Computing the fuzzy topological relations of spatial objects based on
induced fuzzy topology

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For modeling the topological relations between spatial objects, the concepts of a
bound on the intersection of the boundary and interior, and the boundary and
exterior are defined in this paper based on the newly developed computational
fuzzy topology. Furthermore, the qualitative measures for the intersections are
specified based on the $\alpha$-cut induced fuzzy topology, which are $(A_{\alpha} \triangle \partial A)(x)<1-\alpha$ and
$((A')_{\alpha} \triangle \partial A)(x)<1-\alpha$. In other words, the intersection of the interior and
boundary or boundary and exterior are always bounded by $1-\alpha$, where $\alpha$ is a
value of a level cutting. Specifically, the following areas are covered: (a) the
homeomorphic invariants of the fuzzy topology; (b) a definition of the
connectivity of the newly developed fuzzy topology; (c) a model of the fuzzy
topological relations between simple fuzzy regions in GIS; and (d) the
quantitative values of topological relations can be calculated.

Keywords: Closure; Interior; Integration matrix; Homeomorphism; Supported
connected; Topological relations

1. Introduction

Topological relations between spatial objects are fundamental information used in
GIS, along with positional and attribute information. Information on topological
relations can be used for spatial queries, spatial analyses, data quality control (e.g.
checking for topological consistency), and others. Topological relations can be crisp
or fuzzy depending on the certainty or uncertainty of spatial objects and the nature
of their relations. When the spatial objects concerned are uncertain, or their
relations are not certain, the issue of uncertain topological relations emerges. There
are many uncertain relations that need to be modeled among spatial objects in GIS.
For example, the topological relation between two islands can be connected or
separated depending on the level of the tides at the time. The two islands are
connected when the tide is low, and are separated at high tide. The relations between
the two islands are thus uncertain. In this type of spatial analysis, it is essential to
understand the uncertain topological relations between spatial objects.

Mathematically, fuzzy topology, which is a generalization of ordinary topology by
introducing the concept of membership value, can be adapted to the modelling of
spatial objects with uncertainties. Zadeh (1965) introduced the concept of the fuzzy set.
Fuzzy theory has been developed since 1965, and the theory of fuzzy topology (Zadeh
1965, Wong 1974, Wu and Zheng 1991, Lin and Luo 1997) has been developed based

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on the fuzzy set. Topological relations are one of the concerns in modelling spatial objects, besides their geometric and attribute aspects. Uncertain topological relations need to be modelled due to the existence of the indeterminate and uncertain boundaries between spatial objects in GIS. Fuzzy topology theory can potentially be applied to the modelling of fuzzy topological relations among spatial objects.

Research on topological relations between spatial objects has attracted a great deal of attention in the past few decades. The major research outcomes on topological relations between regions have included, for example, 4-intersection models (Egenhofer and Franzosa 1991, Winter 2000) and 9-intersection models (Egenhofer and Franzosa 1991, Clementini and Di Felice 1996, Cohn and Gotts 1996, Smith 1996, Shi and Guo 1999, Tang and Kainz 2002).

The models (Egenhofer 1993, Clementini and Di Felice 1996, Cohn and Gotts 1996, Smith 1996, Tang and Kainz 2002) were developed under the concepts of interior, boundary, and exterior which makes it potentially possible to model uncertainty relations between spatial objects conceptually. However, it is still difficult to implement these concepts (such as interior, boundary, and exterior) in a computer system, even with a membership function of a fuzzy set (or uncertain spatial object). Therefore, at this stage, while the conceptual framework on modeling topological relations between spatial objects is well developed (Egenhofer 1993, Clementini and Di Felice 1996, Cohn and Gotts 1996, Smith 1996, Tang and Kainz 2002), there is room to further enhance the mechanisms to implement these conceptual models. Furthermore, Tang et al. (2005) applied fuzzy setting method for reasoning land cover changes.

In order to develop methods for computing the uncertain spatial relations defined by the conceptual spatial relation models, such as the 9-intersection model, we have initiated a two-stage research framework, as illustrated in figure 1.

- Stage One: to construct a computational fuzzy topology; and
- Stage Two: to develop methods for computing fuzzy topological relations.

![Figure 1](image.png)

Figure 1. A summary of the two-stage research initiatives on computational fuzzy topological relations.
It is the aim in Stage One of the research to construct a computational fuzzy topology, which is a theoretical basis for computational fuzzy topological relations. The focus here is to develop algorithms for computing the interior, boundary, and exterior of uncertain spatial objects. A fuzzy topology can be defined by each interior or each closure operator (Liu and Luo 1997). Along this direction, we can define a fuzzy topology for spatial objects, where the interior and closure operators are defined by suitable level cutting (Liu and Shi, submitted). The algorithm for calculating the interior, boundary, and closure can thus be developed, and form the basis for implementing the conceptual models on spatial relations.

In Stage Two of the research, the aim is to develop methods for computing fuzzy topological relations based on the conceptual spatial relation models. This aim can be realized by: (a) defining the connectivity for the newly developed fuzzy topology; (b) modeling homeomorphic invariants (the topological relations that are unchanged under a homeomorphic mapping) of the new fuzzy topology; and (c) providing fuzzy topological relations between simple fuzzy regions.

The research outcomes of Stage One of the development are given in detail in Liu and Shi (submitted). However, a brief summary of the major findings of Stage One of the research is given in section 2 of this paper, which covers several basic concepts of fuzzy topology and the newly defined computational fuzzy topology. Section 3 presents the preserving properties of the computational fuzzy topology. In section 4, we introduce the concept of connectivity of fuzzy topology in GIS. In section 5, we define a simple fuzzy region, a simple fuzzy line segment, and a fuzzy point. These form the basis for describing qualitative fuzzy topological relations between simple fuzzy elements, which is the focus of section 6. Finally, we compare this newly developed model with the existing ones.

2. A brief summary of computational fuzzy topology

We know that each interior operator corresponds to one fuzzy topology and that each closure operator also corresponds to one fuzzy topology (Liu and Luo 1997). In this section, we will review the coherent fuzzy topologies, which are induced by interior and closure operators.

An I-fuzzy subset on X is a mapping (called the membership function of A) \( \mu_A : X \rightarrow I \); i.e. the family of all the \([0, 1]\)-fuzzy or I-fuzzy subsets on X is just \( I^X \) consisting of all the mappings from X to I.

Let A and B be fuzzy sets in X with membership functions \( \mu_A(x) \) and \( \mu_B(x) \), respectively. Then

\[
\begin{align*}
(i) \quad A = B & \iff \mu_A(x) = \mu_B(x) \text{ for all } x \in X. \\
(ii) \quad A \subseteq B & \iff \mu_A(x) \leq \mu_B(x) \text{ for all } x \in X. \\
(iii) \quad C = A \lor B & \iff \mu_C(x) = \max[\mu_A(x), \mu_B(x)] \text{ for all } x \in X. \\
(iv) \quad D = A \land B & \iff \mu_D(x) = \min[\mu_A(x), \mu_B(x)] \text{ for all } x \in X. \\
v) \quad E = X \setminus A & \iff \mu_E(x) = 1 - \mu_A(x) \text{ for all } x \in X.
\end{align*}
\]

Fuzzy topology is an extension topology of ordinary topology. Here we adopt the definition of Chang (1968), which is let X be a non-empty ordinary set and \( \delta \subseteq I^X \). \( \delta \) is called a I-fuzzy topology on X, and \( (I^X, \delta) \) (or \( (X, I^X, \delta) \) for detail; or \( I^X \) for short) is called an I-fuzzy topological space (I-fts) and \( I^X \) for short, if \( \delta \) satisfies the following conditions:

\[
\begin{align*}
(i) \quad 0, 1 & \in \delta;
\end{align*}
\]
(ii) If A, B ∈ δ, then A ⊖ B ∈ δ.
(iii) Let \( \{A_i : i \in J\} \subset \delta \), where J is an index set, then \( \bigvee_{i \in j} A_i \in \delta \).

For any fuzzy set A in \( I^x \), the interior of A is defined as the join of all of the open subsets contained in A, denoted by \( A^0 \). The closure of A is the meet of all of the closed subsets containing A, denoted by \( \overline{A} \). The complement of A is defined as \( 1 - A(x) \) and denoted by \( A^c \). The boundary of fuzzy set A is defined as \( \partial A = \overline{A} \triangle \overline{A^c} \).

An operator \( \varphi : I^x \rightarrow I^x \) is a fuzzy closure operator if it satisfies the following conditions: (i) \( \varphi(0) = 0 \); (ii) \( A \subseteq \varphi(A) \); (iii) \( \varphi(A \cup B) = \varphi(A) \cup \varphi(B) \); and (iv) \( \varphi(\varphi(A)) = \varphi(A) \), for all \( A \in I^x \). Moreover, an operator \( \varphi : I^x \rightarrow I^x \) is a fuzzy interior operator if it satisfies the following conditions: (i) \( \varphi(1) = 1 \); (ii) \( \varphi(A) \subseteq A \); (iii) \( \varphi(A \cap B) = \varphi(A) \cap \varphi(B) \); and (iv) \( \varphi(\varphi(A)) = \varphi(A) \), for all \( A \in I^x \).

For any fixed \( \alpha \in [0, 1] \), two operators, interior and closure, are defined as:

\[
A_{\alpha}(x) = \begin{cases} 
A(x) & \text{if } A(x) > \alpha \\
0 & \text{if } A(x) \leq \alpha 
\end{cases}
\]

and \( A^{1-\alpha}(x) = \begin{cases} 
1 & \text{if } A(x) \geq 1 - \alpha \\
A(x) & \text{if } A(x), + - \alpha 
\end{cases} \).

which can induce an I-fuzzy topology \( (X, \tau_{\alpha}, t^{1-\alpha}) \) in X, where \( t^{1-\alpha} = \{A^{1-\alpha} : A \in I^x \} \) and \( \tau_{\alpha} = \{A_{\alpha} : A \in I^x \} \) (Liu and Shi, submitted). The details of how these two operators induced an I-fuzzy topology are in Stage One of this research, and one may refer to Liu and Shi (submitted).

The fuzzy topological relations that include Equal, Contained In, Covered By, Contain, Cover, Overlap, and Disjoint are elementary relations in the study of topological relations between spatial objects in GIS. Many researchers develop their models in this area based on these critical relations (Egenhofer and Franzosa 1991, Clementini and Di Felice 1996, Cohn and Gotts 1996, Smith 1996, Shi and Guo 1999, Tang and Kainz 2002).

(i) **Equal**: \( A = B \) iff \( A(x) = B(x) \) for all \( x \in X \).
(ii) **Contained In**: \( A \subseteq B \) iff \( A(x) \subseteq B(x) \) for all \( x \in X \) such that \( A(x) > 0 \).
(iii) **Covered By**: \( A \leq B \) iff \( A(x) \leq B(x) \) for all \( x \in X \) such that \( A(x) > 0 \) and there exist some \( x_1 \in X \) such that \( A(x_1) = B(x_1) > 0 \).
(iv) **Contain**: \( A \supset B \) iff \( A(x) \supset B(x) \) for all \( x \in X \) such that \( B(x) > 0 \).
(v) **Cover**: \( A \supset B \) iff \( A(x) \supset B(x) \) for all \( x \in X \) such that \( B(x) > 0 \) and there exist some \( x_1 \in X \) such that \( A(x_1) = B(x_1) > 0 \).
(vi) **Overlap**: \( A \cap B \neq 0 \) iff \( A \cap B(x_0) > 0 \), \( A(x_1) \cap B(x_1) \) and \( B(x_2) \cap A(x_2) \), for some \( x_0, x_1 \) and \( x_2 \in X \),
(vii) **Disjoint**: \( A \cup B = 0 \) iff \( A \cup B(x) = 0 \) for all \( x \in X \).

The properties of topological spaces that are preserved under homeomorphic mappings are called the topological invariants of the spaces. To study the topological relations, we first need to investigate the properties of a fuzzy mapping, especially homeomorphic mapping. The topological relations are invariants under homeomorphic mappings. With these, we can thus guarantee the properties that will remain unchanged in a GIS transformation, such as the maintenance of topological consistency when digitizing a map or transferring a map from a system to another system. Therefore, in the next section we start with a study of the preserving properties of new developed fuzzy topology.

### 3. Several properties of computational fuzzy topology

In the following two sections, we would like to develop the preserving properties of the computational fuzzy topology and the connectivity of this fuzzy topology in
GIS. The main objective of this section is to prove the open and closed sets that are preserved by fuzzy mapping and fuzzy reverse mapping (see proposition 3.6). Furthermore, the connectivity of a fuzzy topology is fundamental in any study of the topological relations between spatial objects in GIS. Therefore, the properties of connection in the new induced fuzzy topology will be studied in section 4.

**Definition 3.1 (Fuzzy mapping):** Let $I^X$, $I^Y$ be I-fuzzy topological spaces, $f:X \rightarrow Y$ an ordinary mapping. Based on $f:X \rightarrow Y$, define I-fuzzy mapping $f^-:I^X \rightarrow I^Y$ and its I-fuzzy reverse mapping $f^+:I^Y \rightarrow I^X$ by

$$
f^- : I^X \rightarrow I^Y, f^-(A)(y) = \begin{cases} \bigvee \{A(x)\} & \text{if } x \in f^{-1}(y), \forall A \in I^X, \forall y \in Y, \\ 0 & \text{otherwise} \end{cases}
$$

$$
f^+ : I^Y \rightarrow I^X, f^+(B)(x) = B(f(x)), \forall B \in I^Y, \forall x \in X,
$$

where $I=[0, 1]$.

**Definition 3.2 (Injective, surjective and bijective):** Let $(I^X, \delta)$, $(I^Y, \mu)$ be I-fts’s and let $f^-:(I^X, \delta) \rightarrow (I^Y, \mu)$ be an I-fuzzy mapping. Then,

(i) $f^-$ is called an injective mapping if whenever $f^-(A)=f^-(B)$, then $A=B$.

(ii) $f^-$ is called a surjective mapping if all $B \in \mu$, then there exists $A \in \delta$, such that $B=f^-(A)$.

(iii) $f^-$ is called a bijective mapping if $f^-$ is both injective and surjective.

**Theorem 3.3:** Let $I^X$, $I^Y$ be I-fuzzy spaces, $f:X \rightarrow Y$ an ordinary mapping. Then $f^-:(I^X, \delta) \rightarrow (I^Y, \mu)$ is bijective if and only if $f:X \rightarrow Y$ is bijective.

The proof was derived by Liu and Luo (1997).

**Definition 3.4 (Fuzzy continuous mapping):** Let $(I^X, \delta)$, $(I^Y, \mu)$ be I-fts’s, $f^-:(I^X, \delta) \rightarrow (I^Y, \mu)$ is called an I-fuzzy continuous mapping if $f^-$ maps every open subset in $(I^Y, \mu)$ as an open subset in $(I^X, \delta)$, i.e. for all $U \in \mu$, $f^-(U) \in \delta$.

**Definition 3.5 (Fuzzy homeomorphism):** Let $(I^X, \delta)$, $(I^Y, \mu)$ be I-fts’s, $f^-:(I^X, \delta) \rightarrow (I^Y, \mu)$ is called an I-fuzzy homeomorphism, if it is bijective, continuous, and open.

**Proposition 3.6:** Let $A \in I^X$, $B \in I^Y$, let $(I^X, \delta)$, $(I^Y, \mu)$ be I-fts’s induced by the interior operator and closure operator, $f^-:(I^X, \delta) \rightarrow (I^Y, \mu)$ and $f^-:(I^Y, \mu) \rightarrow (I^X, \delta)$. The following then holds:

(i) $f^-(A_x) = \lceil f^-(A) \rceil_x$\[1]

(ii) $f^-(A^{1-x}) = \lceil f^-(A) \rceil^{1-x}$

(iii) $f^+(B_x) = \lceil f^+(B) \rceil_x$

(vi) $f^+(B^{1-x}) = \lceil f^+(B) \rceil^{1-x}$

**Proof:**

(i) For all $y \in Y$, if $\{f^-(y)\} = \phi$, the result is obvious as both sides are zero.

Suppose $\{f^-(y)\} \neq \phi$.

If there exists $x_o \in \{f^-(y)\}$ such that $A(x_o) > x$, then

$$f^-(A_x)(y) = \bigvee_{x \in \{f^-(y)\}} A(x) > x$$

and

$$\lceil f^-(A) \rceil_x(y) = \bigvee_{x \in \{f^-(y)\}} A(x) > x.$$
If for all \( x \in \{ f^{-1}(y) \} \) such that \( A(x) \leq \alpha \), then \( f^-(A_2)(y) = 0 \) and
\[
f^-(A)(y) = \bigvee_{x \in \{ f^{-1}(y) \}} A(x) \leq \alpha, \text{ i.e. } [f^-(A)]_\alpha(y) = 0.
\]
Thus, we have \( f^-(A_2)(y) = [f^-(A)]_\alpha(y) \) for all \( y \in Y \); hence, \( f^-(A_2) = [f^-(A)]_\alpha \).

(ii) For all \( y \in Y \), if \( \{ f^{-1}(y) \} = \phi \), the result is obvious as both sides are zero.
Suppose \( \{ f^{-1}(y) \} \neq \phi \).
If there exists \( x_0 \in \{ f^{-1}(y) \} \) such that \( A(x_0) \geq 1 - \alpha \), then
\[
f^-(A_2)(y) = \bigvee_{x \in \{ f^{-1}(y) \}} A(x) > \alpha \quad \text{and} \quad [f^-(A)]_\alpha(y) = \bigvee_{x \in \{ f^{-1}(y) \}} A(x) > 1.
\]

(ii) If for all \( x \in \{ f^{-1}(y) \} \) such that \( A(x) < 1 - \alpha \), then
\[
f^-(A^{1-\alpha})(y) = \bigvee_{x \in \{ f^{-1}(y) \}} A^{1-\alpha}(x)
\]
\[
= \bigvee_{x \in \{ f^{-1}(y) \}} A(x)
\]
\[
= \begin{cases} 
1 & \text{if } \bigvee_{x \in \{ f^{-1}(y) \}} A(x) < 1 - \alpha \\
\bigvee_{x \in \{ f^{-1}(y) \}} A(x) & \text{if } \bigvee_{x \in \{ f^{-1}(y) \}} A(x) = 1 - \alpha 
\end{cases}
\]
\[
f^-(A)(y) = \bigvee_{x \in \{ f^{-1}(y) \}} A(x)
\]
\[
= \begin{cases} 
1 & \text{if } \bigvee_{x \in \{ f^{-1}(y) \}} A(x) < 1 - \alpha \\
\bigvee_{x \in \{ f^{-1}(y) \}} A(x) & \text{if } \bigvee_{x \in \{ f^{-1}(y) \}} A(x) = 1 - \alpha 
\end{cases}
\]
Hence, \([f^-(A)]^{1-\alpha}(y) = \begin{cases} 
1 & \text{if } \bigvee_{x \in \{ f^{-1}(y) \}} A(x) < 1 - \alpha \\
\bigvee_{x \in \{ f^{-1}(y) \}} A(x) & \text{if } \bigvee_{x \in \{ f^{-1}(y) \}} A(x) = 1 - \alpha 
\end{cases}\)

(iii) \( f^-(B_2)(x) = B_2(f(x)) = \begin{cases} 
B(f(x)) & \text{if } B(f(x)) > \alpha \\
0 & \text{if } B(f(x)) \leq \alpha 
\end{cases}\).
\[
[f^-(B)]_\alpha(x) = \begin{cases} 
f^-(B)(x) & \text{if } f^-(B)(x) > \alpha \\
0 & \text{if } f^-(B)(x) \leq \alpha 
\end{cases}
\]
\[
= \begin{cases} 
B(f(x)) & \text{if } B(f(x)) > \alpha \\
0 & \text{if } B(f(x)) \leq \alpha 
\end{cases}
\]

(iv) \( f^-(B^{1-\alpha})(x) = B^{1-\alpha}(f(x)) = \begin{cases} 
1 & \text{if } B(f(x)) \geq 1 - \alpha \\
B(f(x)) & \text{if } B(f(x)) < 1 - \alpha 
\end{cases}\).
\[
[f^-(B)]^{1-\alpha}(x) = \begin{cases} 
f^-(B)(x) & \text{if } f^-(B)(x) \geq \alpha \\
1 & \text{if } f^-(B)(x) < \alpha \\
B(f(x)) & \text{if } B(f(x)) \geq \alpha \\
0 & \text{if } B(f(x)) < \alpha 
\end{cases}
\]

Q.E.D.
4. Connectivity of spatial objects in GIS based on fuzzy topology

Fuzzy topology can be applied to describe and analyze the structure of neighborhoods and leveling of spaces (Liu and Luo 1997). In other words, the general fuzzy topological space consists of two concepts of spaces, which are ‘levelling space’ and ‘neighborhood space’. Connectivity is a preserving property of fuzzy topology. The usual definition of the connection of fuzzy subset, A, in fuzzy topology is that A cannot be separated by two non-zero open or closed fuzzy sets, called open connected and closed connected, respectively. As the natural character of fuzzy topology, this kind of connection also contains two types of structures—neighborhood and leveling, respectively. In GIS, the connectivity of spatial objects depends on the neighbourhoood structure of the objects themselves, rather than on the leveling structure. Thus, the ordinary definition of the connection of fuzzy topology is not suitable for describing relations between spatial objects in GIS. In $f^R$, figure 2 shows two spatial fuzzy objects in GIS that are considered to be connected. Figure 3 shows two spatial fuzzy objects in GIS that are considered to be supported disconnected.
As a result, it is necessary to develop a new definition of connectivity due to the fact that the existing definition of connectivity in fuzzy topology is not applicable to GIS. In traditional GIS, we only consider neighborhood relations between spatial objects. The connectivity, based on fuzzy topology, involves both neighborhood relations and leveling relations. Therefore, in the application of fuzzy topology to GIS, we need only consider neighborhood relations. Based on this understanding, among the two fuzzy spatial objects in figure 2, one is fuzzy connected (figure 2(a)) while the other is not fuzzy connected (figure 2(b)). On the other hand, in GIS we often consider the connectivity based on its support sets. According to this, both examples of fuzzy spatial objects in figure 2 are considered as connected in GIS. Based on this fact, the concept of connectivity in GIS must be defined based on the background topological space.

The background set $X$ also has its topology, therefore, we may let $\beta$ be a topology of $X$ and $(X, \beta)$ be this background topological space. Thus, we have two kinds of notations: (a) the fuzzy topological space $(I^X, \delta)$ or written as $(X, I^X, \delta)$; and (b) its background topological space $(X, \beta)$. These two topologies may not be related. But under certain assumptions, there are many nice results about their relations (Martin 1980, Luo 1988, Liu and Luo 1997). We denoted this topology by $(X, I^X, \delta, \beta)$.

Figure 2. Two examples of connected fuzzy spatial objects in $\mathbb{R}$.

Figure 3. Two examples of disconnected fuzzy spatial objects in $\mathbb{R}$.
Definition 4.1 (support of A): Support(A) or Supp(A) is equal to the set \( \{ x \in X : A(x) > 0 \} \). The closure of Supp(A) in background topology is denoted by \( \overline{\text{Supp}(A)} \).

Definition 4.2 (supported connected fuzzy set): Let \((X, I^X, \delta)\) be an I-fts and \((X, \beta)\) be its background topological space, \(A, B \in I^X\). \(A\) and \(B\) are called supported separated, if

\[
\overline{\text{Supp}(A)} \cap \text{Supp}(B) = \text{Supp}(A) \cap \overline{\text{Supp}(B)} = \emptyset.
\]

\(A\) is called supported connected in \((X, I^X, \delta, \beta)\), if there does not exist supported separated \(C, D \in I^X\{0\}\) such that \(A=C\cup D\) and \(\text{Supp}(A)=\text{Supp}(C)\cup \text{Supp}(D)\).

Definition 4.3 (Supported connected component): Let \((X, I^X, \delta)\) be an I-fts and \((X, \beta)\) be its background topological space, \(A \in I^X\). \(A\) is called a supported connected component of \((X, I^X, \delta, \beta)\), if \(A\) is a maximal supported connected subset in \((X, I^X, \delta, \beta)\); i.e. if \(B \in I^X\) is a supported connected component and \(B \supseteq A\), then \(B = A\).

Proposition 4.4: Let \((X, I^X, \delta)\) be an I-fts and \((X, \beta)\) be its background topological space, \(A \in I^X\). Every fuzzy point in \(A\) belongs to one and only one supported component of \(A\).

Proposition 4.5: Let \((X, I^X, \delta)\) be an I-fts and \((X, \beta)\) be its background topological space. Then, different supported components of \((X, I^X, \delta, \beta)\) are separated.

Theorem 4.6: Let \((X, I^X, \delta, \beta)\) and \((Y, I^Y, \mu, \gamma)\) be two I-fuzzy spaces, \(f : X \to Y\) an ordinary continuous mapping. If \(A \in I^X\) is a supported connected fuzzy set, then \(f^{-1}(A)\) is also a supported connected fuzzy set.

Proof: Suppose \(A \in I^X\) is a supported connected fuzzy set. Since \(f : X \to Y\) is continuous, \(f(\text{Supp}(A))\) is connected. The next step is to prove that \(f(\text{Supp}(A))=\text{Supp}(f^{-1}(A))\). If this is true, we can conclude that \(f^{-1}(A)\) is a supported connected fuzzy set. But by definition, \(f(\text{Supp}(A))=\{ f(x) : A(x) > 0 \}\) and \(f^{-1}(A)(y) = \begin{cases} \lor \{ A(x) \} & \text{if } x \in f^{-1}(y) \\ 0 & \text{otherwise} \end{cases}\).

Thus, \(\text{Supp}(f^{-1}(A))=\{ y = f(x) : \lor A(x) > 0 \} = \{ y = f(x) : A(x) > 0 \}\).

Q.E.D.

Remark 3: From Theorem 4.6, we can see that the supported connectivity of a fuzzy topological space \((X, I^X, \delta, \beta)\) does not depend on the topological space \((X, I^X, \delta)\), but only on its background topological space \((X, \beta)\). This makes it easy to model fuzzy topological relations in spatial querying and analysis. Moreover, the concept of connectivity in this paper is used for defining simple fuzzy region, line and line segment.

5. Modeling simple fuzzy objects in GIS

Within the framework of crisp object modeling in GIS, we first need to model simple objects before we can model the topological relations between the objects. The basic simple objects in GIS may include simple points, simple line segments,
and simple regions. Simple points, lines, and regions have been discussed widely (Egenhofer and Franzosa 1991, Clementini and Di Felice 1996). Similarly, within the framework of fuzzy object modeling in GIS, we first need to model fuzzy simple objects before we can model the fuzzy topological relations between the objects. Many researchers have been working on modeling fuzzy topological relations in GIS, for example, Tang and Kainz (2002) defines a simple fuzzy region based on fuzzy topology.

In order to give a generic framework on the number of topological relations between these simple spatial objects, simple fuzzy points, simple fuzzy line segments and simple fuzzy regions are defined as follows.

**Definition 5.1 (fuzzy point):** An I-fuzzy point on X is an I-fuzzy subset \( x_a \subseteq X \), defined as: 

\[
    x_a(y) = \begin{cases} 
        a & \text{if } y = x \\
        0 & \text{otherwise}
    \end{cases}
\]

(see figure 4).

**Definition 5.2 (Non-fuzzy line segment in X):** Let P and Q be two points in X. The line segment joining PQ is defined as the image of a map \( \alpha: [0, 1] \rightarrow X \) by \( \alpha(t) = P + t(Q - P) \), where \([0, 1] \) is a closed interval in \( R \).

**Definition 5.3 (Non-fuzzy line in X):** The line in X (or \( R^2 \)) can be described as an embedding of a connected interval from \( R \) to X (or \( R^2 \)), which does not have an intersection, i.e.:

\[
    \alpha : [0, 1] \rightarrow R^2 (or X),
\]

where \([0, 1]\) is a closed interval in \( R \) and \( \alpha(t_1) \neq \alpha(t_2) \) for all \( t_1 \neq t_2, t_1, t_2 \in [0, 1] \).

**Definition 5.4 (Simple fuzzy line segment):** The simple fuzzy line segment (L) is a fuzzy subset in X with

(i) for any \( \alpha \in (0, 1) \), the fuzzy line \( L_\alpha \) (the interior of fuzzy line segment L) is a supported connected line segment (i.e. a non-fuzzy line segment in the background topological space) in the background topological space and

(ii) \( \partial L = L_\alpha \setminus L^{1-\alpha} \) has at most two supported connected components.

**Definition 5.5 (Simple fuzzy line):** A fuzzy subset in X is called a simple fuzzy line (L) if L is a supported connected line in the background topological space (i.e. a non-fuzzy line in the background topological space).
Remark 4: Geometrically, GIS features can be classified as points, lines, and polygons. Actually, a simple fuzzy line segment is the basic element of a fuzzy line. Indeed, any fuzzy line can be represented by a composition of simple fuzzy line segments in GIS. Moreover, simple fuzzy points, simple fuzzy line segments, and simple fuzzy regions are defined and serve to model topological relations between spatial objects. Figure 5(a) shows a simple fuzzy line segment, \( \ell \). Indeed for any \( 0 < \alpha < 1 \), the interior set \( I_\alpha \) is a supported connected line segment and \( \partial \ell \) has at most two supported connected components (figure 5(b)). Figure 6(a) shows a non-simple fuzzy line segment, \( L \). Indeed, there exists \( \alpha \) such that the interior set \( L_\alpha \) is NOT a supported connected line segment, it has two supported connected components (Figure 6(b)).

**Definition 5.6 (Fuzzy region in X):** A fuzzy set \( A \) in \( X \) is called a fuzzy region if \( \text{supp}(A) \) has a non-empty interior in the background topology.

**Definition 5.7 (Simple fuzzy region):** A simple fuzzy region is a fuzzy region in \( X \) with

(i) for any \( \alpha \in (0, 1) \), the fuzzy sets \( A_\alpha \) and \( \partial A(=A_\alpha \cap A^{1-\alpha}) \) are two supported connected regular bounded open sets in the background topological space.

(ii) in the background topological space, any outward ray from \( \text{Supp}(A_\alpha) \) must meet \( \text{Supp}(\partial A) \) and have only one component.
Remark 5: Figure 7(a) shows a simple fuzzy region, since for any \( \alpha \in (0, 1) \), the fuzzy sets \( A_\alpha \) and \( \partial A = A_\alpha \cap A^{1-\alpha} \) are two supported connected, regular bounded open sets in the background topological space. Any outward ray from \( \text{Supp}(A_\alpha) \) must meet \( \text{Supp}(\partial A) \) and have only one component (see figure 7(b)).

Figure 8(a) shows a non-simple fuzzy region, since for some \( \alpha_0 \in (0, 1) \), the fuzzy set \( A_{\alpha_0} \) is not a supported connected set (which has two components) (see figure 8(b)).

Proposition 5.8: A simple fuzzy set (region or line) is supported by a connected fuzzy set.

Proof: If not, \( \text{Supp}(A) \) has two supported connected components. Let \( x_1 \in \text{Supp}(A_1) \) and \( x_2 \in \text{Supp}(A_2) \) be these two supported connected components. Take \( \alpha_0 = \frac{1}{2} \min \{ A(x_1), A(x_2) \} \); then, \( \text{Supp}(A_{\alpha_0}) \) has two supported connected components.

Q.E.D.

Proposition 5.9: Let \((X, I^X, \delta, \beta)\) and \((Y, I^Y, \mu, \gamma)\) be two I-fuzzy spaces, \( f : X \to Y \) an ordinary continuous mapping. If \( A \in I^X \) is a simple fuzzy set, then \( f^{-1}(A) \) is also a simple fuzzy set.

6. Quantitative fuzzy topological relations between simple fuzzy regions

Based on the definitions of simple fuzzy regions in section 5, and a further definition of quantitative method in this study, we now provide a method for determining fuzzy topological relations between spatial objects. The framework is based on the
9-intersection model; however, the method for computing the quantitative fuzzy topological relations between spatial objects has been newly developed in this study. The framework of the fuzzy topological relations between two objects $A$ and $B$ is defined as follows:

$$
\begin{pmatrix}
\int_X A \cap B \, dV \\
\int_X A \cap \partial B \, dV \\
\int_X A \cap (B^c) \, dV \\
\int_X A \cup B \, dV \\
\int_X A \cup \partial B \, dV \\
\int_X A \cup (B^c) \, dV
\end{pmatrix}
$$

First, we should explain the meaning of the notations $\int_X A \cap B \, dV$ and $\int_X A \cap \partial B \, dV$, etc., which define the quantitative relations between spatial objects.

**Definition 6.1:** For any two fuzzy spatial objects (fuzzy sets) $A, B \in \mathcal{I}^X$, defined

$$
\int_X A \cap B \, dV = \frac{\int_X (A \cap B)(x) \, dx}{\int_X (A \cup B)(x) \, dx}
$$

The geometric meaning of $\int_X A \cap B \, dV$, illustrated in figure 9, is the ratio of the area (or volume) of the meet of two fuzzy spatial objects to the join of two fuzzy spatial objects. Here, join of two fuzzy objects means 'union' of two fuzzy objects. In previously studying the topological relations between spatial objects, researchers (Egenhofer and Franzosa 1991, Clementini and Di Felice 1996, Cohn and Gotts 1996, Smith 1996, Shi and Guo 1999, Tang and Kainz 2002) used the intersection of sets to give quantitative relations. Meet of two fuzzy objects means 'intersection' of two fuzzy objects. Here, we use volume ratio (the volume of the intersection to the volume of the meet of two fuzzy spatial objects, see figure 9) to describe the quantitative value of two fuzzy spatial objects. Obviously, the former models can only give a local quantitative value (or Boolean value) at different points, while the model in this paper can give a global quantitative value to each spatial object.

Since for any spatial object $A \in \mathcal{I}^X$, the three components, interior ($A_x$), boundary ($\partial A$), and exterior (($A^c)_x$) are not disjoint (see figure 9), there is double counting of
the integration. For example, \( \int_X A \wedge B \, dV \) and \( \int_X A' \wedge \partial A \, dV \) will be double counted on the part \( \partial A \wedge B \) and \( \partial B \wedge A \). But we have the bound of the overlapping part; that is, for any \( \alpha \), \( (A \wedge \partial A)(x) < 1 - \alpha \) and \( ((A')_d \wedge \partial A)(x) < 1 - \alpha \); and for \( \alpha \geq \frac{1}{2} \), \( A_d \wedge (A')_d = \phi \), respectively (Liu and Shi, submitted). This means that we can control the size of the overlapping between the interior to the boundary and the exterior to the boundary by choosing a large value for \( \alpha \). We can see that if the value of \( \alpha \) is very close to zero, the uncertainty is very large, while if the value of \( \alpha \) is very close to one, the uncertainty is very small. Figure 10 shows how the sizes of \( \text{Supp}(A \wedge \partial A) \) and \( \text{Supp}((A')_d \wedge \partial A) \) change. That is, when a smaller \( \alpha \) is chosen, larger values of \( A \wedge \partial A \) and \( (A')_d \wedge \partial A \) are obtained; when a larger \( \beta \) is chosen, smaller values of \( A' \wedge \partial A \) and \( (A')_d \wedge \partial A \) are obtained. Moreover, if the topology is discrete, for a suitable \( \alpha \), the uncertainty of spatial objects can be controlled to zero. Indeed, an \( \alpha \) is chosen such that \( A_d \wedge \partial A = 0 \) and \( (A')_d \wedge \partial A = 0 \). Therefore, in modeling topological relations between simple spatial regions, we may neglect the effect of \( A \wedge \partial A \) and \( (A')_d \wedge \partial A \).

### 6.1 Identification by a 3 \times 3 integration matrix

By directly using the value of zero and non-zero, which are topological invariants, the 3 \times 3 integration matrix

\[
\begin{pmatrix}
\int_X A \wedge B \, dV & \int_X \partial A \wedge B \, dV & \int_X (A')_d \wedge B \, dV \\
\int_X A \wedge \partial B \, dV & \int_X \partial A \wedge \partial B \, dV & \int_X (A')_d \wedge \partial B \, dV \\
\int_X A \wedge (B')_d \, dV & \int_X \partial A \wedge (B')_d \, dV & \int_X (A')_d \wedge (B')_d \, dV
\end{pmatrix}
\]

gives a total of \( 2^9 = 512 \) different cases of topological relations between two simple fuzzy regions. However, for a simple fuzzy region in \( \mathbb{R}^2 \), it is not possible for all of these topological relations to occur.
Induced fuzzy topology

(i) Let A and B be two simple fuzzy regions in $\mathbb{R}^2$. The $3 \times 3$ integration matrix will then satisfy the following conditions. Due to the fact that simple fuzzy regions are bounded, $\int_X (A^c)^2 \wedge (B^c)^2 \, dV$ is non-zero for all cases.

(ii) Each part of A ($A_x$, $\partial A$) and $(A^c)_x$ must intersect with at least one part of B ($B_x$, $\partial B$ and $(B^c)_x$), and vice versa.

(iii) If $\int_X A_x \wedge B_x \, dV$ and $\int_X A_x \wedge (B^c)_x \, dV$ are non-zero, then $\int_X A_x \wedge \partial B \, dV$ must be non-zero, and vice versa.

(iv) If $\int_X \partial A \wedge \partial B \, dV$ is zero, then either $\int_X (A^c) \wedge \partial B \, dV$ or $\int_X \partial A \wedge (B^c)_x \, dV$ is non-zero.

(v) If both $\int_X A_x \wedge B_x \, dV$ and $\int_X \partial A \wedge B_x \, dV$ are non-zero, then $\int_X \partial A \wedge \partial B \, dV$ must be non-zero.

(vi) If $\int_X A_x \wedge (B^c)_x \, dV$ is non-zero, then $\int_X \partial A \wedge (B^c)_x \, dV$ must be non-zero, and vice versa.

(vii) If $\int_X A_x \wedge B_x \, dV$ is zero and $\int_X \partial A \wedge B_x \, dV$ is non-zero, then $\int_X \partial A \wedge \partial B \, dV$ must be non-zero, and vice versa.

(viii) If $\int_X A_x \wedge B_x \, dV$ and $\int_X A_x \wedge \partial B \, dV$ are non-zero, then $\int_X \partial A \wedge B_x \, dV$ and $\int_X \partial A \wedge \partial B \, dV$ are non-zero, and vice versa.

(ix) If $\int_X A_x \wedge B_x \, dV$ and $\int_X A_x \wedge \partial B \, dV$ are zero, then $\int_X A_x \wedge (B^c)_x \, dV$ and $\int_X \partial A \wedge (B^c)_x \, dV$ are non-zero, and vice versa.

(x) If $\int_X \partial A \wedge B_x \, dV$ and $\int_X \partial A \wedge (B^c)_x \, dV$ are non-zero, then $\int_X \partial A \wedge \partial B \, dV$ must be non-zero, and vice versa.

(xi) If $\int_X A_x \wedge B_x \, dV$ and $\int_X \partial A \wedge B_x \, dV$ are non-zero, then $\int_X (A^c) \wedge B_x \, dV$ is non-zero, and vice versa.

Based on the above conditions, 44 relations between simple fuzzy regions in $\mathbb{R}^2$ were identified by using the $3 \times 3$ integration matrix. This result is similar to the two results from previous studies (Clementini and Di Felice 1996, Tang and Kainz 2002). However, the method used here is totally different. In fact, the method proposed in this study is a generalization of the previous models. This will be discussed in detail in the next section. The 44 relations between two simple fuzzy regions are listed in Appendix I, which includes the value of the matrix, the support view, and membership value of the fuzzy topological relations between two simple fuzzy regions.

The topological relations among simple fuzzy regions with simple line segments, simple fuzzy regions to fuzzy points, simple fuzzy line segments to simple fuzzy line segments and simple fuzzy line segments to fuzzy points will be investigated and reported in a separate paper, because of limitations on the length of this paper.

7. A comparison with the existing models

In dealing with fuzzy spatial objects, Cohn and Gotts (1996) proposed the egg-yolk model and suggested using two concentric sub-regions, indicating the degree of ‘membership’ in a vague/fuzzy region, where ‘yolk’ represents the precise part and ‘egg’ represents the vague/fuzzy part of the region. Based on Region Connection Calculus (RCC) theory (Randell et al. 1992), eight basic relations can be defined. They are: DC (Disconnected), EC (Externally Connected), PO (Partially Overlapping), TPP (Tangential Proper Part), NTTP (Non-tangential Proper Part), EQ (Equal), PPI (Proper Part Inverse), and TPPI (Tangential Proper Part Inverse), respectively (see table 1).

The egg-yolk model is an extension of the RCC theory into the vague/fuzzy region. A total of 46 relations can be identified (Cohn and Gotts 1996).
In dealing with spatial objects with indeterminate boundaries, based on Egenhofer’s nine-intersection model and Di Felice (1996) defined a region with a broad boundary, by using two simple regions. This broad boundary is denoted by $\Delta A$. More precisely, the broad boundary is a simple connected subset of $\mathbb{R}^2$ with a hole. The shaded region in figure 11 is region $A$ with a broad boundary. Based on the empty and non-empty invariance, Clementini and Di Felice’s Algebraic model, gave a total of 44 relations between two spatial regions with a broad boundary.

To investigate the topological relations between fuzzy regions, Tang and Kainz (2002) decomposed a fuzzy set $A$ into several topological parts, as follows:

(i) the core, $A^\circ$, which is the subset of the closure fuzzy set $A$ with $(A^\circ \setminus A^c)(x) = 0$, for all $x \in X$;

![Figure 11. A region A with a broad boundary.](image)
(ii) the fringe, $\ell A$, which is the subset of the closure fuzzy set $A$ with
$(A^\infty \cap A^\varepsilon)(x) > 0$, for all $x \in X$;

(iii) the outer, $A^\circ$, the complement of the support of the closure of fuzzy set $A$.

By using the nine-intersection matrix,

$$
\begin{pmatrix}
A^\circ \cap B^\circ & A^\circ \cap \ell B & A^\circ \cap B^= \\
\ell A \cap B^\circ & \ell A \cap \ell B & \ell A \cap B^=
\end{pmatrix}
$$

are a total of 44 relations between two simply fuzzy regions.

Different from the general topology, when decomposing a fuzzy set into interior, boundary, and exterior, the intersection of two (interior and boundary or boundary and exterior or interior and exterior) may not be empty (Liu and Shi, submitted). Actually, Tang (2004) considered this by introducing more topological invariants. Here, we created a computational fuzzy topology and calculated the intersecting values and obtained bounds which are $(A_\varepsilon \cap \partial A)(x) < 1 - \alpha$ and $(A^\infty \cap \partial A)(x) < 1 - \alpha$. The existing models did not take this fact into consideration, which may lead to unexpected effects in modeling topological relations between spatial objects. In our research, we not only considered this factor, but also gave a significant bound on the overlapping parts that actually can be controlled by varying the level cutting.

Furthermore, the relations between objects have been quantified based on the ratio between integrations, which varies between 0 and 1. The advantage of this method not only provides the existence of the intersection between two parts, but also provides a quantitative value for this intersection. This is a step beyond the existing methods, which can only provide topological relations by giving a value of 1 (with intersection) or 0 (without intersection).

With a different method and based on the $3 \times 3$ integration matrix, there are 44 relations between the simple fuzzy regions in $\mathbb{R}^2$. This result agrees with two previous results (Clementini and Di Felice 1996, Tang and Kainz 2002). Other new findings are on the number of topological relations, there are 16 relations between simple fuzzy region and simple fuzzy line segment in $\mathbb{R}^2$; 46 topological relations between two simple fuzzy line segments; three relations between simple fuzzy region and fuzzy point in $\mathbb{R}^2$ and three relations between simple fuzzy line segment and fuzzy point in $\mathbb{R}^2$.

Table 2 is a summary of the number of relations identified based on the existing models and our newly developed models in this study.

From this table, we can see that the number of topological relations between simple fuzzy regions from this study is similar to the results from previous studies (Clementini and Di Felice 1996, Tang and Kainz 2002). However, the method used

<table>
<thead>
<tr>
<th>Authors</th>
<th>Region–Region</th>
<th>Region–Line</th>
<th>Region–Point</th>
<th>Line–Line</th>
<th>Line–Point</th>
</tr>
</thead>
<tbody>
<tr>
<td>Our model</td>
<td>44</td>
<td>16</td>
<td>3</td>
<td>46</td>
<td>3</td>
</tr>
<tr>
<td>Tang and Kainz’s model</td>
<td>44</td>
<td>30</td>
<td>3</td>
<td>97</td>
<td>3</td>
</tr>
<tr>
<td>Cohn and Gott’s model</td>
<td>46</td>
<td>Nil</td>
<td>Nil</td>
<td>Nil</td>
<td>Nil</td>
</tr>
<tr>
<td>Clementini and Di Felice’ model</td>
<td>44</td>
<td>Nil</td>
<td>Nil</td>
<td>Nil</td>
<td>Nil</td>
</tr>
</tbody>
</table>
in this study is different from earlier studies. Furthermore, the findings on regions to
line and line to line relations are different from Tang’s model. This is due to the
different definition of fuzzy line.

8. Conclusion and discussion

Fuzzy topological relations are elementary relations in studying the topological
relations between spatial objects in GIS, especially for uncertain spatial objects in
GIS. In this paper, we presented a study on developing methods for computing the
fuzzy topological relations of spatial objects, based on the recently developed
computational fuzzy topology. Our contributions include the following: (a)
proposing the homeomorphic invariants of the fuzzy topology; (b) defining the
connectivity based on the newly developed fuzzy topology; (c) modeling the fuzzy
topological relations between simple fuzzy regions in GIS; (d) introducing the
concepts of the intersection of the boundary and interior, the boundary and the
exterior; and (e) obtaining an upper bound of these two intersections.

The preserving properties and the connectivity of the newly developed fuzzy
topology, based on which the topological relations are invariants under home-
omorphic mappings, were studied. With such a development, we can guarantee the
unchanged properties in a GIS transformation, such as the maintenance of
topological consistency in transferring a map from one system to another.

Since the intersection of the interior and boundary or boundary and exterior or
interior and exterior may not be empty, there is double counting of the integration.
In this paper, we have carefully studied the relations among the interior, boundary
and exterior and hence have achieved an approximation of these overlapping parts.
That is, for any \( z \) the values of the intersection of the interior and boundary or
boundary and exterior are always bounded by \( 1 - z \), where \( z \) is the value of the level
cutting. Based on a previous finding that the intersection of interior and exterior is
empty when the value of the level cutting is greater than 0.5 (Liu and Shi,
submitted), we can control the uncertainty (the size of the overlap) between the
interior and exterior to the boundary by controlling the value of \( z \). This is a new
finding which has never been mentioned in any previous studies on topological
relations. Moreover, the value of \( z \) provides an accurate control of the modeling of
topological relations between spatial objects in GIS.

For computing the topological relations between spatial objects, the intersection
concepts and the integration method are applied, and a computational 9-intersection
model is thus developed. The computational topological relations between spatial
objects are defined based on the ratio of the area/volume of the meet to the join of
two fuzzy spatial objects. This is a step ahead of the existing topological relation
models: from the conceptual definition of topological relations to the computable
definition of topological relations. As a result, the quantitative value of topological
relations can be calculated. With a different method and based on the \( 3 \times 3 \)
integration matrix, there are 44 relations between the simple fuzzy regions in \( \mathbb{R}^2 \).
This result agrees with two previous results (Clementini and Di Felice 1996, Tang
and Kainz 2002).

Besides the above theoretical developments, there were a number of new findings
on spatial relations in this study, namely: (a) there is no intersection between interior
and exterior; (b) the upper bound of the intersections between interior and
boundary, boundary and exterior is \( 1 - z \); and (c) there are a total of 44 topological
relations between simple fuzzy regions.
Further research based on this work will focus on investigating the quantitative topological relations among simple fuzzy regions and simple line segments, simple fuzzy regions and fuzzy points, simple fuzzy line segments and simple fuzzy line segments, and simple fuzzy line segments and fuzzy points.

Acknowledgments
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References


LIU, K.F. and Shi, W.Z., submitted, A fuzzy topology for computing the interior, boundary, and exterior of spatial objects quantitatively in GIS.


Appendix 1

Table 3. The 44 relations between two simple fuzzy regions in $\mathbb{R}^2$.

<table>
<thead>
<tr>
<th>Matrix</th>
<th>Illustration support view</th>
<th>Illustration in membership value</th>
</tr>
</thead>
</table>
| 1. \[
\begin{pmatrix}
\phi & \phi & \sim \phi \\
\phi & \sim \phi & \sim \phi \\
\sim \phi & \sim \phi & \sim \phi
\end{pmatrix}
\] | ![Illustration support view](image1.png) | ![Illustration in membership value](image2.png) |
| 2. \[
\begin{pmatrix}
\phi & \sim \phi & \sim \phi \\
\sim \phi & \sim \phi & \sim \phi
\end{pmatrix}
\] | ![Illustration support view](image3.png) | ![Illustration in membership value](image4.png) |
| 3. \[
\begin{pmatrix}
\phi & \sim \phi & \sim \phi \\
\sim \phi & \sim \phi & \sim \phi
\end{pmatrix}
\] | ![Illustration support view](image5.png) | ![Illustration in membership value](image6.png) |
| 4. \[
\begin{pmatrix}
\phi & \sim \phi & \phi \\
\sim \phi & \sim \phi & \sim \phi
\end{pmatrix}
\] | ![Illustration support view](image7.png) | ![Illustration in membership value](image8.png) |
| 5. \[
\begin{pmatrix}
\phi & \sim \phi & \phi \\
\sim \phi & \sim \phi & \phi
\end{pmatrix}
\] | ![Illustration support view](image9.png) | ![Illustration in membership value](image10.png) |
| 6. \[
\begin{pmatrix}
\sim \phi & \sim \phi & \sim \phi \\
\sim \phi & \sim \phi & \sim \phi
\end{pmatrix}
\] | ![Illustration support view](image11.png) | ![Illustration in membership value](image12.png) |
| 7. \[
\begin{pmatrix}
\sim \phi & \sim \phi & \phi \\
\phi & \sim \phi & \sim \phi
\end{pmatrix}
\] | ![Illustration support view](image13.png) | ![Illustration in membership value](image14.png) |
Table 3. (Continued)

<table>
<thead>
<tr>
<th>Matrix</th>
<th>Illustration support view</th>
<th>Illustration in membership value</th>
</tr>
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<tbody>
<tr>
<td>8. ( \begin{pmatrix} \sim \phi &amp; \sim \phi &amp; \phi \ \phi &amp; \sim \phi &amp; \sim \phi \ \phi &amp; \phi &amp; \sim \phi \end{pmatrix} )</td>
<td><img src="image1" alt="Diagram" /></td>
<td><img src="image2" alt="Diagram" /></td>
</tr>
<tr>
<td>9. ( \begin{pmatrix} \sim \phi &amp; \phi &amp; \phi \ \phi &amp; \sim \phi &amp; \phi \ \phi &amp; \phi &amp; \sim \phi \end{pmatrix} )</td>
<td><img src="image3" alt="Diagram" /></td>
<td><img src="image4" alt="Diagram" /></td>
</tr>
<tr>
<td>10. ( \begin{pmatrix} \phi &amp; \phi &amp; \sim \phi \ \phi &amp; \sim \phi &amp; \phi \ \phi &amp; \phi &amp; \sim \phi \end{pmatrix} )</td>
<td><img src="image5" alt="Diagram" /></td>
<td><img src="image6" alt="Diagram" /></td>
</tr>
<tr>
<td>11. ( \begin{pmatrix} \phi &amp; \phi &amp; \sim \phi \ \phi &amp; \sim \phi &amp; \sim \phi \ \phi &amp; \phi &amp; \sim \phi \end{pmatrix} )</td>
<td><img src="image7" alt="Diagram" /></td>
<td><img src="image8" alt="Diagram" /></td>
</tr>
<tr>
<td>12. ( \begin{pmatrix} \sim \phi &amp; \sim \phi &amp; \phi \ \sim \phi &amp; \phi &amp; \phi \ \sim \phi &amp; \phi &amp; \phi \end{pmatrix} )</td>
<td><img src="image9" alt="Diagram" /></td>
<td><img src="image10" alt="Diagram" /></td>
</tr>
<tr>
<td>13. ( \begin{pmatrix} \phi &amp; \phi &amp; \phi \ \phi &amp; \phi &amp; \phi \ \phi &amp; \phi &amp; \phi \end{pmatrix} )</td>
<td><img src="image11" alt="Diagram" /></td>
<td><img src="image12" alt="Diagram" /></td>
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<tr>
<td>14. ( \begin{pmatrix} \phi &amp; \sim \phi &amp; \phi \ \phi &amp; \sim \phi &amp; \phi \ \phi &amp; \sim \phi &amp; \phi \end{pmatrix} )</td>
<td><img src="image13" alt="Diagram" /></td>
<td><img src="image14" alt="Diagram" /></td>
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Table 3. (Continued)

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<tr>
<th>Matrix</th>
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<th>Illustration in membership value</th>
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<tbody>
<tr>
<td>15. ( \begin{array}{ccc} \phi &amp; \phi &amp; \sim \phi \ \sim \phi &amp; \sim \phi &amp; \sim \phi \ \phi &amp; \phi &amp; \sim \phi \end{array} )</td>
<td>![Image 1]</td>
<td>![Image 2]</td>
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<tr>
<td>16. ( \begin{array}{ccc} \phi &amp; \sim \phi &amp; \phi \ \phi &amp; \sim \phi &amp; \phi \ \sim \phi &amp; \sim \phi &amp; \phi \end{array} )</td>
<td>![Image 3]</td>
<td>![Image 4]</td>
</tr>
<tr>
<td>17. ( \begin{array}{ccc} \phi &amp; \sim \phi &amp; \sim \phi \ \sim \phi &amp; \sim \phi &amp; \sim \phi \ \phi &amp; \sim \phi &amp; \sim \phi \end{array} )</td>
<td>![Image 5]</td>
<td>![Image 6]</td>
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<tr>
<td>18. ( \begin{array}{ccc} \phi &amp; \sim \phi &amp; \sim \phi \ \sim \phi &amp; \sim \phi &amp; \sim \phi \ \sim \phi &amp; \sim \phi &amp; \sim \phi \end{array} )</td>
<td>![Image 7]</td>
<td>![Image 8]</td>
</tr>
<tr>
<td>19. ( \begin{array}{ccc} \phi &amp; \sim \phi &amp; \sim \phi \ \sim \phi &amp; \sim \phi &amp; \sim \phi \ \phi &amp; \sim \phi &amp; \sim \phi \end{array} )</td>
<td>![Image 9]</td>
<td>![Image 10]</td>
</tr>
<tr>
<td>20. ( \begin{array}{ccc} \phi &amp; ~ \phi &amp; ~ \phi \ ~ \phi &amp; ~ \phi &amp; ~ \phi \ ~ \phi &amp; ~ \phi &amp; ~ \phi \end{array} )</td>
<td>![Image 11]</td>
<td>![Image 12]</td>
</tr>
<tr>
<td>21. ( \begin{array}{ccc} ~ \phi &amp; ~ \phi &amp; ~ \phi \ ~ \phi &amp; ~ \phi &amp; ~ \phi \ ~ \phi &amp; ~ \phi &amp; ~ \phi \end{array} )</td>
<td>![Image 13]</td>
<td>![Image 14]</td>
</tr>
</tbody>
</table>
Table 3. (Continued)

<table>
<thead>
<tr>
<th>Matrix</th>
<th>Illustration support view</th>
<th>Illustration in membership value</th>
</tr>
</thead>
<tbody>
<tr>
<td>22. $\begin{pmatrix} \phi &amp; \sim \phi &amp; \sim \phi \ \sim \phi &amp; \sim \phi &amp; \sim \phi \ \phi &amp; \phi &amp; \sim \phi \end{pmatrix}$</td>
<td><img src="image" alt="Illustration 22" /></td>
<td><img src="image" alt="Illustration in membership value for 22" /></td>
</tr>
<tr>
<td>23. $\begin{pmatrix} \sim \phi &amp; \sim \phi &amp; \sim \phi \ \sim \phi &amp; \sim \phi &amp; \sim \phi \ \phi &amp; \phi &amp; \sim \phi \end{pmatrix}$</td>
<td><img src="image" alt="Illustration 23" /></td>
<td><img src="image" alt="Illustration in membership value for 23" /></td>
</tr>
<tr>
<td>24. $\begin{pmatrix} \sim \phi &amp; \sim \phi &amp; \phi \ \sim \phi &amp; \sim \phi &amp; \sim \phi \ \phi &amp; \phi &amp; \sim \phi \end{pmatrix}$</td>
<td><img src="image" alt="Illustration 24" /></td>
<td><img src="image" alt="Illustration in membership value for 24" /></td>
</tr>
<tr>
<td>25. $\begin{pmatrix} \sim \phi &amp; \sim \phi &amp; \sim \phi \ \sim \phi &amp; \sim \phi &amp; \sim \phi \ \phi &amp; \phi &amp; \sim \phi \end{pmatrix}$</td>
<td><img src="image" alt="Illustration 25" /></td>
<td><img src="image" alt="Illustration in membership value for 25" /></td>
</tr>
<tr>
<td>26. $\begin{pmatrix} \sim \phi &amp; \sim \phi &amp; \phi \ \sim \phi &amp; \sim \phi &amp; \sim \phi \ \phi &amp; \phi &amp; \sim \phi \end{pmatrix}$</td>
<td><img src="image" alt="Illustration 26" /></td>
<td><img src="image" alt="Illustration in membership value for 26" /></td>
</tr>
<tr>
<td>27. $\begin{pmatrix} \sim \phi &amp; \sim \phi &amp; \phi \ \sim \phi &amp; \sim \phi &amp; \sim \phi \ \phi &amp; \phi &amp; \sim \phi \end{pmatrix}$</td>
<td><img src="image" alt="Illustration 27" /></td>
<td><img src="image" alt="Illustration in membership value for 27" /></td>
</tr>
<tr>
<td>28. $\begin{pmatrix} \sim \phi &amp; \sim \phi &amp; \phi \ \sim \phi &amp; \sim \phi &amp; \sim \phi \ \phi &amp; \phi &amp; \sim \phi \end{pmatrix}$</td>
<td><img src="image" alt="Illustration 28" /></td>
<td><img src="image" alt="Illustration in membership value for 28" /></td>
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</table>
Table 3. (Continued)

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<tr>
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</tr>
</thead>
<tbody>
<tr>
<td>29. (\begin{pmatrix} \sim \phi &amp; \sim \phi &amp; \sim \phi \ \phi &amp; \sim \phi &amp; \sim \phi \ \phi &amp; \sim \phi &amp; \sim \phi \end{pmatrix})</td>
<td><img src="image" alt="Illustration support view" /></td>
<td><img src="image" alt="Illustration in membership value" /></td>
</tr>
<tr>
<td>30. (\begin{pmatrix} \sim \phi &amp; \sim \phi &amp; \sim \phi \ \sim \phi &amp; \sim \phi &amp; \sim \phi \ \sim \phi &amp; \sim \phi &amp; \sim \phi \end{pmatrix})</td>
<td><img src="image" alt="Illustration support view" /></td>
<td><img src="image" alt="Illustration in membership value" /></td>
</tr>
<tr>
<td>31. (\begin{pmatrix} \sim \phi &amp; \sim \phi &amp; \sim \phi \ \phi &amp; \sim \phi &amp; \sim \phi \ \phi &amp; \sim \phi &amp; \sim \phi \end{pmatrix})</td>
<td><img src="image" alt="Illustration support view" /></td>
<td><img src="image" alt="Illustration in membership value" /></td>
</tr>
<tr>
<td>32. (\begin{pmatrix} \sim \phi &amp; \sim \phi &amp; \sim \phi \ \sim \phi &amp; \sim \phi &amp; \sim \phi \ \sim \phi &amp; \sim \phi &amp; \sim \phi \end{pmatrix})</td>
<td><img src="image" alt="Illustration support view" /></td>
<td><img src="image" alt="Illustration in membership value" /></td>
</tr>
<tr>
<td>33. (\begin{pmatrix} \sim \phi &amp; \sim \phi &amp; \sim \phi \ \phi &amp; \sim \phi &amp; \sim \phi \ \phi &amp; \sim \phi &amp; \sim \phi \end{pmatrix})</td>
<td><img src="image" alt="Illustration support view" /></td>
<td><img src="image" alt="Illustration in membership value" /></td>
</tr>
<tr>
<td>34. (\begin{pmatrix} \sim \phi &amp; \sim \phi &amp; \sim \phi \ \phi &amp; \sim \phi &amp; \sim \phi \ \phi &amp; \sim \phi &amp; \sim \phi \end{pmatrix})</td>
<td><img src="image" alt="Illustration support view" /></td>
<td><img src="image" alt="Illustration in membership value" /></td>
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<tr>
<td>35. (\begin{pmatrix} \sim \phi &amp; \sim \phi &amp; \sim \phi \ \phi &amp; \sim \phi &amp; \sim \phi \ \phi &amp; \sim \phi &amp; \sim \phi \end{pmatrix})</td>
<td><img src="image" alt="Illustration support view" /></td>
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<td><img src="image1" alt="Illustration support view" /></td>
<td><img src="image2" alt="Illustration in membership value" /></td>
</tr>
<tr>
<td>$\begin{pmatrix} \sim \phi &amp; \sim \phi &amp; \sim \phi \ \sim \phi &amp; \sim \phi &amp; \sim \phi \ \phi &amp; \sim \phi &amp; \sim \phi \end{pmatrix}$</td>
<td><img src="image3" alt="Illustration support view" /></td>
<td><img src="image4" alt="Illustration in membership value" /></td>
</tr>
<tr>
<td>$\begin{pmatrix} \sim \phi &amp; \sim \phi &amp; \sim \phi \ \sim \phi &amp; \sim \phi &amp; \sim \phi \ \phi &amp; \sim \phi &amp; \sim \phi \end{pmatrix}$</td>
<td><img src="image5" alt="Illustration support view" /></td>
<td><img src="image6" alt="Illustration in membership value" /></td>
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<tr>
<td>$\begin{pmatrix} \sim \phi &amp; \sim \phi &amp; \sim \phi \ \sim \phi &amp; \sim \phi &amp; \sim \phi \ \phi &amp; \sim \phi &amp; \sim \phi \end{pmatrix}$</td>
<td><img src="image7" alt="Illustration support view" /></td>
<td><img src="image8" alt="Illustration in membership value" /></td>
</tr>
<tr>
<td>$\begin{pmatrix} \sim \phi &amp; \sim \phi &amp; \sim \phi \ \sim \phi &amp; \sim \phi &amp; \sim \phi \ \phi &amp; \sim \phi &amp; \sim \phi \end{pmatrix}$</td>
<td><img src="image9" alt="Illustration support view" /></td>
<td><img src="image10" alt="Illustration in membership value" /></td>
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<tr>
<td>$\begin{pmatrix} \sim \phi &amp; \sim \phi &amp; \sim \phi \ \sim \phi &amp; \sim \phi &amp; \sim \phi \ \phi &amp; \sim \phi &amp; \sim \phi \end{pmatrix}$</td>
<td><img src="image11" alt="Illustration support view" /></td>
<td><img src="image12" alt="Illustration in membership value" /></td>
</tr>
<tr>
<td>$\begin{pmatrix} \sim \phi &amp; \sim \phi &amp; \sim \phi \ \sim \phi &amp; \sim \phi &amp; \sim \phi \ \phi &amp; \sim \phi &amp; \sim \phi \end{pmatrix}$</td>
<td><img src="image13" alt="Illustration support view" /></td>
<td><img src="image14" alt="Illustration in membership value" /></td>
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<tr>
<td>$\begin{pmatrix} \sim \phi &amp; \sim \phi &amp; \sim \phi \ \sim \phi &amp; \sim \phi &amp; \sim \phi \ \phi &amp; \sim \phi &amp; \sim \phi \end{pmatrix}$</td>
<td><img src="image15" alt="Illustration support view" /></td>
<td><img src="image16" alt="Illustration in membership value" /></td>
</tr>
</tbody>
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Appendix 2: The proof on being an induced fuzzy topology

Both interior and closure operators can actually define a fuzzy topology respectively (Liu and Luo 1997). Furthermore, we can prove these two fuzzy topological spaces are actually coherent with each other. That is to say, they define the same fuzzy topological space. The following is the proof.

**Definition 1 (Interior and closure operators):** Let \( A \) be a fuzzy set in \([0, 1]^x_1\). For any fixed \( x \in [0, 1] \), define the interior and closure operators on \([0, 1]^x_1\) as

\[
A \rightarrow A^x_1 \quad \text{and} \quad A^x_1 \rightarrow A^x_2
\]

respectively, where the fuzzy sets \( A^x_1 \) and \( A^x_2 \) in \( X \) are defined by:

\[
A^x_1(x) = \begin{cases} 
A(x) & \text{if } A(x) > x \\
0 & \text{if } A(x) \leq x 
\end{cases}
\]

\[
A^x_2(x) = \begin{cases} 
1 & \text{if } A(x) \geq x \\
A(x) & \text{if } A(x) < x 
\end{cases}
\]

**Proposition 2:** Let \( A, B, \) and \( A_i \) \((i \in \Lambda)\) be fuzzy sets of \( I^x_1 \). Then the following holds for all \( x \in [0, 1] \):

(i) \( 0_x = 0 = 0_x \) and \( 1_x = 1 = 1_x \);

(ii) \( A \leq B \Rightarrow A^x \leq B^x \) and \( A^x \leq B^x \);

(iii) \( A \leq B \Rightarrow (A^x)^c = (A^c)^{1-x} \) and \( (A^x)^c = (A^c)^{1-x} \);

(iv) \( A^x \leq A^x \) and \( (A^x)^c = (A^c)^{1-x} \);

(v) \( A \leq B \Rightarrow A^x \leq A^x \) and \( (A^x)^c = (A^c)^{1-x} \);

(vi) \( x \leq \beta \Rightarrow A^x \leq A^x \) and \( (A^x)^c = (A^c)^{1-x} \);

(vii) \( x \leq \beta \Rightarrow A^x \leq A^x \) and \( (A^x)^c = (A^c)^{1-x} \);

(viii) If \( \Lambda \) is finite, then \( \bigvee_{i \in \Lambda} A_i = \bigvee_{i \in \Lambda} A_i^x \) and \( \bigwedge_{i \in \Lambda} A_i = \bigwedge_{i \in \Lambda} A_i^x \);

(ix) \( \bigwedge_{i \in \Lambda} A_i^x = \bigwedge_{i \in \Lambda} A_i^x \) and \( \bigvee_{i \in \Lambda} A_i^x = \bigvee_{i \in \Lambda} A_i^x \);

(x) \( A^x \leq A \leq A^1 \).

**Proposition 3:** The mappings \( \bar{x} \) and \( \bar{x} \) are the interior and closure operators, respectively.

**Remark A:** By this proposition, for any fuzzy set \( A \in I^x_1 \), \( A \in I^x_1 \) is closed if and only if \( \bar{x}(A) = A \). That is, the fuzzy topology induced by \( \bar{x} \) is the collection of \( \tau_{\bar{x}} = \{ A^x : A \in I^x_1 \} \). On the other hand, \( A \in I^x_1 \) is open if and only if \( \bar{x}(A) = A \). That is, the fuzzy topology induced by \( \bar{x} \) is the collection of \( \tau_{\bar{x}} = \{ A^x : A \in I^x_1 \} \). According to Liu and Luo (1997), the family of all the fuzzy topologies on \( X \) is one-one corresponding with the family of all interior and closure operators, respectively. However, these two operators only define two fuzzy topologies separately, and they may not define a coherent topological space.
Remark B: The results in proposition 2 allow us to define a new fuzzy topological space. Indeed, a fuzzy topological space $(\Gamma, \delta)$ on $X$ satisfies the conditions (a) $0, 1 \in \delta$; (b) if $A, B \in \delta$, then $A \cup B \in \delta$, (c) let $\{A_i: i \in J\} \subseteq \delta$, where $J$ is an index set, then $\bigvee_{i \in J} A_i \in \delta$. The elements in $\delta$ are called open elements and the elements in the complement of $T$ are called closed elements. The result in proposition 2(i) allows (a) to be defined, proposition 2(viii) allows (b) to be defined, and proposition 2(ix) allows (c) to be defined. Moreover, 2(v) makes the interior and closure operators coherent with each other.

Remark C: Poscali and Ajmal (1997) also defined two similar operators, interior and closure operators. In their definitions, these two operators may not further define a coherent fuzzy topology, and they give a necessary and sufficient condition for these two operators to be coherent with each other. These are the necessary and sufficient conditions for these two operators to define an identical fuzzy topology.

Definition 4: For $1 \geq x > 0$, define $\tau^x = \{A^x: A \in \Gamma^x\}$ and $\tau_x = \{A_x: A \in \Gamma^x\}$.

Proposition 5: The triple $(X, \tau_x, \tau^{1-x})$ is an I-fts, where $\tau_x$ is the open sets and $\tau^{1-x}$ is the closed sets that satisfy $(A_x)^c = (A^c)^{1-x}$, i.e. the complement of the elements in the $\tau_x$ closed set.

Proof: $0^{1-x} = 0_x$ and $1^{1-x} = 1_x$ show that 0 and 1 are elements of $\tau_x$ and $\tau^{1-x}$. That $\left( \bigwedge_{i \in \Lambda} A_i \right)_x = \bigwedge_{i \in \Lambda} (A_i)_x$, where $\Lambda$ is finite, shows that the finite intersection of $\delta$ is also in $\delta$. Finally, $\left( \bigvee_{i \in \Lambda} A_i \right)_x = \bigvee_{i \in \Lambda} (A_i)_x$ shows that the union of $\delta$ is also in $\delta$.

Q.E.D.