^2} \tag{1.5}$$

for the KdV equation. If $c > 0$ and $a > 0$, then Theorem 1.1 shows that there is no locally holomorphic solution around the origin. However, if $c = -12$, it can be easily checked that this function *is* a time-independent solution of the

KdV equation. In the latter case, if $a > 0$, all the Taylor coefficients are real and negative. This example shows that we cannot enlarge the hypothesis to include purely negative coefficients in Theorem 1.1.

The initial data described in the above theorems are extensions of the usual initial value problems studied for the KdV and KP equations. For the KdV equation, inverse scattering theory [4, 5] shows that there exists a unique solution for real u_0 such that

$$\int_{-\infty}^{\infty} (1 + |x|)|u_0(x)| < \infty. \tag{1.6}$$

In fact, there are well known exact solutions (such as in example (1.5)) which do not satisfy this condition. But these solutions have poles on $\mathbb{R}$. To our knowledge, there has been no study made of whether the restriction (1.6) is necessary for analytic data without poles on $\mathbb{R}$. One motivation for our study is to consider initial values that may be bounded on the real line but do not necessarily satisfy (1.6) because of possible growth at infinity. Part of standard PDE theory is to deduce information such as the admissible order and type of growth of initial data at infinity. This information is not known for KdV-type equations.

Existence of solution for the KdV equation for growing initial data have been studied by Kenig *et al* [6] in the class of smooth functions on the real x-line and for a half-line in t which in our variables is $(-\infty, 0]$. In particular, a classical solution has been shown to exist [6] for $t \geq 0$ (in their variables), when u_0 is given by $p(x) + f(x)$ where p is a polynomial of odd degree with nonnegative coefficients and f is in the Schwartz class. Our result shows that this classical solution cannot be holomorphic around the origin except if the degree of p is unity and f is identically zero.

Finally, we remark here that our method can also be extended to other PDEs, such as Burgers' equation

$$u_t = uu_x + u_{xx},$$

the modified KdV equation

$$u_t = u^2 u_x + u_{xxx},$$

and the modified KP equation

$$u_{yy} = (u_{xxx} + u^2 u_x + u_t)_x.$$

Similar results hold for these equations with the only change being the degree of the initial data for which there exists a holomorphic solution.

The proof of Theorem 1.1 and 1.2 are given in Sections 2 and 3 respectively. We also give an alternative proof of the case of example (1.5) in Section 2.

2 Proof of Theorem 1.1

Here we prove Theorem 1.1 by studying the formal solution

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x) t^n, \tag{2.7}$$

in particular, the recursion relation satisfied by its coefficients. We divide the proof into three cases. The first is when $u_0(x)$ is polynomial. We show that if $\deg(u_0) \geq 2$ then u_n must grow factorially in n. Then we show that the proof extends simply to the case of nonpolynomial $u_0(x)$. Finally we show that there does exist a holomorphic solution, in a neighbourhood of the origin in $\mathbb{C}^2$, when $u_0(x)$ is a linear function of x.

Note that the coefficients $u_n(x)$ satisfy

$$(n+1)u_{n+1} = \sum_{k=0}^{n} u_k u'_{n-k} + u'''_n \tag{2.8}$$

2.1 The Polynomial Case

Here we consider the case of polynomial $u_0(x)$ of degree $d_0 \geq 2$. Clearly, u_n must be polynomial if u_0 is. Let d_n be its degree and write

$$u_n = \sum_{k=0}^{d_n} c_{n,k} x^{d_n - k}$$

Assume that $c_{0,0} > \epsilon$ for some $0 < \epsilon < 1$.

Lemma 2.1 *The degree d_n and coefficient $c_{n,0}$ of the largest degree term in u_n satisfy*

$$d_n = (n+1)d_0 - n \tag{2.9}$$

$$c_{n,0} > \epsilon^{n+1}, \tag{2.10}$$

Proof: The proof is by induction. First note that the third derivative term on the right side of (2.8) acts to decrease degree, whereas the convolution term increases it. We have at $n = 1$

$$\begin{aligned} d_1 &= 2d_0 - 1 && (2.11) \\ c_{1,0} &= d_0 c_{0,0}^2 > 2\epsilon^2 > \epsilon^2 && (2.12) \end{aligned}$$

Now suppose that the results hold for $1, \ldots, n$. Consider the $(n+1)$-st case. Then the maximal degree of the convolution term is given by

$$(k+1)d_0 - k - 1 + (n-k+1)d_0 - (n-k) = (n+2)d_0 - (n+1)$$

for $0 \leq k \leq n$, which proves the result for d_n for all $n \geq 1$. The defining equation for the coefficient $c_{n,0}$ is

$$\begin{aligned} (n+1)c_{n+1,0} &= \sum_{k=0}^{n} d_k c_{k,0} c_{n-k,0} \\ &> \sum_{k=0}^{n} c_{k,0} c_{n-k,0} \\ &> (n+1)\epsilon^{n+2} \end{aligned}$$

by the induction hypothesis. Hence the result holds for c_n for all $n \geq 1$.

□

We now focus on the third derivative term in Eqn(2.8) to show divergence. Its contribution to lower-degree terms can be estimated as follows. First we identify the degree of its contribution by using

$$\begin{aligned} d_{3n} - 3 &= d_{3n+1} - (d_0 + 2) && (2.13) \\ &\vdots \\ d_{3n+m} - l_m - 3 &= d_{3n+m+1} - l_{m+1} && (2.14) \end{aligned}$$

where $l_{m+1} = l_m + d_0 + 2$ which implies $l_m = m(d_0 + 2)$. Therefore, we get:

Lemma 2.2 *For $3(m+1) \leq d_{3n}$, $c_{3n+m+1,l_{m+1}}$ is lowerbounded by*

$$c_{3n+m+1,l_{m+1}} > \frac{(d_{3n+m} - l_m + 3m)!(3n)!}{(d_{3n+m} - l_m - 3)!(3n+m+1)!} c_{3n,0}$$

Proof: From the third derivative term, by using nonnegativity, we get

$$c_{3n+m+1,l_{m+1}} > \frac{(d_{3n+m} - l_m)(d_{3n+m} - l_m - 1)(d_{3n+m} - l_m - 2)}{(3n+m+1)} c_{3n+m,l_m}.$$

The desired result follows from a recursive use of this inequality and the relations (2.13–2.14) which give

$$\frac{(d_{3n+m-i} - l_{m-i})!}{(d_{3n+m-i} - l_{m-i} - 3)!} = \frac{(d_{3n+m-i+1} - l_{m-i+1} + 3)!}{(d_{3n+m-i+1} - l_{m-i+1})!}$$

for $0 \leq i \leq m$.

Now choose $m = n - 1$ (which satisfies $3(m+1) \leq d_{3n}$ because $d_0 \geq 2$). Then we get

$$c_{4n,l_n} > \frac{\big(3n(d_0 - 1) + d_0\big)!(3n)!}{\big(3n(d_0 - 2) + d_0\big)!(4n)!} \epsilon^{3n+1}.$$

Since this grows factorially with n, for $d_0 \geq 2$, we get divergence for the formal series (2.7).

□

2.2 The Nonpolynomial Case

Now suppose u_0 is nonpolynomial. Then $\{u_n\}$ are no longer polynomial, so we write

$$u_n = \sum_{p=0}^{\infty} b_{n,p} x^p.$$

Assume, as before, that $b_{0,d_0} > \epsilon$ for some $0 < \epsilon < 1$ and $d_0 \geq 2$. (Note that d_0 is no longer the degree of u_0.) Then from the recursion relation (2.8) we again get that, for all $n \geq 1$ and $d_n := n(d_0 - 1) + d_0$, the coefficients b_{n,d_n} have a lower bound

$$b_{n,d_n} > \epsilon^{n+1}.$$

The remainder of the argument for the polynomial case now follows to give a factorially growing lowerbound for b_{4n,q_n}, where $q_n := d_{4n} - l_n = 3n(d_0 - 2) + d_0$.

2.3 The Linear Case

Here we consider the case $u_0(x) = a_0 + a_1 x$. In this case, there exists an exact solution:

$$u(x,t) = \frac{a_0 + a_1 x}{1 - a_1 t}.$$

This solution is clearly holomorphic in the polydisk

$$\{(x,t)|x \in \mathbb{C}, |t| < 1/|a_1|\}.$$

Note that this result holds even if a_0, a_1 are not nonnegative.

2.4 The Case of an Isolated Double Pole

Consider the example (1.5) as initial datum with $a = 1$. (We can use the translational invariance of the KdV equation to assume this without loss of generality.) We give an alternate, simple, proof of nonexistence for this case here.

We claim that the coefficients $u_n(t)$ of 2.7 can be written as

$$u_n(t) = \frac{a_n}{(1-x)^{3n+2}}.$$

The inductive proof follows from the substitution of this form into the right side of (2.8):

$$\begin{aligned} & \sum_{k=0}^{n} u_k u'_{n-k} + u'''_n \\ = & \sum_{k=0}^{n} \frac{\big(3(n-k)+2\big) a_k a_{n-k}}{(1-x)^{3n+5}} \\ & \quad + \frac{(3n+2)(3n+3)(3n+4)a_n}{((1-x)^{3n+5}} \end{aligned}$$

So the recursion relation satisfied by a_n is

$$\begin{aligned} (n+1)a_{n+1} &= \sum_{k=0}^{n} \big(3(n-k)+2\big) a_k a_{n-k} \\ &\quad + (3n+2)(3n+3)(3n+4)a_n \\ &\geq (3n+2)(3n+3)(3n+4)a_n \end{aligned}$$

Hence the a_n are lower bounded by

$$a_n \geq \frac{(3n+1)!c}{n!}.$$

Clearly these give rise to a divergent series (2.7).

3 Proof of Theorem 1.2

For the KP equation, we consider the formal solution

$$u(t,x,y) = \sum_{j=0}^{\infty} u_j(t,x) y^j, \tag{3.15}$$

whose coefficients satisfy the recurrence relation:

$$(n+2)(n+1)u_{n+2} = u_{n,4x} + \sum_{j=0}^{n} (u_j u_{n-j,x})_x + u_{n,tx} \tag{3.16}$$

for $n \geq 2$. We follow a similar argument to that of the previous section and give the details of the proof for polynomial u_0, u_1 of respective degree $d_0 \geq 3$, $d_1 \geq 3$ in x. The argument for the nonpolynomial u_0 or u_1 is the same under the assumption that d_0 and d_1 are no longer their degrees but the degrees of some terms in their Taylor expansion that we track in the recursive estimate of coefficients.

Suppose that u_0 and u_1 are both non-negative polynomials in x (that is, polynomials with all coefficients non-negative for all t), and are analytic functions of t, so that

$$u_j(t,x) = \sum_{k=0}^{d_j} c_{j,k}(t) x^{d_j - k} \tag{3.17}$$

for $j = 0, 1$, where $c_{j,k}(t)$ are analytic functions of t with all Taylor coefficients non-negative.

Then u_n, for $n \geq 2$, will likewise be a non-negative polynomial in x for all t. We assume that u_j for $j \geq 2$ have the form given by (3.17). Since we have such non-negativity, we can generally ignore the term u_{tx} in (1.2) and (3.16) for purposes of calculating lower bounds.

The first question is, what degree are the polynomials u_j? The degree d_j follows different patterns (with respect to j) depending on whether $3d_0$ is greater than or less than $2d_1 + 2$. We consider these cases separately, and also describe what happens when u_0 or u_1 is identically zero.

Lemma 3.1 *(i). If $3d_0 \geq 2d_1 + 2$, then for all $n \geq 0$, $d_{2n} = n(d_0 - 2) + d_0$ and $d_{2n+1} = n(d_0 - 2) + d_1$.*

(ii). If $3d_0 < 2d_1 + 2$, then for all $n \geq 0$, $d_{3n} = n(d_1 - 2) + d_0, d_{3n+1} = n(d_1 - 2) + d_1$ and $d_{3n+2} = n(d_1 - 2) + 2d_0 - 2$.

(iii). If $u_1 = 0$, then for all $n \geq 0$, $d_{2n} = n(d_0 - 2) + d_0$ and $u_{2n+1} = 0$.

(iv). If $u_0 = 0$, then $u_2 = 0$ and for all $n \geq 1$, $d_{3n} \leq n(d_1 - 2) + 1$, $d_{3n+1} = n(d_1 - 2) + d_1$ and $d_{3n+2} \leq n(d_1 - 2)$.

Proof: In all cases, we obtain from (3.16) the following relation:

$$d_k = \max\{d_0 + d_{k-2} - 2, \ldots, d_{k-2} + d_0 - 2, d_{k-2} - 1\} \tag{3.18}$$

or equivalently:

$$d_k = \max_{j=0,\ldots,k-2}\{d_j + d_{k-2-j} - 2, d_{k-2} - 1\}$$

where the last term is only present if $c_{k-2,0}(t)$ is not constant, and is not necessary (and not mentioned) below, with the exception of (iv).

(i) The claim is true for $k = 0$ by definition of d_0 and d_1. Assume that it is true for all $n \leq k$, for some k. Then we use (3.18) as follows:

$$\begin{aligned}
d_{2k+2} &= \max\{d_{2j} + d_{2k-2j} - 2, d_{2j+1} + d_{2k-2j-1} - 2\} \\
&= \max\{k(d_0 - 2) + 2d_0 - 2, (k-1)(d_0 - 2) + 2d_1 - 2\} \\
&= k(d_0 - 2) + 2d_0 - 2 \\
d_{2k+3} &= \max\{d_{2j} + d_{2k-2j+1} - 2, d_{2j+1} + d_{2k-2j} - 2\} \\
&= \max\{k(d_0 - 2) + d_0 + d_1 - 2, k(d_0 - 2) + d_1 + d_0 - 2\} \\
&= k(d_0 - 2) + d_0 - 2 + d_1.
\end{aligned}$$

In the first series of equations, we use the assumption that $3d_0 \geq 2d_1 + 2$; also, j ranges from 0 to k.

(ii) For $k=0$, we need only check that $d_2 = 2d_0 - 2$; but this follows since $2u_2 = u_{0,4x} + u_{0,tx} + (u_0 u_{0,x})_x$ (by (3.16)). Assume that the claim is true for all $n \leq k$, for some k. Then we again use (3.18) as follows:

$$
\begin{aligned}
d_{3k+3} &= \max\{d_{3j} + d_{3(k-j)+1} - 2, d_{3j+1} + d_{3(k-j)} - 2, d_{3j+2} + d_{3(k-j-1)+2} - 2\} \\
&= \max\{k(d_1-2) + d_0 + d_1 - 2, k(d_1-2) + d_1 + d_0 - 2, \\
&\quad (k-1)(d_1-2) + 4d_0 - 6\} \\
&= (k+1)(d_1-2) + d_0 \\
d_{3k+4} &= \max\{d_{3j} + d_{3(k-j)+2} - 2, d_{3j+1} + d_{3(k-j)+1} - 2, d_{3j+2} + d_{3(k-j)} - 2\} \\
&= \max\{k(d_1-2) + 3d_0 - 4, k(d_1-2) + 2d_1 - 2, k(d_1-2) + 3d_0 - 4\} \\
&= (k+1)(d_1-2) + d_1 \\
d_{3k+5} &= \max\{d_{3j} + d_{3(k-j+1)} - 2, d_{3j+1} + d_{3(k-j)+2} - 2, d_{3j+2} + d_{3(k-j-1)+1} - 2\} \\
&= \max\{(k+1)(d_1-2) + 2d_0 - 2, k(d_1-2) + d_1 + 2d_0 - 4, \\
&\quad k(d_1-2) + 2d_0 + d_1 - 4\} \\
&= (k+1)(d_1-2) + 2d_0 - 2
\end{aligned}
$$

Throughout, j varies between 0 and $k-1$, and we use the assumption that $3d_0 < 2d_1 + 2$ in the first two series of equations.

Thus, by induction, the claim is true for all integers $n \geq 0$.

(iii) The claim is trivially true for $n=0$. Suppose it is true for all $n \leq k$. Then by (3.16), $(2n+3)(2n+1)u_{2n+3} = \sum_{j=0}^{2n+1}(u_j u_{2n+1-j,x})_x$; but each of the terms in the sum is identically zero (either j is odd, or $2n+1-j$ is odd). Also by (3.16), we have that

$$d_{2n+2} = \max\{d_0 + d_{2n} - 2, d_2 + d_{2n-2} - 2, \ldots, 2d_n - 2\}.$$

But each of these numbers is just $(n+1)(d_0-2) + d_0$. Thus, by induction, the claim is true for all $k \geq 0$.

(iv) In this case, the degree can take values below the maximum (and in some cases, u_{3n+2} can be 0) if, for example, $c'_{1,0}(t)$ is identically zero.

Since $u_0 = 0$, we have that $u_2 = 0$. Assume that the claim is true for all $n \leq k$, for some k. Then we again use (3.18) as follows:

$$
\begin{aligned}
d_{3k+3} &\leq \max\{d_{3j} + d_{3(k-j)+1} - 2, d_{3j+1} + d_{3(k-j)} - 2, d_{3j+2} + d_{3(k-j-1)+2} - 2, \\
&\quad d_{3k+1} - 1\} \\
&\leq \max\{k(d_1-2) + d_1 - 1, k(d_1-2) + d_1 - 1, (k-1)(d_1-2) - 2, \\
&\quad k(d_1-2) + d_1 - 1\} \\
&= (k+1)(d_1-2) + 1
\end{aligned}
$$

$$
\begin{aligned}
d_{3k+4} &= \max\{d_{3j} + d_{3(k-j)+2} - 2, d_{3j+1} + d_{3(k-j)+1} - 2, d_{3j+2} + d_{3(k-j)} - 2, \\
&\quad d_{3k+2} - 1\} \\
&= \max\{k(d_1 - 2) - 1, k(d_1 - 2) + 2d_1 - 2, k(d_1 - 2) - 1, k(d_1 - 2) - 1\} \\
&= (k+1)(d_1 - 2) + d_1
\end{aligned}
$$

$$
\begin{aligned}
d_{3k+5} &\leq \max\{d_{3j} + d_{3(k-j+1)} - 2, d_{3j+1} + d_{3(k-j)+2} - 2, d_{3j+2} + d_{3(k-j-1)+1} - 2 \\
&\quad d_{3k+3} - 1\} \\
&= \max\{(k+1)(d_1 - 2), k(d_1 - 2) + d_1 - 2, \\
&\quad k(d_1 - 2) - 1, (k+1)(d_1 - 2)\} \\
&= (k+1)(d_1 - 2)
\end{aligned}
$$

Throughout, j varies between 0 and $k-1$.

□

From the above lemma, we see that the degree of the polynomial u_n in x will only grow (as n increases) if either $d_0 \geq 3$ or $d_1 \geq 3$. If so, and if $c_{0,0}(t)$ or $c_{1,0}(t)$ (where they exist) have a positive lower bound, the leading coefficients of certain of the terms u_n grow exponentially, as follows.

Lemma 3.2 *(i). If $3d_0 \geq 2d_1 + 2$, $d_0 \geq 3$ and for all $t > 0$, $c_{0,0}(t) > \epsilon$ and $c_{1,0}(t) > \epsilon$, then for all $n \geq 0$, $c_{2n,0}(t) > \epsilon^{n+1}$ and $c_{2n+1,0}(t) > \epsilon^{n+1}$.*

(ii). If $3d_0 < 2d_1 + 2$, $d_1 \geq 3$ and for all $t > 0$, $c_{1,0}(t) > \epsilon$, then for all $n \geq 0$, $c_{3n+1,0}(t) > \epsilon^{n+1}$.

(iii). If $u_1 = 0$, $d_0 \geq 3$ and for all $t > 0$, $c_{0,0}(t) > \epsilon$, then for all $n \geq 0$, $c_{2n,0}(t) > \epsilon^{n+1}$.

(iv). If $u_0 = 0$, $d_1 \geq 3$ and for all $t > 0$, $c_{1,0}(t) > \epsilon$, then for all $n \geq 0$, $c_{3n+1,0}(t) > \epsilon^{n+1}$.

Proof: In all cases, we begin by deducing the following relation from (3.16):

$$
k(k-1)c_{k-1,0}(t) \geq (d_k + 1) \sum_{j=0}^{k-2} d_j b_{j,k} c_{j,0}(t) c_{k-2-j,0}(t). \tag{3.19}
$$

Here $b_{j,k}$ is simply a constant, either 0 or 1 depending on which terms of the form $(u_j u_{k-2-j,x})_x$ contribute to the highest-order term x^{d_k}.

(i) Assume that the claim is true for all $n \leq k$. Using all our assumptions, we have that

$$\begin{aligned}
(2k+2)(2k+1)c_{2k+2,0}(t) &> (d_{2k+2}+1)\epsilon^{k+2}\sum_{j=0}^{k} d_{2j} \\
&= \epsilon^{k+2}((k+1)(d_0-2)+d_0+1) \\
&\quad (k+1)\left(\frac{1}{2}k(d_0-2)+d_0\right) \\
&\geq (2k+2)(2k+1)\epsilon^{k+2} \text{ since } \mathrm{d}_0 \geq 3
\end{aligned}$$

$$\begin{aligned}
(2k+3)(2k+2)c_{2k+3,0}(t) &> (d_{2k+3}+1)\epsilon^{k+2}\sum_{j=0}^{2k} d_j \\
&= \epsilon^{k+2}((k+1)(d_0-2)+d_1+1) \\
&\quad (k+1)(k(d_0-2)+d_0+d_1) \\
&\geq (2k+3)(2k+2)\epsilon^{k+2}.
\end{aligned}$$

Note that all terms contribute in the second case, but only odd terms in the first.

(ii) Assume that the claim is true for all $n \leq k$. Then we have that

$$\begin{aligned}
(3k+4)(3k+3)c_{3k+4,0}(t) &> (d_{3k+4}+1)\epsilon^{k+2}\sum_{j=0}^{k} d_{3j+1} \\
&= \epsilon^{k+2}((k+1)(d_1-2)+d_1+1) \\
&\quad (k+1)\left(\frac{1}{2}k(d_1-2)+d_1\right) \\
&\geq (3k+4)(3k+3)\epsilon^{k+2} \text{ if } \mathrm{d}_1 \geq 4 \text{ or } \mathrm{k} \geq 6.
\end{aligned}$$

The cases $d_1 = 3$ and $k = 1, \ldots, 5$ can be calculated explicitly (and fully, using exact values) to show that in these cases also, $c_{3k+4,0}(t)$ is greater than ϵ^{k+2}.

So the claim is true for all integers $n \geq 0$.

(iii) Assume that the claim is true for all $n \leq k$. Then we have that

$$\begin{aligned}
(2k+2)(2k+1)c_{2k+2,0}(t) &> (d_{2k+2}+1)\epsilon^{k+2}\sum_{j=0}^{k} d_{2j} \\
&= \epsilon^{k+2}((k+1)(d_0-2)+d_0+1) \\
&\quad (k+1)\left(\frac{1}{2}k(d_0-2)+d_0\right) \\
&\geq (2k+2)(2k+1)\epsilon^{k+2} \text{ if } \mathrm{d}_0 \geq 3
\end{aligned}$$

(iv) The proof here is identical to that in (ii), since the degree of u_{3k+1} is the same in both cases (and since only terms of the form $c_{3j+1,0}(t)$ contribute to $c_{3k+1}(t)$ in both cases).

□

We now follow the method used for the KdV equation.

Lemma 3.3 *(i). If $3d_0 \geq 2d_1+2$, $d_0 \geq 3$ and $c_{0,0}(t) > \epsilon$ etc., then for all positive integers q,*

$$c_{10q,q(d_0+2)} > \frac{(4q(d_0-2)+d_0)!(8q)!}{(4q(d_0-3)+d_0)!(10q)!}\epsilon^{4q+1}.$$

(ii). If $3d_0 < 2d_1+2$, $d_1 \geq 3$ and $c_{0,0}(t) > \epsilon$ etc., then for all positive integers q,

$$c_{42q+1,2q(d_1+4)} > \frac{(12q(d_1-2)+d_1)!(36q+1)!}{12q(d_1-3)+d_1)!(42q+1)!}\epsilon^{12q+1}.$$

(iii). If $u_1 = 0$, $d_0 \geq 3$ and $c_{0,0}(t) > \epsilon$ etc., then for all positive integers q,

$$c_{10q,q(d_0+2)} > \frac{(4q(d_0-2)+d_0)!(8q)!}{(4q(d_0-3)+d_0)!(10q)!}\epsilon^{2q+1}.$$

(iv). If $u_0 = 0$, $d_1 \geq 3$ and $c_{1,0}(t) > \epsilon$ etc., then for all positive integers q,

$$c_{42q+1,2q(d_1+4)} > \frac{(12q(d_1-2)+d_1)!(36q+1)!}{12q(d_1-3)+d_1)!(42q+1)!}\epsilon^{12q+1}.$$

Proof: From (3.16), we see due to the $u_{n-2,xxxx}$ term and the non-negativity of the coefficients that the coefficient of x_{d_p} in u_p, $c_{p,0}(t)$, adds to the coefficient of x^{d_p-4} in u_{p+2}, $c_{p+2,d_{p+2}-d_p+4}$. That is,

$$c_{p+2,d_{p+2}-d_p+4} > \frac{d_p(d_p-1)(d_p-2)(d_p-3)}{(p+2)(p+1)} c_{p,0}.$$

Similarly,

$$c_{p+4,d_{p+4}-d_p+8} > \frac{(d_p-4)(d_p-5)(d_p-6)(d_p-7)}{(p+4)(p+3)} c_{p+2,d_{p+2}-d_p+4}.$$

Continuing in this way gives the inequality

$$c_{p+2m,d_{p+2m}-d_p+4m} > \frac{d_p! p!}{(d_p-4m)!(p+2m)!} c_{p,0} \tag{3.20}$$

for m any positive integer such that $d_p \geq 4m$.

(i) Set $p := 8q$ and $m := q$ for some positive integer q. Then from Lemma 3.1(i), we have that $d_{8q} = 4q(d_0 - 2) + d_0$ and $d_{10q} = 5q(d_0 - 2) + d_0$; from Lemma 3.2(i), we have that $c_{8q,0} > \epsilon^{4q+1}$. Substituting these expressions into (3.20) completes the proof.

(ii) Set $p := 36q + 1$ and $m := 3q$ for some positive integer q. Then from Lemma 3.1(ii), we have that $d_{36q+1} = 12q(d_1 - 2) + d_1$ and $d_{42q+1} = 14q(d_1 - 2) + d_1$; from Lemma 3.2(ii), we have that $c_{36q+1,0} > \epsilon^{12q+1}$. Substituting these expressions into (3.20) completes the proof.

(iii) Set $p := 8q$ and $m := q$ for some positive integer q. Then from Lemma 3.1(iii), we have that $d_{8q} = 4q(d_0 - 2) + d_0$ and $d_{10q} = 5q(d_0 - 2) + d_0$; from Lemma 3.2(iii), we have that $c_{8q,0} > \epsilon^{2q+1}$. Substituting these expressions into (3.20) completes the proof.

(iv) Set $p := 36q + 1$ and $m := 3q$ for some positive integer q. Then from Lemma 3.1(iv), we have that $d_{36q+1} = 12q(d_1 - 2) + d_1$ and $d_{42q+1} = 14q(d_1 - 2) + d_1$; from Lemma 3.2(iv), we have that $c_{36q+1,0} > \epsilon^{12q+1}$. Substituting these expressions into (3.20) completes the proof.

$\square$

These quantities grow factorially with q; thus the formal series (3.15) diverges.

There are also holomorphic solutions in the neighbourhood of the origin of the KP equation; for example, if the initial conditions are linear or quadratic, then we can find exact solutions. These include the following solutions, which are constant in t:

$$u(x,y) = a_0 + a_1 x + (b_0 + b_1 x)y + \frac{1}{2}a_1^2 y^2 + \frac{1}{3}a_1 b_1 y^3 + \frac{1}{12}b_1^2 y^4,$$

for u_0 and u_1 linear functions of x (i.e. a_0, a_1, b_0 and b_1 constants), which is holomorphic for all x and y, and

$$u(x,y) = \frac{(x+A)^2}{(y+k)^2} + \frac{B}{y+k} + C(y+k)^2,$$

for u_0 and u_1 quadratic in x (and k, A, B, C constants, $k \neq 0$), which is holomorphic for $|y| < |k|$.

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References

[1] M. J. Ablowitz, A. Ramani, and H. Segur. Nonlinear evolution equations and ordinary differential equations of Painlevé type. *Lett. Nuovo Cim.*, 23:333–338, 1978.

[2] M. J. Ablowitz, A. Ramani, and H. Segur. A connection between nonlinear evolution equations and ordinary differential equations of P-type I and II. *J. Math. Phys.*, 21:715–721, 1006–1015, 1980.

[3] J. Weiss, M. Tabor, and G. Carnevale. The Painlevé property for partial differential equations. *J. Math. Phys.*, 24:522–526, 1983.

[4] M. J. Ablowitz and H. Segur. *Solitons and the Inverse Scattering Transform*. SIAM, Philadelphia, 1981.

[5] M. J. Ablowitz and P. A. Clarkson. *Solitons, Nonlinear Evolution Equations and Inverse Scattering*, volume 149 of *London Mathematical Society Lecture Notes in Mathematics*. Cambridge University Press, Cambridge, 1991.

[6] C. E. Kenig, G. Ponce, and L. Vega. Global solutions for the kdv equation with unbounded data. *J. Diff. Eqns*, 139:339–364, 1997.