On the moral hazard problem without the first-order approach

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Abstract

We study the moral hazard problem without the first-order approach or other common structure. We present sufficient conditions under which the shadow value of simultaneously tightening the minimum payment and individual rationality constraints has a simple and intuitive expression. We then show how this expression can be used to perform comparative statics exercises in which we study (i) the effect of a change in the agent’s wealth on the well-being of the principal; and (ii) the effects of the outside option and minimum payment on the effort level optimally implemented.

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1. Introduction

Moral hazard problems are endemic in the economy. Most analysis of moral hazard problems, beginning with Holmström [6], and Mirrlees [16,17] and continuing through a large theoretical and applied literature, hinges on the first-order approach (FOA), in which the agent’s incentive constraints are replaced by a corresponding first-order condition. To guarantee that contracts which solve the relaxed problem are increasing, one imposes the Monotone Likelihood Ratio
Property (MLRP), which requires that effort and outcomes are positively correlated in a strong sense. One then imposes another layer of structure to ensure that the agent’s utility with respect to her effort choice is single peaked when facing an increasing contract. The FOA allows one to directly characterize optimal contracts using calculus methods. However, the restrictions it imposes on the structure of the problem are not merely technical, casting doubt on the generality of the conclusions that one can draw.

Two main issues are raised when dispensing with the FOA. First is whether an optimal contract exists. Second, and equally important, is the character of the optimal contract and its comparative statics properties. Both issues are studied in Grossman and Hart [5] assuming a finite and one-dimensional set of signals and effort levels. But, once one departs from a finite set of signals, or when the set of signals and rewards is multi-dimensional, the analysis of the problem becomes more involved. Kadan, Reny, and Swinkels [13] establish the existence of an optimal contract regardless of the validity of the FOA in a general setting. In this paper we focus on the second issue. We derive a general property of optimal contracts that does not hinge on the FOA, and use this property to study how the solution to a moral hazard problem varies with the wealth of the agent, the agent’s outside option and the level of a required minimum payment (as, for example, a minimum wage, or a limited liability constraint).

Our core observation is that regardless of the validity of the FOA, the shadow value as one simultaneously tightens the minimum payment constraint and the individual rationality constraint is equal to the expectation of the marginal cost of providing a util to the agent, evaluated at the optimal contract. Intuitively, start from an optimal contract, and suppose that both the outside option and the minimum payment are increased by one util. A naïve way to fix the optimal contract is to increase the utility of the agent by a util at each outcome. Since utility goes up by the same amount at each outcome, incentive compatibility still holds. An envelope argument implies that this fix is in fact optimal to the first order.

This argument only applies if the cost to the principal of implementing the optimal contract is appropriately differentiable in the parameters of the problem. We define an effort level as boundedly implementable if it can be implemented (not necessarily optimally) using a bounded contract, and show that bounded implementability is a sufficient condition for the requisite differentiability. We also provide an easily verifiable condition on the primitives of the model which guarantees that all effort levels under consideration are boundedly implementable. This leads to simple examples where the FOA cannot be applied but our results hold.

When the FOA holds, our result is a straightforward consequence of the standard characterization of the optimal contract. Thiele and Wambach [20, p. 257] derive a result similar to ours in the setting of Grossman and Hart [5] (and hence not relying on the FOA), but assuming a finite number of outcomes, a finite number of actions, and no minimum payment constraint. We show that this result holds much more broadly than has previously been understood. We need neither the FOA nor a finite number of outcomes or actions. In common with Sinclair-Desgagné [19], and Conlon [4], the signal can be multiple dimensional. For example, a retail employee might be compensated both on sales and a customer satisfaction rating. In common with Holmström and Milgrom [7,8] we also allow the agent’s effort to be multi-dimensional, reflecting, for example,

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1 See Rogerson [18], Jewitt [9], Sinclair-Desgagné [19], and Conlon [4], for examples of this approach.

2 For example, the conditions in Jewitt [9] imply that optimal contracts must be nearly concave. Such conditions, while useful, fail whenever one deals with a binding minimum payment (such as limited liability) due to the convex kink in payments introduced by the constraint. Conlon [4, p. 276] provides a useful discussion of the restrictiveness of the FOA.
different tasks being taken. No particular assumptions, such as the monotone likelihood ratio property, are imposed on the statistical structure of the problem.

We turn next to applications. We begin by generalizing and modifying the analysis of Thiele and Wambach [20], who study how the principal’s surplus depends on the wealth of the agent. In practice firms are often able to choose between different candidates for management positions. It is important to understand how the existing wealth of the candidate affects the well-being of the firm. It is typical for top management to accumulate significant wealth over time. A natural question is how this wealth affects the well-being of the firm they manage. Similarly, a firm may be interested in the depth of the pockets of a supplier, and how this affects the firm’s well being.

The wealth of the agent affects both his outside option and his level of risk aversion and hence his responsiveness to incentives. Thiele and Wambach [20] consider a case with no minimum payment and provide a sufficient condition on the curvature of the agent’s utility function under which the principal is better off with a less wealthy agent.3 We generalize this result, and then add to it in that we point out that depending on the economic context, there are two sensible interpretations of the effect of wealth on the minimum payment constraint. With limited liability, it may well be that an increase in wealth also increases the amount one can take away from the agent in the event of poor performance. In that context, when the (IR) constraint binds, we have results that largely parallel those of Thiele and Wambach. In particular, whether the principal is better or worse off with a richer agent depends on a curvature condition on the utility of the agent. But, if the (IR) constraint is non-binding (as can be true if the minimum payment constraint binds), then an increase in the wealth of the agent is unambiguously good for the principal regardless of the curvature of the utility function.

In other settings, the minimum payment one can give the agent does not decrease when the agent becomes wealthier. This is true of minimum wages and the “no negative compensation” constraint for executive pay. It is also true when, for example, a court will not enforce damages on a supplier that it finds unreasonable. But then, as the agent becomes richer, the lowest utility that can be imposed rises. In this context, different from Thiele and Wambach’s result, if the smallest allowable transfer to the agent is some non-negative amount then regardless of the curvature of the utility function the principal is worse off with a richer agent.

Our second application is to study how the optimally implemented effort level varies with the outside option and the imposed minimum payment. For example, how does an increase in the minimum wage affect the productivity that firms will optimally ask of workers in sectors with both minimum wages and incentive pay? Examples are production workers paid performance premiums, sales representatives, and the service sector. The results apply equally in a supplier relationship when the level of damages that is enforceable changes. Kadan and Swinkels [11] study this issue assuming that the FOA is valid. They provide sufficient conditions on the agent’s utility and the statistical structure implying that an increase in the minimum payment leads to a lower implemented effort. Here we take a less traditional approach. For a given effort level we consider the distribution over pay induced by the optimal contract implementing this effort level. Our results relate properties of this distribution as effort changes to the effect of changes in the minimum wage or outside option on the induced effort.4

3 Chade and Vera de Serio [3] show that the Thiele and Wambach sufficient conditions are “tight” in the sense that no weaker sufficient conditions can apply to all moral hazard problems.

4 In Kadan and Swinkels [12] we study a moral hazard problem in which the principal can choose the optimal number of agents and the optimal implemented effort within a market setting.
The downside of the approach we follow here is that the pay distribution, which is determined by both the optimal contract and by the statistical distribution over outcomes, is endogenous. On the other hand, the conditions we present do not rely on the validity of the FOA or on the associated statistical structure, and they are often more easily interpretable than conditions on primitives such as those used in Kadan and Swinkels [11].

Consider first a setting where, when the principal chooses to induce a higher effort level, he does so by adding to previous performance based payments. Then, the agent is being paid at least as much at each outcome, and, because he is working harder, better outcomes are more likely. Hence, the pay-distribution is raised in a first-order stochastic dominance sense when the principal chooses to induce a higher effort level. Our first result shows that this implies that as either the minimum payment or the outside option are increased, it becomes more expensive on the margin for the principal to induce effort. As a result, the optimally induced effort level falls.

When the individual rationality constraint binds, inducing more effort will often involve lowering pay at bad outcomes, and raising it at good ones. Because of this, first-order stochastic dominance may or may not hold. Assume first that the pay-distribution at a higher effort level single crosses the pay-distribution at a lower effort level, with both relatively high payments and relatively low payments more likely. For this case, we show that if we impose a restriction on the curvature of the agent’s utility function, then, once again, an increase in the minimum payment or outside option increases the marginal cost to the principal of inducing effort. Finally, with a slightly stronger curvature condition on utility, we reach the same conclusion allowing the distribution over pay for different effort levels to cross any number of times, as long as the pay-distribution changes in effort in a way which is similar to second-order stochastic dominance.

Section 2 presents the model. Section 3 establishes our main result, studies the concept of bounded implementability, and offers examples. Section 4 presents applications of the result to the study of the effect of the wealth of the agent on the well-being of the principal, and to comparative statics for implemented effort. Section 5 concludes.

2. Model and preliminaries

An agent exerts effort $e \in E$, where $E$ is a compact metric space. Thus, $e$ can be multi-dimensional reflecting different aspects of the actions taken by the agent, where some dimensions may be discrete and others continuous. A signal $x \in X$ is generated according to a probability measure $P(\cdot|e)$ on $X$, where $X$ is a Polish space.5 The space $X$ can be multi-dimensional, and no order structure on $X$ is assumed. We require that $P(\cdot|e)$ be continuous in $e$ in the weak topology.6

Thus, we dispense with the typical structure and assumptions used to justify the FOA.

A contract is a measurable real function specifying a payment $\pi(x)$ for each $x$. The agent has separable disutility of effort $c(e)$, outside option $u_0$, and utility of income $u(w)$ that is three times continuously differentiable, increasing, and strictly concave. We allow for a minimum payment constraint, restricting the payment given each signal to be no lower than some value. Such a constraint seems relevant in many cases of interest as limited liability or a minimum wage. We assume that there is a $p_* > -\infty$ where $p_*$ is in the domain of the utility function (and so

5 That is, $X$ is a complete and separable metric space. As an example, any Euclidean space is Polish.

6 That is, for any continuous and bounded function $h : X \to \mathbb{R}$, $\int_X h(x) d P(x|e)$ is continuous in $e$. This is a very mild requirement, which is satisfied, for example, if $P(\cdot|e)$ can be represented by a distribution function $F(\cdot|e)$, which is continuous in $e$. However, given that $X$ does not necessarily possess an order structure, a distribution function may not exist. And, even if a distribution does exist, we do not require the existence of a density.
satisfies $u(p_*) > -\infty$ such that all minimum payments constraints under consideration involve a payment greater than $p_*$. We parametrize the minimum payment in utility terms, so that for every $x$, we require $u(\pi(x)) \geq u > u_*$, where $u_*$ is defined as $u(p_*)$. We also assume that as $w \to \infty$, $u(w) \to \infty$ and $u'(w) \to 0$.

The principal’s revenue, before the costs associated with hiring and motivating the agent, is $B(e)$. We assume that $B(e)$ is continuous in $e$. If $x$ is the gross terminal value of a project, then $B(e) = \int_X x \, dP(x|e)$, but since our signal $x$ may be more general, we make no such restriction to $B$ in general.

The principal’s problem is to maximize by choice of $\pi(\cdot)$ and $e \in E$,

$$B(e) - \int_X \pi(x) \, dP(x|e)$$

subject to individual rationality, incentive compatibility, and minimum payment constraints. As is typical in these settings, we can begin with the cost minimization problem (CM), where for any given $e$, $u$, and $u_0$ the principal chooses a contact $\pi(\cdot)$ that solves

$$\min_{\pi(\cdot)} \int_X \pi(x) \, dP(x|e) \quad \text{(CM)}$$

$$\int u(\pi(x)) \, dP(x|e) - c(e) \geq u_0 \quad \text{(IR)}$$

$$e \in \arg \max_{e' \in E} \int u(\pi(x)) \, dP(x|e') - c(e') \quad \text{(IC)}$$

$$u(\pi(x)) \geq u \quad \forall x. \quad \text{(MP)}$$

Say that $e$ is implementable given $(u, u_0)$ if there is any finite expected cost contract satisfying (IR), (IC), and (MP) given $e$, $u$, and $u_0$. Under mild conditions on the statistical structure of the problem, in Appendix B we build on Kadan, Reny, and Swinkels [13] to show that for any given $(u_0, u)$, there is a non-empty closed set $E^* \subseteq E$ of implementable effort levels, and that for each implementable $e$, there is a unique least cost contract $\pi(\cdot, e, u, u_0)$ implementing $e$. We shall henceforth assume the existence of such an $E^*$ and such a contract for each $e \in E^*$.

Let $C(e, u, u_0) = \int_X \pi(\cdot, e, u, u_0) \, dP(x|e)$ be the cost of implementing $\pi(\cdot, e, u, u_0)$. Then, one can write the principal’s full optimization problem as simply

$$\max_{e \in E^*} B(e) - C(e, u, u_0). \quad \text{(F)}$$

For given $(e, u, u_0)$, define

$$G(p|e, u, u_0) = P(\{x: \pi(x, e, u, u_0) \leq p\}|e).$$

This is the cumulative distribution of payments when the principal implements $e$. It depends on both the statistical structure of the problem given by $P(\cdot|e)$ and on the optimal contract implementing $e$. Note that the support of $G(p|e, u, u_0)$ may depend on $(e, u, u_0)$ but is always a subset of $(p_*, \infty)$.

7 From now on, when the domain of integration is the entire signal space $X$, we omit it.
3. A generalized shadow value

In the traditional analysis of the moral hazard problem, one assumes that \( P(\cdot|e) \) can be represented by a density function \( f(\cdot|e) \). Then, assuming MLRP, and that the FOA holds, one can replace (IC) by its first-order condition, and derive that the unique cost minimizing contract for a setting when the minimum payment constraint (MP) is slack is given by

\[
\frac{1}{u'(\pi(x,e,u_0,u_0))} = \lambda + \mu \frac{f_e(x|e)}{f(x|e)},
\]

where \( \lambda \) and \( \mu \) are the multipliers on the (IR) and (IC) constraints, respectively \([6,16]\). Taking the expectation of both sides of (1) gives the standard result

\[
\lambda = \int \frac{1}{u'(\pi(x,e,u,u_0))} f(x|e) \, dx,
\]

(2)

or, noting that \( \lambda = C_{u_0} \) and using the definition of \( G(\cdot|e) \),

\[
C_{u_0} = \int \frac{1}{u'(p)} \, dG(p|e,u,u_0).
\]

(3)

There is another simple way to see (3). Imagine that \( u_0 \) is increased by one util. One way to fix the optimal contract is to increase payments such that the utility of the agent increases by a single util at each outcome, an exercise which has cost \( \frac{1}{u'(p)} \) when \( p \) is being paid. The expected cost of doing so is thus \( E(\frac{1}{u'}) \), the expectation of the marginal cost of providing a util to the agent, evaluated at the optimal contract. And, an argument which is essentially an application of the envelope theorem shows that this is in fact a good local approximation to \( C_{u_0} \). The key is that while the argument leading to (2) depends on the FOA, this argument does not.

When (MP) is binding, this argument fails. In particular, increasing pay at outcomes where pay is governed by the minimum constraint in no way approximates an optimal response to an increase in \( u_0 \) alone. It is however, a sensible way to respond if \( u \) and \( u_0 \) are increased simultaneously. To formalize this, let

\[
C_z(e,u,u_0) \equiv \frac{d}{dz} C(e,u,z,u_0+z)|_{z=0},
\]

(4)

noting that when \( C \) is differentiable in both \( u \) and \( u_0 \), \( C_z(e,u,u_0) \) is just \( C_{u_0} + C_u \). At the heart of the paper is that

\[
C_z(e,u,u_0) = \int \frac{1}{u'(\pi(x,e,u,u_0))} \, dP(x|e)
\]

\[
= \int_{p_*} \frac{1}{u'(p)} \, dG(p|e,u,u_0).
\]

(5)

That is, \( E(\frac{1}{u'}) \) is a generalized shadow value as one increases both \( u \) and \( u_0 \) simultaneously, and this holds without the FOA or indeed without much structure at all.

Part of proving (5) involves showing that its ingredients are well defined. That is, we need to show that \( C(e,u+z,u_0+z) \) is differentiable at \( z = 0 \) and that \( \frac{1}{u'(p)} \) has a well-defined expectation given \( G \). We address these issues in the next section. For now, let us see how the result follows when these properties are assumed.
Define the contract \( \hat{\pi}(x, z) \) by
\[
\hat{u}(\hat{\pi}(x, z)) = u(\pi(x, e, u, u_0)) + z
\]
for any \( z \) small (positive or negative).\(^8\) Since \( \hat{\pi}(x, 0) = \pi(x, e, u, u_0) \), differentiating (6) and evaluating at \( z = 0 \) gives
\[
\hat{\pi}_z(x, 0) = \frac{1}{u'(\pi(x, e, u, u_0))}.
\]
By construction, \( \hat{\pi}(\cdot, z) \) satisfies (IR) for \( u_0 + z \) and (MP) for \( u + z \). And, since we have added a constant to utility at each outcome, the relative attractiveness of any two effort levels is unaffected, and so \( \hat{\pi}(\cdot, z) \) satisfies (IC). Thus, \( \hat{\pi}(\cdot, z) \) is feasible, and so
\[
C(e, u + z, u_0 + z) \leq \int \hat{\pi}(x, z) dP(x|e)
\]
for all \( z \) on a neighborhood of 0. But then, since
\[
C(e, u, u_0) = \int \hat{\pi}(x, 0) dP(x|e),
\]
\( C(e, u + z, u_0 + z) \) and \( \int \hat{\pi}(x, z) dP(x|e) \) have the same slope with respect to \( z \) at \( z = 0 \). That is,
\[
C_z(e, u, u_0) = \frac{d}{dz} \int \hat{\pi}(x, 0) dP(x|e)
= \int \hat{\pi}_z(x, 0) dP(x|e)
= \int \frac{1}{u'(\pi(x, e, u, u_0))} dP(x|e)
= \int \frac{1}{u'(p)} dG(p|e, u, u_0),
\]
as claimed.

3.1. Bounded implementability and the formal result

Let us now turn to the question of when the component parts of (5) are well defined. The following concept will turn out to be crucial.

**Definition 1.** Say that \( e \in E \) is boundedly implementable given \((u, u_0)\) if there exists a bounded contract \( \pi \) that implements \( e \) (i.e., a bounded contract that satisfies (IR), (IC), and (MP) given \( e \)).

Note that there is no requirement here that the optimal contract given \((u, u_0)\) is bounded, merely that some contract implementing \( e \) is. The following lemma shows that bounded implementability of \( e \) is independent of the choice of \((u, u_0)\).

\(^8\) The idea of perturbing contracts in utility space is also used in Jewitt, Kadan, and Swinkels [10, p. 65] in establishing existence and uniqueness of optimal contracts in a setting where the FOA holds.
Lemma 1. If $e$ is boundedly implementable given one $(u, u_0)$, then $e$ is boundedly implementable for all $(u, u_0)$.

This and all other proofs are in Appendix A. Given this lemma we simply refer to an effort level $e$ as boundedly implementable. An immediate implication of this lemma is that if $e$ is boundedly implementable, then $C(e, u, u_0)$ is finite for all $(u, u_0)$. In Section 3.2 we will investigate the notion of bounded implementability in more detail. We will then demonstrate that this is a relatively mild requirement, and that it is in fact satisfied in many settings where the FOA cannot be applied. We will also provide an easily verifiable sufficient condition on the primitives of the problem that guarantees bounded implementability of all effort levels under consideration.

We also need some continuity in $G$ as $u$ and $u_0$ changes. Say that $G$ is weakly continuous if for each boundedly implementable $e$, $G(\cdot|e, u, u_0)$ is continuous in $(u, u_0)$ in the weak topology. Primitive conditions for weak continuity are mild. One easy such primitive is that $X$ is Euclidean, and $P(\cdot|e)$ has a strictly positive and continuous density $f(\cdot|e)$. Another is that $X$ is a finite set and that $P(\{x\}|e)$ is continuous in $e$ for each $x \in X$. In general, weak continuity will hold under the same sets of conditions that Kadan, Reny and Swinkels [13] use to establish existence. See Appendix B for further details.

Our central result is as follows.

Proposition 1. Let $e \in E$ be boundedly implementable, let $G$ be weakly continuous, and fix $(u, u_0)$ with $u > u_\ast$. Then, $\frac{1}{u'}(p)$ is $G(\cdot|e, u, u_0)$-integrable, $C(e, u + z, u_0 + z)$ is differentiable in $z$ at $z = 0$, and

$$C_z(e, u, u_0) = \int_{p_\ast}^{\infty} \frac{1}{u'(p)} dG(p|e, u, u_0).$$

To see that the integral in (5) correctly accounts for the fact that the support of $G$ depends on $(e, u, u_0)$, note that by the choice of $p_\ast$, $[p_\ast, \infty)$ contains the support of $G(\cdot|e, u, u_0)$ for all $(e, u, u_0)$, and that outside of the support of $G(\cdot|e, u, u_0)$, we have $dG(p|e, u, u_0) = 0$. Also, it is important to note that bounded implementability and weak continuity are used to establish the differentiability of $C(e, u + z, u_0 + z)$ only, whereas the integrability of $\frac{1}{u'}(p)$ does not hinge on these requirements. If $\frac{1}{u'(p)}$ was bounded, then the differentiability of $C(e, u + z, u_0 + z)$ would follow from weak continuity alone. The bounded implementability assumption allows us to show that for relevant $(u, u_0)$, the part of the expectation of $\frac{1}{u'}$ that can be attributed to large payments becomes trivial, and so weak continuity can still be invoked.

Finally, two important special cases should be considered. First, where (MP) is slack, $C_z = C_{u_0}$, and we thus have that (3) holds without the FOA or other standard structure. Second, when (MP) is active, but (IR) is not (as can easily occur given (MP)) then $C_z = C_u$, and we find a simple analog to (3) describing the effect of a change in the minimum feasible utility.

3.2. Bounded implementability – interpretation and examples

Proposition 1 applies to effort levels which are boundedly implementable. We now explore this notion in more detail, and provide an easily verifiable sufficient condition. We then use this condition to present examples in which Proposition 1 can be applied whereas the FOA cannot.
It is easy to see that the set of effort levels that are boundedly implemented is non-empty. In particular, choose any continuous and bounded contract \( \pi \). Since \( P(\cdot|e) \) is continuous in the weak topology in \( e \), and since \( E \) is compact, the agent has a best response to \( \pi \). Also, if \( X \) is compact, the FOA is valid, and MLRP holds, then all \( e \in E \) are boundedly implementable. Indeed, in this case the optimal contract implementing \( e \) is given by (1), and is in particular continuous in \( x \), and thus bounded. For a case where \( e \) is boundedly implementable but the optimal contract for \( e \) is not bounded see Example 4 below.

In general, it is possible to construct examples where the set of boundedly implementable effort levels is a strict subset of the implementable effort levels. See Appendix C for (a pretty tortured) such example, in which an effort level is not boundedly implementable, and yet can be implemented at finite cost.

When \( E \) is an interval on the real line, the following result provides an over-sufficient but convenient condition to check for bounded implementability.

**Proposition 2.** Assume \( E \subset \mathbb{R} \) is an interval, that \( c(e) \) is differentiable with \( c' \geq 0 \) and \( c'' \geq 0 \), and that there is some measurable set of signals \( \hat{X} \subset X \) such that \( P(\hat{X}|e) \) is strictly increasing, differentiable, and weakly concave in \( e \). Then, \( e \) is boundedly implementable for all \( e \in E \).

### 3.2.1. Examples

The condition in Proposition 2 is easy to verify and is satisfied in many examples. We now present several simple examples of two kinds. First, we illustrate cases in which the known sufficient conditions for the FOA fail, and yet Proposition 1 holds true. Then, we provide an example in which the FOA must fail, and yet Proposition 1 is intact. For this section, we assume that \( X \subset \mathbb{R}^N \), and that \( P \) admits a cumulative \( F \) and a density \( f \). Weak continuity of \( G \) holds for each example (see Appendix B for details).

Both Rogerson’s [18] and Jewitt’s [9] sufficient conditions for the FOA require that \( f \) satisfy the Monotone Likelihood Ratio Property (MLRP), i.e., \( \frac{d}{dx} \frac{f_e(x|e)}{f(x|e)} \) is non-negative. Rogerson also requires that \( F \) satisfy the Convexity of the Distribution Function Condition (CDFC), i.e., \( F_{ee}(x|e) \geq 0 \) for all \( x \) and \( e \). Jewitt, on the other hand, imposes additional conditions on both the statistical structure and on the utility function. The next two examples show that none of these conditions is required for Proposition 1 to hold.

**Example 1** (Sufficiency of CDFC and FOSD without MLRP). Let \( X = [0, 1] \). Assume that CDFC holds. Assume also that \( F_e(x|e) < 0 \) for all \( e \in E \) and \( x \in (0, 1] \); namely, effort moves the distribution of signals in the sense of First Order Stochastic Dominance (FOSD). Choose \( \hat{x} \in (0, 1) \) arbitrarily and set \( \hat{X} = [\hat{x}, 1] \). Then, \( P(\hat{X}|e) = 1 - F(\hat{x}|e) \), which, by FOSD and CDFC, is increasing and concave. Hence, by Proposition 2 all \( e \) are boundedly implementable.

For a specific example set \( E = [\frac{2}{3}, 1] \) and let

\[ F(x|e) = x + (x^2 - x^3)(1 - 2e^2 + e^3). \] (7)

Then, it is easy to verify that FOSD and CDFC are satisfied. Yet, MLRP fails,\(^9\) and so Rogerson’s and Jewitt’s conditions for the FOA fail.

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\(^9\) Indeed, \( \frac{d}{dx} \frac{f_e(x|e)}{f(x|e)} \) switches signs at \( x = \frac{1}{3} \) for all \( e \).
**Example 2** (No MLRP, CDFC, or FOSD). Let $X = E = [0, 1]$. Let $f$ be given by

$$
f(x|e) = \begin{cases} 
(1 - e)(1 - e(1 - e)), & x \in [0, \frac{1}{2}), \\
(1 - e)(1 + e(1 - e)), & x \in [\frac{1}{2}, \frac{1}{4}), \\
(1 + e)(1 - e(1 - e)), & x \in [\frac{1}{2}, \frac{3}{4}), \\
(1 + e)(1 + e(1 - e)), & x \in [\frac{3}{4}, 1]. 
\end{cases}
$$

It is easily checked that $F(\frac{3}{4}|e)$ is non-monotone so that FOSD (and hence MLRP) fails, while $F(\frac{1}{4}|e)$ is non-convex so that CDFC fails. But, the probability of a signal in $[\frac{1}{2}, 1]$ is $1 + e$ which is increasing and concave, and so by Proposition 2 all $e \in E$ are boundedly implementable, and we can apply Proposition 1.

The next example illustrates the applicability of our results when signals are multi-dimensional. Sinclair-Desgagné [19] and Conlon [4] provide sufficient conditions for the FOA to be valid in multi-signal settings, both requiring a multi-dimensional version of MLRP. Namely, they require that if $x, x' \in X$, and $x' \preceq x$ (componentwise), then $f_e(x'|e)/f(x'|e) \geq f_e(x|e)/f(x|e)$. The next example illustrates a case where multi-dimensional MLRP fails, and yet Proposition 1 is intact.

**Example 3** (Two-dimensional signals). Let $X = [0, 1]^2$. Suppose $f_1(x_1|e)$ satisfies FOSD and CDFC, but that $f_2(x_2|e)$ is arbitrary, and that the joint density of signals over $X$ is

$$
f(x_1, x_2|e) = f_1(x_1|e)f_2(x_2|e),
$$
i.e., the signals are drawn independently conditional on $e$. Then, for ill-behaved $f_2$, $f(x_1, x_2|e)$ fails multi-dimensional MLRP, and Sinclair-Desgagné’s and Conlon’s conditions cannot be applied. But, setting $\hat{X} = \{(x_1, x_2): x_1 \geq \frac{1}{2}\}$, $P(\hat{X}|e)$ is increasing and concave in $e$, and so Proposition 2 can be applied.

The previous examples illustrated cases where the known sufficient conditions for the validity of the FOA fail. In the next example, the FOA itself must fail, and yet Proposition 1 is intact. The idea behind the example is that the minimum payment constraint introduces a natural convexity into optimal contracts. This convexity can make the agent’s optimization problem convex rather than concave, invalidating the FOA.

**Example 4** (The FOA fails). Let $X = [0, \infty)$, and $E = [1, 2]$. Suppose that $P(\cdot|e)$ is the exponential distribution with mean $e$ (see Holmström [6] for a similar example). Then, $f(x|e) = \frac{1}{e} \exp(-\frac{x}{e})$, and so $\frac{f_e(x|e)}{f(x|e)} = \frac{x - e}{e^2}$. Let $u(w) = \sqrt{2w}$, $c(e) = e$, and $u, u_0, > 0$. Fix $e > 0$ in $E$. If the FOA were valid then the optimal contract given $e$, denoted by $\hat{\pi}$, satisfies

$$
u(\hat{\pi}(x)) = \max \left( u, \lambda + \mu \frac{x - e}{e^2} \right)
$$

(see, for example, Jewitt, Kadan, and Swinkels [10]). When the (MP) constraint is binding, $u(\hat{\pi}(x))$ is convex (and not linear) in $x$, and so, applying the arguments in Jewitt [9],
\[
\int u(\hat{\pi}(x)) f(x|e) \, dx - c(e) \text{ is strictly convex in } e. \]

The agent is thus at a minimum of his utility, and the FOA is invalid. But, it is immediate that Proposition 2 is valid and hence all \( e \in E \) are boundedly implementable. Indeed, set \( \hat{X} = [2, \infty) \). Then, \( P(\hat{X}|e) = 1 - F(2|e) = \exp(-\frac{e}{2}) \), which is increasing and concave in \( e \) for \( e \geq 1 \). Finally, note that for \( (e, u, u_0) \) where (MP) is not binding, the Jewitt [9] conditions do apply and so the FOA approach is valid. Thus, this is a case where \( e \) is boundedly implementable and yet the optimal contract is not bounded.

3.3. An extension

We end this section with an extension to Proposition 1. Imagine that, in addition to the minimum payment constraint, there is also a constraint \( u(\pi(x)) \geq \tau + u_L(x) \) specifying a lower bound on utility as a function of the outcome. Then, similar to Proposition 1,

\[
\frac{d}{dz} C(e, u + z, u_0 + z, \tau + z)|_{z=0} = \int \frac{1}{u'(p)} \, dG(p|e, u, u_0, \tau). \]

But, since increasing \( \tau \) restricts the feasible set of contracts, \( C(e, u, u_0, \tau) \) increases in \( \tau \), and thus

\[
\frac{d}{dz} C(e, u + z, u_0 + z, \tau)|_{z=0} \leq \int \frac{1}{u'(p)} \, dG(p|e, u, u_0, \tau). \]

Thus, even when there is a more complicated lower bound on payments, we are still able to derive a useful upper bound on the shadow value as the minimum payment and outside option increase simultaneously.

4. Applications

We now turn to exploring two applications of Proposition 1. We first study how the wealth of the agent affects the well-being of the principal. Then we explore how changes in the minimum payment affect the agent’s effort. Throughout this section, we assume that any implementable \( e \) is boundedly implementable, and that \( G \) is weakly continuous.

---

10 This follows from Karlin [14], since the exponential distribution is totally positive, and satisfies \( \int x f(x|e) \, dx = e \), which is linear (and hence weakly concave) in \( e \), while \( -u(\hat{\pi}(x)) \) is concave in \( x \). Hence, \( -\int u(\hat{\pi}(x)) f(x|e) \, dx \) is concave in \( e \), and is strictly so since \( u(\hat{\pi}) \) is not linear (because of the region over which (MP) is binding).

11 For example, imagine that in addition to the constraint that he receives minimum utility \( u \), a manager can be paid no less than 10% of the value of the firm.

12 Shifting \( u_L(\cdot) \) up by a constant in utility space may or may not have a natural economic interpretation. The importance of (8) is in its implications.

13 This derivative is not guaranteed to exist. If it does not, then

\[
\limsup_{z \downarrow 0} \frac{C(e, u + z, u_0 + z, \tau) - C(e, u, u_0, \tau)}{z}
\]

is bounded by the RHS.

14 Similarly, if (MP) sets the only minimum payment constraint, but there is also a constraint of the form \( u(\pi(x)) \leq \tau + u_H(x) \) then (8) again holds, but now \( C \) decreases in \( \tau \), and so the sign of (9) is reversed.
4.1. Wealth and the well-being of the principal

Consider an agent who has outside wealth \( w_0 \). How the agent’s wealth affects the principal has important implications in a variety of contexts. A key point in the analysis is that different contexts result in different ways by which wealth enters into the problem.

Let \( \hat{\pi}(x) \) be the amount that the principal pays the agent given a signal \( x \). Wealth affects the problem faced by the principal in three ways. First, it enters into the (IR) constraint, as a wealthier agent has a better outside option. For concreteness, we assume that the next best alternative to the relationship with the principal is to be unemployed and consume outside wealth, so that the agent has outside option \( u(w_0) \). Second, the wealth of the agent is added to whatever compensation the principal gives the agent. So, given effort level \( e \) and signal \( x \), the agent has utility \( u(\hat{\pi}(x) + w_0) - c(e) \). We assume \( c(e) \geq 0 \), so that the agent, fiscal considerations aside, prefers not to work.

Third, wealth may affect the minimum utility constraint. Here different contexts lead to different models. Consider first a setting in which, for example, the agent cannot be driven to a negative wealth. Then the constraint is \( w_0 + \hat{\pi}(x) \geq 0 \), and so as \( w_0 \) rises, \( \hat{\pi}(x) \) can become smaller. A richer supplier can be sued for more if things go wrong.

In other settings, the relevant constraint is directly on the transfer, so that \( \hat{\pi}(x) \geq m \), where \( m \) is some minimum pay level. This is, for example, true of a minimum wage \( m \). For executive compensation, it seems difficult as a practical matter to write a contract that has the manager paying the firm in any state, even if the manager has outside wealth. And, courts may be unwilling to enforce penalties beyond some amount regardless of the wealth of the agent.

4.1.1. The case of a final wealth constraint

Consider first a constraint on the final wealth of the agent. That is, if we let \( \pi(\cdot) = w_0 + \hat{\pi}(\cdot) \) be the final wealth of the agent then the constraint is that \( \pi(x) \geq m \) for all \( x \). The cost minimization problem of the principal can then be written

\[
\begin{align*}
\min_{\pi(\cdot)} & \int \pi(x) dP(x|e) - w_0 \\
\text{s.t.} & \int u(\pi(x)) dP(x|e) - c(e) \geq u(w_0), \\
& e \in \arg \max_{e' \in E} \int u(\pi(x)) dP(x|e) - c(e'), \\
& \pi(x) \geq m \quad \text{for all } x.
\end{align*}
\]

Note in particular that \( \int \pi(x) dP(x|e) - w_0 = \int \hat{\pi}(x) dP(x|e) \). Let \( C_{F^W}(e, m, w_0) \) (with \( FW \) mnemonic for final wealth) be the value of this program. Comparing this program to (CM), we have

\[
C_{F^W}(e, m, w_0) = C(e, u(m), u(w_0)) - w_0, \quad (10)
\]

and so

\[
C_{F^W}^{w_0}(e, m, w_0) = C_{w_0}(e, u(m), u(w_0))u'(w_0) - 1. \quad (11)
\]

Thus, when (IR) does not bind,

\[
C_{F^W}^{w_0}(e, m, w_0) = -1.
\]
and so an increase in the wealth of the agent benefits the principal. We thus have the following generalization of Proposition 3 of Thiele and Wambach [20].

**Proposition 3.** Consider the case of a final wealth constraint. At any \((e, m, w_0)\) where (IR) does not bind, \(C_{w_0}^{FW}(e, m, w_0)\) is negative.

Thus, if (IR) does not bind at the optimal \(e\) given \(m\) and \(w_0\), the principal is better off in the face of a small increase in the agent’s wealth. The intuition is simple. When (IR) does not bind, an increase in the wealth of the agent can be fully expropriated, dollar-for-dollar in each state by the principal, and yet the agent does not quit. This result does not depend on Proposition 1.

When (IR) does bind but the final wealth constraint, \(\pi(x) \geq m\), does not, then from (5),

\[
C_{w_0}(e, u(m), u(w_0)) = \int \frac{1}{u'(p)} dG(p|e, u(m), u(w_0)),
\]

and so from (11)

\[
C_{w_0}^{FW}(e, m, w_0) = u'(w_0) \int \frac{1}{u'(p)} dG(p|e, u(m), u(w_0)) - 1. \tag{12}
\]

Thus \(C_{w_0}^{FW}\) has the same sign as

\[
\int \frac{1}{u'(p)} dG(p|e, u(m), u(w_0)) - \frac{1}{u'(w_0)}. \tag{13}
\]

Observe from this that if it turns out that transfers \(\hat{\pi}(\cdot)\) are always positive, then trivially, \(C_{w_0}^{FW} > 0\), since then on the support of \(G\), \(\frac{1}{u'(p)} \geq \frac{1}{u'(w_0)}\).

Now, from (IR),

\[
\int u(p) dG(p|e, u(m), u(w_0)) - u(w_0) \geq 0
\]

(and strictly so if \(c(e) > 0\)). Thus, as in Thiele and Wambach [20], if the marginal cost of providing a util at \(p\), \(\frac{1}{u'(p)}\), is a convex function of \(u(p)\),\(^{15}\) then by Jensen’s inequality \(C_{w_0}^{FW}(e, m, w_0) \geq 0\), with conflicting forces otherwise. We thus have the following generalization to Thiele and Wambach’s Proposition 1.

**Proposition 4.** Assume that (IR) binds at \((e, m, w_0)\) but the final wealth constraint does not. If transfers are always weakly positive, or if \(\frac{1}{u'(p)}\) is a convex function of \(u(p)\), then \(C_{w_0}^{FW}(e, m, w_0) \geq 0\).

So, if \(e\) was the unique optimal effort level for the principal given \(m\) and \(w_0\), then the principal is harmed by a small increase in the agent’s wealth.\(^{16}\) To see the intuition for Proposition 4, consider raising the agent’s outside wealth one dollar, so that his utility from not working increases

\(^{15}\) That is, if we define \(h\) implicitly by \(\frac{1}{u'(p)} = h(u(p))\), then \(h\) is convex. This is equivalent to \(\frac{1}{u'(p)}\) being convex in \(p\), or to the prudence \(\frac{u''}{u'}\) being less than 3 times the absolute risk aversion \(\frac{u''}{u'}\). A CRRA utility function \(u(w) = w^{1-\gamma}/(1-\gamma)\) qualifies if \(\gamma \geq \frac{1}{2}\), so that relative risk aversion is not too low. This condition is less stringent than condition U1\(^*\) and is the complement of condition U2 in Kadan and Swinkels [11].

\(^{16}\) It is possible that at other \(e\), (IR) does not bind, and hence costs at those \(e\) fall with \(w_0\). Indeed, examples can be constructed where as \(w_0\) rises, the principal eventually jumps to a higher implemented effort level.
by \( u'(w_0) \). By Proposition 1, when the final wealth constraint does not bind, to the first order, the optimal way to respond to this is to add the same utility, \( u'(w_0) \), at each outcome, which has a cost equal to the expectation of \( \frac{u'(w_0)}{u'(p)} \). On the other hand, the effect of increasing \( w_0 \) by a dollar is to effectively provide a free extra dollar of compensation at each outcome. Hence, whether the principal is hurt or helped depends on whether the expectation of \( \frac{u'(w_0)}{u'(p)} \) is bigger or smaller than 1. When transfers are positive, the answer to this is simple, because \( u'(w_0) \) is bigger than 1 at all relevant \( p \). When transfers can also be negative, the condition that \( \frac{1}{u'(p)} \) is convex as a function of \( u(p) \) gives the extra structure needed to sign the relevant difference.

Given that the previous two propositions have opposite predictions, we have no specific prediction when both the (IR) and final wealth constraints bind. The principal is worse off because the agent has a higher outside option, but better off because the smallest feasible payment is reduced.

4.1.2. The case of a minimum transfer constraint

Consider instead the setting in which the constraint is on the transfer to the agent. Then, the (MP) constraint becomes \( \hat{\pi}(x) \geq m \), or \( \pi(x) \geq m + w_0 \). Thus, the cost of implementing action \( e \) to an agent with initial wealth \( w_0 \) and given a minimum transfer \( m \) is given by

\[
C^T(e, m, w_0) = C(e, u(m + w_0), u(w_0)) - w_0. \tag{14}
\]

Assume for a moment that \( C \) is differentiable in \( u \) and \( u_0 \). Then,

\[
C^T_{w_0}(e, m, w_0) = C_u(e, u(m + w_0), u(w_0))u'(m + w_0)
+ C_{u_0}(e, u(m + w_0), u(w_0))u'(w_0) - 1.
\tag{15}
\]

Assume also that \( m \geq 0 \). Then, we have

\[
C^T_{w_0}(e, m, w_0) \geq \left[ C_u(e, u(m + w_0), u(w_0)) + C_{u_0}(e, u(m + w_0), u(w_0)) \right]u'(m + w_0) - 1. \tag{15}
\]

Thus, by (5)

\[
C^T_{w_0}(e, m, w_0) \geq \int \frac{u'(m + w_0)}{u'(p)} dG(p | e, u(m + w_0), u(w_0)) - 1. \tag{16}
\]

But then, as before, since \( \pi(x) \geq m + w_0 \) for all \( x \), we have \( \frac{u'(m + w_0)}{u'(p)} \geq 1 \) everywhere on the support of \( G \). Hence, the RHS of (16) is at least 0, with strict inequality for any positive effort level (where \( \pi(\cdot) \) cannot pay a constant \( m + w_0 \) and satisfy (IC)). We conclude that \( C^T(e, m, w_0) \) increases in \( w_0 \).

**Proposition 5.** Consider the case of a non-negative constraint on the transfer to the agent. Then, \( C^T_{w_0}(e, m, w_0) > 0 \) for all positive \( e \).

Hence, different from Thiele and Wambach [20], when the relevant constraint involves a minimum transfer and the minimum is weakly positive, we have that unambiguously, the principal
is worse off with a wealthier agent.\footnote{The requirement that the minimum transfer is non-negative is realistic in some scenarios but may be restrictive in others.} Intuitively, imagine that the wealth of the agent increases by a dollar, and that, when the agent is at his minimum pay, where he consumes $m + w_0$, this increases his utility by $\delta$. Since $m$ is non-negative, the utility of the agent at his outside option, where he consumes $w_0$ increases by at least $\delta$. But then, by Proposition 1, the cost to the principal of restoring feasibility is at least the cost of adding $\delta$ to the utility of the agent at all outcomes, and, anytime that $\hat{F}(x) > m$, the agent was paid more than the minimum, doing so costs more than a dollar.

Note that there are parallels between Propositions 5 and 4 since in both cases the principal can only benefit from an increase in $w_0$ if transfers are sometimes negative. Indeed, the final steps of the proofs of Proposition 5 and the first part of Proposition 4 are similar. But, the results are quite different, in that in the first part of Proposition 4, we needed to assume that (IR) was binding, but the final wealth constraint was not, while Proposition 5 was agnostic on the issue. The key is that in the setting of Proposition 4, $w_0$ enters the problem only through the (IR) constraint, while here, $w_0$ enters both the (IR) constraint and the minimum payment constraint (compare, in particular, (10) and (14)). The shadow value identified in Proposition 1 is about the effect of a simultaneous change in both constraints. Therefore, in the setting of Proposition 4 we can only give a definitive result when the minimum wealth constraint is irrelevant, while in the setting of Proposition 5 the issue of which constraint binds is irrelevant.

4.2. Comparative statics for effort

We now turn to showing how (5) can be used to perform a natural and important comparative statics exercise. We examine how the effort that the principal chooses to impose varies in the minimum payment and the outside option of the agent. We assume that $E = [0, \bar{e}]$.

Intuitively, we are looking for conditions under which the marginal cost of implementing effort $e \in E$ at $(u + z, u_0 + z)$ is increasing in $z$. A complication is that our general structure does not guarantee the existence of the cross-derivative $C_{ze}$. Instead, we will explore conditions under which $C(e, u + z, u_0 + z)$ is supermodular in $e$ and $z$. When this is so, then when $z$ is increased, the principal will optimally choose a (weakly) lower $e$.\footnote{In particular, the profit of the principal given $(e, u + z, u_0 + z)$ is $B(e) - C(e, u + z, u_0 + z)$ and is hence submodular. Thus, our comparative statics argument does not hinge on the principal’s first-order condition being satisfied.} As before, where (MP) does not bind, the implication is that $C$ is supermodular in $u_0$ and $e$, and when (IR) does not bind, that $C$ is supermodular in $u$ and $e$.\footnote{In Section 4.1, we established tight links between $C$ and each of $C^T$ and $C^{FW}$. Hence, statements about the effect of the minimum payment and outside option on optimally induced effort have important implications for understanding the effects of the wealth of the agent on optimally induced effort as well.}

Note that because we have assumed that any implementable $e$ is boundedly implementable and that $G$ is weakly continuous, $C_z$ is well defined, and thus supermodularity of $C(e, u + z, u_0 + z)$ holds if and only if for any $e' > e$, where $e$ and $e' \in E^*$

\[ C(e', u + z, u_0 + z) - C(e, u + z, u_0 + z) \]

is increasing in $z$ or equivalently,
\[ C_z(e', u, u_0) - C_z(e, u, u_0) \geq 0 \quad (17) \]
for all \((u, u_0)\). Our results will consist of various ways of guaranteeing that (17) holds. Our first result makes no demands on \(u\), but a fairly strong demand on \(G\).

**Proposition 6.** Assume that increasing \(e\) shifts payments in a first-order stochastic dominance sense. That is, assume that for all \(e' > e\) and for all \((u, u_0)\)

\[ G(p|e, u, u_0) - G(p|e', u, u_0) \geq 0. \]

Then, \(C(e, u + z, u_0 + z)\) is supermodular in \(e\) and \(z\).

This follows immediately from Proposition 1, since \(\frac{1}{u'(p)}\) is an increasing function and so has an expectation which increases if an increase in effort increases pay in the sense of first-order stochastic dominance.

First-order stochastic dominance is fairly intuitive when (IR) is not binding, as the only point in payments above the minimum is for incentive reasons, and so when more effort is desired, these payments are likely to rise.\(^{21}\) When (IR) binds, first-order stochastic dominance becomes less intuitive, as a high effort contract may make low payment outcomes more likely. Imposing a restriction on preferences allows for a more permissive condition on \(G\).

**Proposition 7.** Assume that \((i)\) \(\frac{1}{u'(p)}\) is a convex function of \(u(p)\), \((ii)\) for all \(e' > e\),

\[ \int_{p^*}^{\infty} u(p) dG(p|e', u, u_0) \geq \int_{p^*}^{\infty} u(p) dG(p|e, u, u_0), \]

and \((iii)\) \(G(p|e', u, u_0) - G(p|e, u, u_0)\) crosses zero at most once, and does so from above. Then, \(C(e, u + z, u_0 + z)\) is supermodular in \(e\) and \(z\).

Recall that condition \((i)\) also played a role in Proposition 4. This condition says that the marginal cost to the principal of providing a util to the agent at \(p\) is a convex function of the utility provided at \(p\). Under condition \((ii)\), as the induced \(e\) is increased, the agent is no worse off before effort costs are accounted for.\(^{22}\) Condition \((iii)\) essentially says that under \(e'\), rewards are “higher powered”, in that both relatively high payments and relatively low payments become more likely (unless the two distributions do not cross at all). Intuitively, since the distribution of payments given \(e'\) is more spread-out than given \(e\), the distribution of \(u(p)\) is also more spread out. Thus, the expectation of the convex function \(\frac{1}{u'(p)}\) of \(u(p)\) goes up similar to second-order stochastic dominance.

Finally, with a somewhat more restrictive conditions on \(u\), a less restrictive condition on \(G\) is possible, allowing distributions over pay for different effort levels to cross any number of times.

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\(^{21}\) As \(e\) changes, the informativeness of one signal versus another can also change, which can overwhelm the general need to provide stronger incentives.

\(^{22}\) This is automatic if the (IR) constraint is binding and \(c(e)\) is increasing.
Proposition 8. Assume that (i) \( \frac{1}{u'(p)} \) is convex in \( p \) and (ii) that for \( e' > e \)
\[
\int_t^\infty \left[ G(p|e, u, u_0) - G(p|e', u, u_0) \right] dp \geq 0
\]
for all \( t \). Then, \( C(e, u + z, u_0 + z) \) is supermodular in \( e \) and \( z \).

Convexity of \( \frac{1}{u'} \) in \( p \) is more demanding than convexity in \( u(p) \).\textsuperscript{23} Equivalent to (ii) is that the distribution for \( -p \) given \( e' \) is second-order stochastically dominated by the distribution for \( -p \) given \( e \). Hence, we are again asking that the agent be paid more and be exposed to more risk when he works harder, but in a weaker sense than before. Then, since \( \frac{1}{u'} \) is assumed convex, its expected value rises when effort goes up, leading to supermodularity.

4.2.1. Discussion

Using properties of the pay-distribution \( G \) certainly does not follow the traditional approach to comparative statics. Indeed, \( G \) is an endogenous construct relying on both \( P(\cdot|e) \) and the optimal contract. However, we think this approach also offers some significant advantages. To see this, it is useful to compare the results in this subsection to our related work in Kadan and Swinkels \[11\] (hereafter KS).

In KS we study the effect of changing the minimum pay on the implemented effort level. Different from here, in KS we rely heavily on the FOA, which gives rise to a full characterization of the optimal contract. Using this characterization, we can provide “traditional” comparative statics results in which we provide sufficient conditions on the primitives of the model implying that an increase in minimum pay results in lower implemented effort. Specifically, Proposition 4 in KS broadly takes an approach similar to Propositions 6, 7, and 8 here. It provides sufficient conditions on the statistical structure of the model implying that when the (IR) constraint is not binding, an optimal contract associated with high effort either never crosses or single crosses an optimal contract associated with low effort. Such single crossing can be either from below or from above implying that the pay distribution of high effort contracts can be either more or less spread-out than that of low effort contracts. Then, we match these conditions with assumptions on the utility function, requiring that \( \frac{1}{u'} \) be either convex or concave. Applying Proposition 1 (which is straightforward when the FOA holds), we obtain a conclusion similar to that obtained here.

The first advantage of the approach taken in this paper is, of course, that we dispense with the onerous structure needed to justify the FOA. So, for example, we do not need MLRP, CDFC or indeed any particular order structure on signals. A second advantage can be seen by considering the details of Proposition 4 in KS. The proposition prescribes conditions on the distribution function which are stated in terms of what are essentially super-modularity restrictions on the likelihood ratios of the density of signals for different effort levels. The economic interpretation of these conditions is not always straightforward. By contrast, the conditions on \( G \) used in Propositions 6, 7, and 8 are easily interpreted in terms of familiar first- and second-order stochastic dominance, albeit on an object which, because it includes the optimal contract, is further removed from the primitives of the model. Furthermore, the conditions used in KS often

\textsuperscript{23} It is equivalent to prudence being less than twice absolute risk aversion (recall Footnote 15). It is also equivalent to condition \textit{UI}" in Kadan and Swinkels \[11\]. For CRRA utilities this requires relative risk aversion at least 1 (log utility).
lead to first- and second-order stochastic dominance of the pay-distribution.\footnote{For example, Proposition 4 (and in particular Lemma 7) in KS allows for cases in which a high effort contract pays more than a low effort contract for any signal, implying that the pay distribution of the high effort dominates that of the low effort by first-order stochastic dominance.} Thus, the results presented here using the pay-distribution are helpful, since different sets of (harder to interpret) sufficient conditions on primitives lead to these (easier to interpret) properties of the pay distribution.

5. Conclusion

We study a moral hazard problem that has been stripped of the restrictive structure normal to such analysis. In particular, we dispense entirely with the first-order approach. We find a central role for $E(\frac{1}{\pi})$, the expectation of the marginal cost to the principal of providing a util to the agent, showing that it is the shadow value of tightening the individual rationality and minimum payment constraints simultaneously.

The analysis requires attention to the fact that it is not \textit{a priori} obvious that the value of the optimization problem faced by the principal is sufficiently differentiable so that a shadow value is well defined. We show that a sufficient condition for this problem not to arise is that the effort level in question can be implemented by some bounded contract (although the optimal contract may well be unbounded). This condition – bounded implementability – is mild, and is satisfied in a variety of cases in which the first-order approach cannot be applied.

The fact that this shadow value has a simple form even in this general environment has a number of applications. We show first that $E(\frac{1}{\pi})$ is closely related to the effect of a change in the wealth of an agent on the well-being of the principal, building on and extending results by Thiele and Wambach [20] and highlighting that it matters whether the change in the agent’s wealth affects only his outside option or also the amount that the agent can be penalized in the event of a bad outcome.

The behavior of $E(\frac{1}{\pi})$ as effort increases also tells us a great deal about how the marginal cost of effort is affected by changes in the minimum payment and in the outside option of the agent. In particular, we find simple properties on how the distribution over pay changes with induced effort implying that an increase in the minimum pay or outside option of the agent will lead to a higher marginal cost to the principal of inducing effort.

Appendix A. Proofs

\textbf{Proof of Lemma 1.} Fix $e$. Assume that for some $(\hat{u}, \hat{u}_0)$ there is a contract $\hat{\pi}$ implementing $e$ and $\hat{b} < \infty$ for which $\hat{\pi}(x) \leq \hat{b}$ for all $x$. For any $\tau \in [0, \infty)$ define $\tilde{b} < \infty$ by $u(\tilde{b}) = u(\hat{b}) + \tau$. Then, the contract $\pi$ defined by $u(\pi(x)) = u(\hat{\pi}(x)) + \tau$ is bounded by $\tilde{b}$ and implements $e$ given $(\hat{u} + \tau, \hat{u}_0 + \tau)$. Of course, a contract which implements $e$ for $(u, u_0)$ automatically also implements $e$ for any lower $(\hat{u}, \hat{u}_0)$. So, any given $e$ can either be implemented with a bounded contract for all $(u, u_0)$ or none. \hfill $\Box$

\textbf{Proof of Proposition 1.} We first prove the following elementary lemma, which establishes a sufficient condition for differentiability of a real-valued function.
Lemma 2. Let $Z$ be an open subset of $\mathbb{R}$, and let $h: Z \to \mathbb{R}$ be continuous, and let $H: Z \to \mathbb{R}$ be continuous and satisfy that at any $z \in Z$,
\begin{equation}
\limsup_{\varepsilon \downarrow 0} \frac{H(z + \varepsilon) - H(z)}{\varepsilon} \leq h(z),
\end{equation}
and
\begin{equation}
\liminf_{\varepsilon \downarrow 0} \frac{H(z) - H(z - \varepsilon)}{\varepsilon} \geq h(z).
\end{equation}
Then, $H$ is differentiable on $Z$, with $H'(z) = h(z)$ for all $z \in Z$.

Proof. Suppose there is $z_0 \in Z$ for which either $H(z_0)$ is not differentiable or that $H'(z_0) \neq h(z_0)$. Then, by (19) and (20) there is an $a > 0$ such that either
\begin{equation}
\liminf_{\varepsilon \downarrow 0} \frac{H(z_0 + \varepsilon) - H(z_0)}{\varepsilon} < h(z_0) - a
\end{equation}
or
\begin{equation}
\limsup_{\varepsilon \downarrow 0} \frac{H(z_0) - H(z_0 - \varepsilon)}{\varepsilon} > h(z_0) + a.
\end{equation}
Assume wlog that the first inequality holds. By continuity of $h(z)$, one can choose $\delta > 0$ such that for $z \in (z_0, z_0 + \delta)$, $h(z) > h(z_0) - a$. Choose $z_1 \in (z_0, z_0 + \delta)$ such that
\begin{equation}
\frac{H(z_1) - H(z_0)}{z_1 - z_0} < h(z_0) - a.
\end{equation}
Denote the LHS of (21) by $B$, and let
\[ \phi(z) = H(z) - Bz. \]
Then $\phi(\cdot)$ is continuous and satisfies $\phi(z_0) = \phi(z_1)$. Let $z_2$ be the point at which $\phi(\cdot)$ is minimized in $(z_0, z_1]$. Such a $z_2$ exists since $z = z_0$ minimizes $\phi(\cdot)$ in $[z_0, z_1]$, then so does $z_1$. Now, for all $z \in (z_0, z_2)$,
\begin{equation}
\frac{H(z_2) - H(z)}{z_2 - z} \leq \frac{H(z_2) - \phi(z_2) - Bz}{z_2 - z} \leq \frac{H(z_2) - \phi(z_2) - Bz}{z_1 - z_0} = B.
\end{equation}
Thus, rewriting $z$ as $z_2 - \varepsilon$,
\begin{equation}
\liminf_{\varepsilon \downarrow 0} \frac{H(z_2) - H(z_2 - \varepsilon)}{\varepsilon} \leq B < h(z_0) - a < h(z_2),
\end{equation}
where the first inequality follows from (22), the second from (21), and the third from the fact that $z_2 \in (z_0, z_0 + \delta)$. But this contradicts (20). □

Fix a boundedly implementable $e$, and $u$ and $u_0$. For any $z$ (positive or negative), define the contract $\hat{\pi}(x, z)$ implicitly by
\begin{equation}
u(\hat{\pi}(x, z)) = u(\pi(x, e, u, u_0)) + z.
\end{equation}

Lemma 3. There is an interval $Z = (z_l, z_h)$ with $z_l < 0$ and $z_h > 0$ such that for $z \in Z$, $\hat{\pi}(x, z)$ is well defined and $\hat{\pi}(\cdot, z)$ is $P(\cdot | e)$-integrable. On $Z$, $\hat{\pi}(x, z)$ is differentiable in $z$ and convex in $z$ for each $x$, with $\hat{\pi}_z(x, z) = \frac{1}{u(\hat{\pi}(x, z))} > 0$ and $\hat{\pi}_z(x, 0) = \frac{1}{u(\pi(x, e, u, u_0))}$.

\[ ^{25} \text{We thank an anonymous referee for providing the proof presented below. This proof is more direct than our original proof.} \]
Proof. Since lim\textsubscript{w→∞} u(w) = ∞, \( \hat{\pi}(x, z) \) is well defined for any \( z > \tilde{u} - u \), where we note that \( \tilde{u} - u < 0 \) by assumption. For any such \( z \), \( \hat{\pi}(x, z) > p^\# > -\infty \). Hence, \( \hat{\pi}(x, z) \) is \( P(\cdot|e) \)-integrable if \( \int \hat{\pi}(x, z)P(x|e)\,dx < \infty \). Since \( C(e, u, u_0) \) is finite and \( \hat{\pi}(x, z) \) is increasing in \( z \), this is trivially true for \( z < 0 \). Consider the sequence \( z_k = \frac{1}{2^k} \). Then, as \( k \to \infty \), \( \hat{\pi}(x, z_k) \) monotonically decreases to \( \hat{\pi}(x, 0) \). Hence, since \( C(e, u, u_0) \) is finite, and using Lebesgue’s monotone convergence theorem, \( \int \hat{\pi}(x, z_k)P(x|e)\,dx \) is finite for some \( k' < \infty \). But then, \( \hat{\pi}(x, z) \) is integrable for all \( z < \frac{1}{2^k'} \).

Differentiating (23) gives

\[
\hat{\pi}_z(x, z) = \frac{1}{u'(\hat{\pi}(x, z))} > 0,
\]

(24)

and

\[
\hat{\pi}_{zz}(x, z) = -\frac{u''(\hat{\pi}(x, z))}{u'(\hat{\pi}(x, z))^3} > 0.
\]

(25)

Moreover, since \( \hat{\pi}(x, 0) = \pi(x, e, u, u_0) \), we have

\[
\hat{\pi}_z(x, 0) = \frac{1}{u'(\pi(x, e, u, u_0))}.
\]

(26)

The next lemma is necessary since the usual conditions for Leibnitz’s rule may fail here.

Lemma 4. For all \( z \in \mathbb{Z} \), \( \frac{1}{u'(\hat{\pi}(x, z))} \) is \( P(\cdot|e) \)-integrable, and \( \hat{C}(z) = \int \hat{\pi}(x, z)\,dP(x|e) \) is differentiable with

\[
\hat{C}_z(z) = \int \frac{1}{u'(\hat{\pi}(x, z))} \,dP(x|e).
\]

Proof. Fix some \( z \in \mathbb{Z} \), and let \( \varepsilon_k \downarrow 0 \) be any decreasing sequence such that

\[
\tau \equiv \lim_{k \to \infty} \int \frac{\hat{\pi}(x, z + \varepsilon_k)P(x|e) - \hat{\pi}(x, z)P(x|e)}{\varepsilon_k} \,dP(x|e)
\]

is well defined. Since \( \hat{\pi}_z(x, z) > 0, \tau \geq 0 \), but we have not yet ruled out that \( \tau = \infty \). By Lemma 3, \( \hat{\pi}(x, z) \) is convex in \( z \). Hence, for each \( x \), \( \frac{\hat{\pi}(x, z + \varepsilon_k) - \hat{\pi}(x, z)}{\varepsilon_k} \) is decreasing in \( k \), and so, by Lebesgue’s monotone convergence theorem,

\[
\tau = \lim_{k \to \infty} \int \frac{\hat{\pi}(x, z + \varepsilon_k) - \hat{\pi}(x, z)}{\varepsilon_k} \,dP(x|e)
\]

\[
= \int \lim_{k \to \infty} \frac{\hat{\pi}(x, z + \varepsilon_k) - \hat{\pi}(x, z)}{\varepsilon_k} \,dP(x|e)
\]

\[
= \int \hat{\pi}_z(x, z) \,dP(x|e)
\]

\[
= \int \frac{1}{u'(\hat{\pi}(x, z))} \,dP(x|e).
\]

Fix any \( k' \) large enough that \( z + \varepsilon_{k'} \in \mathbb{Z} \). Then, by Lemma 3 \( \int \hat{\pi}(x, z + \varepsilon_{k'})P(x|e)\,dx \) and \( \int \hat{\pi}(x, z)\,dP(x|e) \) are each finite, and thus so is

\[
\int \frac{\hat{\pi}(x, z + \varepsilon_{k'}) - \hat{\pi}(x, z)}{\varepsilon_{k'}} \,dP(x|e).
\]
But then, since \( \int \frac{\hat{\pi}(x,z_\varepsilon) - \hat{\pi}(x,z)}{\varepsilon} \, dP(x|e) \) is decreasing in \( k \), \( \tau < \infty \). Thus \( \frac{1}{u'(\hat{\pi}(x,z))} \) is \( P(\cdot|e) \)-integrable. And, since \( \varepsilon_k \) was any sequence along which \( \int_{\varepsilon_k} \hat{\pi}(x,z) \, dP(x|e) - \int_{\varepsilon_k} \hat{\pi}(x,0) \, dP(x|e) \) converged in the non-negative extended reals, it follows that

\[
\lim_{\varepsilon \downarrow 0} \int_{\varepsilon} \hat{\pi}(x,z_\varepsilon) \, dP(x|e) - \int_{\varepsilon} \hat{\pi}(x,0) \, dP(x|e)
\]

exists and is equal to \( \int_{\varepsilon} \frac{1}{u'(\hat{\pi}(x,z))} \, dP(x|e) \).

Similarly, consider any decreasing sequence \( \varepsilon_k \downarrow 0 \) such that

\[
\hat{\tau} = \lim_{k \to \infty} \int_{\varepsilon_k} \hat{\pi}(x,z) \, dP(x|e) - \int_{\varepsilon_k} \hat{\pi}(x,z - \varepsilon_k) \, dP(x|e)
\]

is well defined, again in the non-negative extended reals. By convexity, \( \hat{\pi}(x,z) - \hat{\pi}(x,z - \varepsilon_k) \) is increasing in \( k \) with pointwise limit \( \hat{\pi}_z(x,z) = \left[ \frac{1}{u'(\hat{\pi}(x,z))} \right] \). As we have already established that \( u'(\hat{\pi}(x,z)) \) is \( P(\cdot|e) \)-integrable, it follows again from Lebesgue’s monotone convergence theorem that \( \hat{\tau} = \int \frac{1}{u'(\hat{\pi}(x,z))} \, dP(x|e) \). Hence, \( \hat{C}_z(z) \) exists and is equal to \( \int \frac{1}{u'(\hat{\pi}(x,z))} \, dP(x|e) \). □

By construction, for \( z \in Z \), \( \hat{\pi}(\cdot,z) \) satisfies (IR) for \( u_0 + z \) and (MP) for \( u + z \) (see (23)). Furthermore, \( \hat{\pi}(\cdot,z) \) satisfies (IC) since we have added a constant to utility at each outcome, leaving the relative attractiveness of any two effort levels unchanged. Thus, \( \hat{\pi}(\cdot,z) \) is feasible at \( (u_0 + z, u_0 + z) \), and so

\[
C(e,u + z, u_0 + z) \leq \hat{C}(z)
\]

for all \( z \in Z \). Also, since \( \hat{\pi}(x,0) = \pi(x,e,u,u_0) \) we have

\[
C(e,u,u_0) = \hat{C}(0),
\]

and thus for \( \varepsilon > 0 \) small enough

\[
\frac{C(e,u + \varepsilon, u_0 + \varepsilon) - C(e,u,u_0)}{\varepsilon} \leq \frac{\hat{C}(\varepsilon) - \hat{C}(0)}{\varepsilon},
\]

and so, taking \( \varepsilon \downarrow 0 \) and using Lemma 4 gives

\[
\limsup_{\varepsilon \downarrow 0} \frac{C(e,u + \varepsilon, u_0 + \varepsilon) - C(e,u,u_0)}{\varepsilon} \leq \hat{C}_z(0) = \int \frac{1}{u'(p)} \, dG(p|e,u,u_0).
\]

Similarly,

\[
\liminf_{\varepsilon \downarrow 0} \frac{C(e,u,u_0) - C(e,u - \varepsilon, u_0 - \varepsilon)}{\varepsilon} \geq \hat{C}_z(0) = \int \frac{1}{u'(p)} \, dG(p|e,u,u_0).
\]

And, since \( u \) and \( u_0 \) were arbitrary, one can in particular consider the case where \( u \) is replaced by \( u + z \) and \( u_0 \) is replaced by \( u_0 + z \) to conclude that

\[
\limsup_{\varepsilon \downarrow 0} \frac{C(e,u + z + \varepsilon, u_0 + z + \varepsilon) - C(e,u + z, u_0 + z)}{\varepsilon} \leq \int \frac{1}{u'(p)} \, dG(p|e,u + z, u_0 + z)
\]
and
\[
\liminf_{\varepsilon \downarrow 0} \frac{C(e, u + z, u_0 + z) - C(e, u + z - \varepsilon, u_0 + z - \varepsilon)}{\varepsilon} \geq \int \frac{1}{u'(p)} dG(p|e, u + z, u_0 + z).
\]

Since \(\pi(x, e, u, u_0) = \hat{\pi}(x, 0)\), by Lemma 4, \(\frac{1}{u'(p)}\) is \(G(\cdot|e, u, u_0)\)-integrable. Thus, if both \(C(e, u + z, u_0 + z)\) and \(\int \frac{1}{u'(p)} dG(p|e, u + z, u_0 + z)\) are continuous in \(z\) then we can rely on Lemma 2 by setting \(H(z) = C(e, u + z, u_0 + z)\) and \(h(z) = \int \frac{1}{u'(p)} dG(p|e, u + z, u_0 + z)\) to conclude that \(C(e, u + z, u_0 + z)\) is differentiable in \(z\), with
\[
C_z(e, u, u_0) = \int \frac{1}{u'(p)} dG(p|e, u, u_0),
\]
which would complete the proof of Proposition 1.26

We first show the continuity of \(C(e, u, u_0, u_0)\). This follows from the concavity of \(u\).

**Lemma 5.** \(C(e, u, u_0)\) is continuous in \((u, u_0)\).

**Proof.** Fix \((u, u_0)\). Note first that for \(z > 0\),
\[
C(e, u, u_0) \leq C(e, u + z, u_0 + z) \leq \hat{C}(z),
\]
where \(\hat{C}(z)\) is continuous, and where by definition, \(C(e, u, u_0) = \hat{C}(0)\). Thus,
\[
\lim_{z \downarrow 0} C(e, u + z, u_0 + z) = C(e, u, u_0). \quad (30)
\]

For \(z > 0\), let \(\tilde{\pi}^z\) be the contract which for each \(x\) gives the agent the certainty equivalent of a lottery putting probability \(\frac{1}{2}\) on \(\pi(x, e, u - z, u_0 - z)\) and \(\frac{1}{2}\) on \(\hat{\pi}(x, z)\). Then, \(u(\tilde{\pi}^z(x)) \geq \frac{1}{2} (u - z) + \frac{1}{2} (u + z) = u\), and
\[
\int u(\tilde{\pi}^z(x)) dP(x|e) - c(e) \geq \frac{1}{2} (u_0 - z) + \frac{1}{2} (u_0 + z) = u_0.
\]

Also, since the agent finds \(e\) optimal relative to both \(\pi(x, e, u - z, u_0 - z)\) and \(\hat{\pi}(x, z)\), \(e\) is a *fortiori* optimal when the agent effectively faces a lottery between the two contracts. Hence \(\tilde{\pi}^z\) is feasible for \((e, u, u_0)\) and so
\[
C(e, u, u_0) \leq \frac{1}{2} C(e, u - z, u_0 - z) + \frac{1}{2} \hat{C}(z),
\]
or, taking limits as \(z \downarrow 0\),
\[
C(e, u, u_0) \leq \frac{1}{2} \lim_{z \downarrow 0} C(e, u - z, u_0 - z) + \frac{1}{2} \hat{C}(0),
\]

26 Note that since \(C(e, u + z, u_0 + z)\) is enveloped from above by \(\hat{C}(z)\), and the latter is continuously differentiable in \(z\), only downward kinks are possible in \(C(e, u + z, u_0 + z)\) considered as a function of \(z\). Thus, even without weak continuity and bounded implementability we would have that \(\int \frac{1}{u'(p)} dG(p|e, u, u_0)\) remains a supergradient of \(C(e, u, u_0)\).
from which, since \( C(e, u, u_0) = \dot{C}(0) \), we have that \( \lim_{z \downarrow 0} C(e, u - z, u_0 - z) \geq C(e, u, u_0) \).

But, for \( z > 0 \), \( C(e, u - z, u_0 - z) \leq C(e, u, u_0) \), and so

\[
\lim_{z \downarrow 0} C(e, u - z, u_0 - z) = C(e, u, u_0). \tag{31}
\]

Now, let \((u^k, u_0^k) \to (u, u_0)\). Then, letting \( \varepsilon^k = \max(|u - u^k|, |u_0 - u_0^k|) \),

\[
C(e, u - \varepsilon^k, u_0 - \varepsilon^k) \leq C(e, u^k, u_0^k) \leq C(e, u + \varepsilon^k, u_0 + \varepsilon^k),
\]

and so from (30) and (31), \( C(e, u^k, u_0^k) \to C(e, u, u_0) \).

Next, we turn to the continuity of \( \int \frac{1}{u'(p)} dG(p|e, u + z, u_0 + z) \). This would be straightforward if we knew that optimal contracts are bounded. However, all we know is that contracts are bounded from below and that the relevant effort levels are boundedly implementable. The following lemma establishes the key implication of bounded implementability. It shows that for any given \( \tau \), uniformly across contracts optimally implementing \( e \) for \((u, u_0)\) below \( \tau \), the part of the expectation of \( \frac{1}{u'} \) that can be attributed to large payments becomes trivial.

**Lemma 6.** For any \( \tau > u_* \), and \( \delta > 0 \). Then, there exists \( r < \infty \) such that for any \((u, u_0) \leq (\tau, \tau)\) with \( u > u_* \),

\[
\int_{\pi(x; e, u, u_0) > r} \frac{1}{u'(\pi(x; e, u, u_0))} dP(x|e) \leq \delta.
\]

**Proof.** Let \( \pi^\tau \) be a bounded contract implementing \( e \) for \((u, u_0) = (\tau, \tau)\), and let \( b^\tau \) be an associated bound. Fix \((\hat{u}, \hat{u}_0) \leq (\tau, \tau)\) with \( \hat{u} > u_* \), and consider \( \pi(\cdot, e, \hat{u}, \hat{u}_0) \). Since \((\hat{u}, \hat{u}_0) \leq (\tau, \tau)\), \( \pi^\tau \) is a feasible contract for \((e, \hat{u}, \hat{u}_0)\). Write \( \hat{u}(x) \) for \( u(\pi(x; e, \hat{u}, \hat{u}_0)) \), and \( u^\tau(x) \) for \( u(\pi^\tau(x)) \). Consider the contract \( \pi(\cdot, \alpha) \) defined for \( \alpha \in [0, 1] \) by

\[
u(\pi(x, \alpha)) = (1 - \alpha)\hat{u}(x) + \alpha u^\tau(x). \tag{32}\]

Then, as before, \( \pi(x, \alpha) \) implements \( e \) given \((\hat{u}, \hat{u}_0)\).

Denote \( \gamma(\cdot) = u^{-1}(\cdot) \). Since \( (\cdot, e, \hat{u}, \hat{u}_0) \) is optimal, it must be weakly cheaper than \( \pi(x, \alpha) \):

\[
\int [\gamma(\hat{u}(x) + \alpha(u^\tau(x) - \hat{u}(x))) - \gamma(\hat{u}(x))] dP(x|e) \geq 0. \tag{33}\]

Break this up as

\[
\int_{\hat{u}(x) > u^\tau(x)} [\gamma(\hat{u}(x) + \alpha(u^\tau(x) - \hat{u}(x))) - \gamma(\hat{u}(x))] dP(x|e)
\]

\[
+ \int_{\hat{u}(x) \leq u^\tau(x)} [\gamma(\hat{u}(x) + \alpha(u^\tau(x) - \hat{u}(x))) - \gamma(\hat{u}(x))] dP(x|e).
\]

The first integrand is negative. So, if we restrict the domain of integration of the first integral to \{\( \hat{u}(x) > u(r) \)\} for some \( r > b^\tau \), then we have weakly increased the sum. The second integrand, using convexity of \( \gamma(\cdot) \), and the fact that \( b^\tau \) is an upper bound for \( \pi^\tau \), is at most

\[
\alpha \gamma'(u(b^\tau))(u^\tau(x) - \hat{u}(x)) \leq \alpha \frac{u(b^\tau) - \hat{u}}{u'(b^\tau)} < \alpha \frac{u(b^\tau) - u_*}{u'(b^\tau)}
\]
and so the second integral is at most $\alpha \frac{u(b^\tau) - u_*}{u'(b^\tau)}$. Thus, by (33)
\[
\int_{\hat{u}(x) > u(r)} \left[ \gamma\left(\hat{u}(x) + \alpha\left(u^\tau(x) - \hat{u}(x)\right)\right) - \gamma\left(\hat{u}(x)\right) \right] dP(x|e) + \alpha \frac{u(b^\tau) - u_*}{u'(b^\tau)} \geq 0,
\]

or
\[
\int_{\hat{u}(x) > u(r)} \left[ \gamma\left(\hat{u}(x) \right) - \gamma\left(\hat{u}(x) + \alpha\left(u^\tau(x) - \hat{u}(x)\right)\right) \right] dP(x|e) \leq \alpha \frac{u(b^\tau) - u_*}{u'(b^\tau)}. \tag{34}
\]

But, again by convexity of $\gamma$, for $\hat{u}(x) > u(r) > u(b^\tau)$,
\[
\gamma\left(\hat{u}(x) \right) - \gamma\left(\hat{u}(x) + \alpha\left(u^\tau(x) - \hat{u}(x)\right)\right) \\
\geq \alpha \left(\hat{u}(x) - u^\tau(x)\right) \gamma'\left[\hat{u}(x) + \alpha\left(u^\tau(x) - \hat{u}(x)\right)\right] \\
\geq \alpha \left(u(r) - u(b^\tau)\right) \gamma'\left(\hat{u}(x) + \alpha\left(u^\tau(x) - \hat{u}(x)\right)\right).
\]

Substituting into (34) and dividing by $\alpha[u(r) - u(b^\tau)]$, we have that
\[
\int_{\hat{u}(x) > u(r)} \gamma'\left(\hat{u}(x) + \alpha\left(u^\tau(x) - \hat{u}(x)\right)\right) dP(x|e) \leq \frac{1}{u(r) - u(b^\tau)} \frac{u(b^\tau) - u_*}{u'(b^\tau)}. \tag{35}
\]

Now, as $\alpha \downarrow 0$, $\gamma'\left(\hat{u}(x) + \alpha\left(u^\tau(x) - \hat{u}(x)\right)\right)$ increases monotonically to
\[
\gamma'\left(\hat{u}(x)\right) = \frac{1}{u'(\pi(x,e,\hat{u},\hat{u}_0))}
\]
and so by Lebesgue’s monotone convergence theorem as $\alpha \downarrow 0$, the LHS of (35) converges to
\[
\int_{\hat{u}(x) > u(r)} \frac{1}{u'(\pi(x,e,\hat{u},\hat{u}_0))} dP(x|e).
\]

Since $\pi(x,e,\hat{u},\hat{u}_0) > r$ if and only if $\hat{u}(x) > u(r)$, we thus have
\[
\int_{\pi(x,e,\hat{u},\hat{u}_0) > r} \frac{1}{u'(\pi(x,e,\hat{u},\hat{u}_0))} dP(x|e) \leq \frac{1}{u(r) - u(b^\tau)} \frac{u(b^\tau) - u_*}{u'(b^\tau)}.
\]

Note that the RHS is independent of $(\hat{u},\hat{u}_0)$, and that since $\lim_{r \to \infty} u(r) = \infty$, the RHS converges to 0 in $r$. □

The continuity of $\int \frac{1}{u'(p)} dG(p|e, u, u_0)$ now follows from Lemma 6 and the weak continuity of $G$ as shown in the next lemma.

**Lemma 7.** $\int \frac{1}{u'(p)} dG(p|e, u, u_0)$ is continuous in $(u, u_0)$.

**Proof.** If $\frac{1}{u'(p)}$ is bounded, then the claim follows immediately from the weak continuity of $G$ and the definition of weak convergence. Assume that $\frac{1}{u'(p)}$ is not bounded, and that the claim in the proposition is false. Then, there is $\tau > u_*$, $(u, u_0)$, a sequence $(u^k, u_0^k) \to (u, u_0)$ with $(u^k, u_0^k) \leq (\tau, \tau)$, and $\delta > 0$ such that
\[ \lim_{k \to \infty} \int_{p_*}^\infty \frac{1}{u'(p)} \, dG(p \mid e, u^k, u_0^k) > \int_{p_*}^\infty \frac{1}{u'(p)} \, dG(p \mid e, u, u_0) + 2\delta \]

(an inequality in the other direction is ruled out by the portmanteau theorem).\(^{27}\) Using Lemma 6, pick \(r\) such that for all \(k\),

\[ \int_{p > r} \left( \frac{1}{u'(p)} - \frac{1}{u'(r)} \right) \, dG(p \mid e, u^k, u_0^k) \leq \int_{p > r} \frac{1}{u'(p)} \, dG(p \mid e, u^k, u_0^k) \leq \delta. \quad (36) \]

Then

\[ \lim_{k \to \infty} \int_{p_*}^\infty \frac{1}{u'(p)} \, dG(p \mid e, u^k, u_0^k) = \lim_{k \to \infty} \int_{p_*}^\infty \min \left( \frac{1}{u'(p)} \right) \, dG(p \mid e, u^k, u_0^k) \]

\[ + \lim_{k \to \infty} \int_{p > r} \left( \frac{1}{u'(p)} - \frac{1}{u'(r)} \right) \, dG(p \mid e, u^k, u_0^k) \]

\[ \leq \int_{p_*}^\infty \min \left( \frac{1}{u'(p)} \right) \, dG(p \mid e, u, u_0) + \delta \]

\[ \leq \int_{p_*}^\infty \frac{1}{u'(p)} \, dG(p \mid e, u, u_0) + \delta, \]

where the replacement of the second term in the first inequality follows from (36), and the replacement of the first term in the first inequality follows from the weak continuity of \(G\) since \(\min(\frac{1}{u'(p)} \land \frac{1}{u'(r)})\) is a bounded continuous function. This is a contradiction. \(\Box\)

This completes the proof of Proposition 1. \(\Box\)

Proof of Proposition 2. Let \(e \in E\). We will show that \(e\) can be implemented using a bounded contract in the form of a step function. Denote \(\hat{u} = \max(u_0, u)\) and for each \(\Delta \geq 0\) consider contract \(\pi(x; \Delta)\) that gives utility \(\hat{u}\) for \(x \notin \hat{X}\) and \(\hat{u} + \Delta\) for \(x \in \hat{X}\). Trivially \(\pi(x; \Delta)\) satisfies (IR) and (MP). And, choosing

\[ \Delta_0 = \frac{c'(e)}{P_e(\hat{X} \mid e)} > 0, \quad (37) \]

it is clear from the concavity of \(P(\hat{X} \mid e)\) in \(e\) and the convexity of \(c(e)\) in \(e\) that (IC) is satisfied. \(\Box\)

Proof of Proposition 5. Define \(z(\varepsilon) = u(m + w_0 + \varepsilon) - u(m + w_0)\). Then, for \(\varepsilon > 0\)

\[ C(e, u(m + w_0 + \varepsilon), u(w_0 + \varepsilon)) \geq C(e, u(m + w_0) + z(\varepsilon), u(w_0) + z(\varepsilon)) \]

\(^{27}\) The relevant part of the portmanteau theorem is that if a sequence of probability measures \(P_n\) weakly converges to a probability measure \(P\), then \(\liminf_{n \to \infty} \int f \, dP_n \geq \int f \, dP\) for all lower semi-continuous functions \(f\) which are bounded from below. See, for example, van der Vaart and Wellner [21, Theorem 1.3.4].
since \( u \) is concave and \( m \geq 0 \). Thus,

\[
\liminf_{\varepsilon \downarrow 0} \frac{C^T(e, m, w_0 + \varepsilon) - C^T(e, m, w_0)}{\varepsilon} \\
\geq \liminf_{\varepsilon \downarrow 0} \frac{C(e, u(m + w_0) + z(\varepsilon), u(w_0) + z(\varepsilon)) - C(e, u(m + w_0), u(w_0))}{z(\varepsilon)} \frac{z(\varepsilon)}{\varepsilon} - 1 \\
= C_z(e, u(m + w_0), u(w_0)) z'(0) - 1.
\]

But, \( z'(0) = u'(m + w_0) \), and so as in main text, the last expression is non-negative. We conclude that \( C^T(e, m, w_0) \) increases in \( w_0 \).

**Proof of Proposition 7.** Let us show first that for all \((e, u, u_0)\),

\[
C_z(e, u, u_0) = \frac{1}{u'(p_*)} + \int_{p_*}^{\infty} \left[ \frac{1}{u'(p)} \right]' \left[ 1 - G(p|e, u, u_0) \right] dp. \tag{38}
\]

To see this, note by Proposition 1 that \( C_z(e, u, u_0) = \lim_{r \to \infty} \int_{p_*}^{r} \frac{1}{u'(p)} dG(p|e, u, u_0) \). As a distribution function, \( G \) is right-continuous on \([p_*, \infty)\), and so, since \( \frac{1}{u'} \) is continuous we can apply Theorem 18.4 in Billingsley [1] and integrate by parts using \( dG = -d[G(r|e, u, u_0) - G(p|e, u, u_0)] \) to get

\[
\int_{p_*}^{r} \frac{1}{u'(p)} dG(p|e, u, u_0) \\
= \frac{1}{u'(p_*)} G(r|e, u, u_0) + \int_{p_*}^{\infty} \left[ \frac{1}{u'(p)} \right]' \left[ G(r|e, u, u_0) - G(p|e, u, u_0) \right] dp.
\]

The integrand is monotone in \( r \), and so (38) follows by Lebesgue’s monotone convergence theorem.

So, let \( e' > e \), and let

\[
\eta(p) = \frac{1}{u'(p)} > 0,
\]

noting that \( \eta(p) \) is increasing because \( \frac{1}{u'(p)} \) is a convex function of \( u(p) \). By assumption,

\[
\int_{p_*}^{\infty} u(p) dG(p|e', u, u_0) \geq \int_{p_*}^{\infty} u(p) dG(p|e, u, u_0),
\]

or, integrating by parts,\footnote{To see this, note that exactly as before,}

\[
\lim_{r \to \infty} \int_{p_*}^{r} u(p) dG(p|e', u, u_0)
\]
\[ \int_{p_*}^{\infty} u'(p) \left[ G(p|e, u_0) - G(p'|e, u_0) \right] dp \geq 0. \]

Since \([G(p|e, u_0) - G(p'|e, u_0)]\) has sign pattern \([-/+]\) in \(p\), it follows that

\[ \int_{p_*}^{\infty} u'(p) \eta(p) \left[ G(p|e, u_0) - G(p'|e, u_0) \right] dp \geq 0, \]

or equivalently,

\[ \int_{p_*}^{\infty} \left[ \frac{1}{u'(p)} \right] \left[ G(p|e, u_0) - G(p'|e, u_0) \right] dp \geq 0, \]

which, using (38) is equivalent to \(C_z(e', u, u_0) \geq C_z(e, u, u_0)\), and we are done.

**Proof of Proposition 8.** For \(s \in (-\infty, -p_*]\) define \(H(s|e') = 1 - G(-s|e', u, u_0)\) and \(H(s|e) = 1 - G(-s|e, u, u_0)\). Define \(\tau(s) = \frac{-p_*}{u'(-s)}\), and note that

\[ C_z(e', u, u_0) - C_z(e, u, u_0) = \int_{-\infty}^{-p_*} \tau(s) dH(s|e) - \int_{-\infty}^{-p_*} \tau(s) dH(s|e'). \]

Since \(\frac{1}{u'(p)}\) is increasing and convex, \(\tau(s)\) is increasing and concave, and from (18), \(H(\cdot|e)\) second-order stochastically dominates \(H(\cdot|e')\). The result is then immediate from standard results regarding second-order stochastic dominance.

**Appendix B. Primitives for existence and weak continuity of \(G\)**

In this appendix, we outline assumptions on the primitives of the model under which both existence and weak continuity of \(G\) are guaranteed. We rely heavily on Kadan, Reny, and Swinkels [13, henceforth KRS] for this, and refer the reader to that paper for details.

KRS show existence of an optimal mechanism in a general setup that allows for both moral hazard and adverse selection. The setup used here is a simple special case of theirs. Still, to conform with the full set of assumptions used in KRS we need to introduce two additional assumptions on the statistical structure of the problem. These assumptions will allow us to apply the existence and uniqueness results in KRS (Corollaries 1 and 2), and also to show that \(G\) is weakly continuous.

\[
= \lim_{r \to \infty} u(p_*) G(r|e', u_0) + \lim_{r \to \infty} \int_{p_*}^{\infty} u'(p) (G(r|e', u_0) - G(p|e', u_0)) \]

\[
= u(p_*) + \int_{p_*}^{\infty} u'(p) (1 - G(p|e', u_0)) \]

and similarly for \(e\). Note also that \(\int_{p_*}^{\infty} u'(p) (1 - G(p|e', u_0)) \leq u'(p_*) C(e', u_0)\) and so is finite.
Recall that in Section 2 we assumed that $P(\cdot|e)$ is continuous in $e$ in the weak topology. We did not, however, make any assumptions regarding the support of $P(\cdot|e)$. In particular, the support can be either fixed or vary with $e$.\footnote{In the presence of a minimum payment constraint supports can change without introducing forcing contracts.} For every $e \in E$ let $S_e$ be the support of $P(\cdot|e)$.

**Assumption 1.** For every $e, e' \in E$, $P(\cdot|e')$ is absolutely continuous with respect to $P(\cdot|e)$ on $S_e$.

Assumption 1 allows the support of signals to either be constant or to shift with $e$, but it prevents situations where a zero probability event under $P(\cdot|e)$ contained in $S_e$ has a positive probability under $P(\cdot|e')$ given some $e'$. Our next assumption imposes some additional continuity restrictions on how information varies with effort. Relying on Assumption 1, for every $e, e' \in E$ let $\xi(\cdot, e', e) : X \to \mathbb{R}$ be the Radon–Nikodym derivative of $P(\cdot|e')$ with respect to $P(\cdot|e)$ on $S_e$.

**Assumption 2.** $\xi(x, e', e)$ is lower semi-continuous whenever $e' \neq e$.

KRS (Section 2.3) discuss a variety of cases where Assumptions 1 and 2 hold. This includes the following two standard cases:

- **Density.** Assume $X$ is Euclidean and $P(\cdot|e)$ can be represented by a continuous and positive density $f(x|e)$.
- **Discrete signals.** Suppose that $X$ is a finite set and that $P(\{x\}|e)$ is continuous in $e$ for all $x \in X$.

Note that all the examples in Section 3.2 satisfy Assumptions 1 and 2. Indeed, in each of Examples 1, 3, and 4 $P(\cdot|e)$ is represented by a positive and continuous density function. And, Example 2 is equivalent to a case where $X$ is finite.

With Assumptions 1 and 2 in hand, we can apply KRS Corollary 1 to obtain existence of a cost minimizing contract for all implementable $e$. As argued before, $E^*$ is non-empty (because under the conditions given, the agent has a best response to, for example, any continuous and bounded contract). It is also easily shown (along the lines described in the proof of Proposition 9 below) that $E^*$ is a closed subset of $E$. Corollary 2 in KRS, which relies on the separability and concavity of the agent’s utility, guarantees that in our setting the cost minimizing contract for each $e$ is deterministic and unique.

We next show that Assumptions 1 and 2 along with bounded implementability of $e$ guarantee that $G$ is weakly continuous. For this we need to delve a bit more deeply into the details of KRS. For any given $(e, u, u_0)$, the (deterministic) optimal contract $\pi(\cdot, e, u, u_0)$ can be identified with a function from $X$ to the Dirac probability measure $\delta_{\pi(\cdot, e, u, u_0)}$, which for each $x \in X$ puts probability 1 on $\pi(x, e, u, u_0) \in [p_*, \infty)$. Then, for each $(e, u, u_0)$, we have that $\pi(\cdot, e, u, u_0)$ and $P(\cdot|e)$ naturally induce a joint probability measure $\mu(\cdot|e, u, u_0)$ on $X \times [p_*, \infty)$, where for any Borel sets $B \subset X$, and $C \subset [p_*, \infty)$,

$$\mu(B \times C|e, u, u_0) = \int_{x \in B} \delta_{\pi(x, e, u, u_0)}(C) dP(x|e).$$
These measures are similar in spirit to the distributional strategies introduced by Milgrom and Weber [15]. We can now establish that under the assumptions considered here, $G$ is weakly continuous.

**Proposition 9.** Assume that Assumptions 1 and 2 hold. Then, for any boundedly implementable $e \in E$, $G(\cdot|e, u, u_0)$ is continuous in $(u, u_0)$ in the weak topology.

**Proof.** Consider a boundedly implementable $e \in E$, and fix $(u, u_0)$. Note that $G(\cdot|e, u, u_0)$ is the projection of $\mu(\cdot|e, u, u_0)$ into payments. Thus, it is sufficient to show that $\mu(\cdot|e, u, u_0)$ is continuous in $(u, u_0)$ in the weak topology.

Pick $\tau > \max(u, u_0)$. Consider a sequence $(u_k, u_{0k}) \to (u, u_0)$ where $(u_k, u_{0k}) \leq (\tau, \tau)$ for all $k$. We have that $C(e, u_k, u_{0k}) \leq C(e, \tau, \tau) < \infty$. Then, Step 1 in the proof of Proposition 1 in KRS shows that the sequence of measures $\{\mu(\cdot|e, u_k, u_{0k})\}$ is tight. By Prohorov’s theorem [2, p. 57], this sequence has a cluster point $\hat{\mu}$ in the weak topology. As in Step 2 of the proof of Proposition 1 in KRS, let $\hat{\pi}$ be the contract obtained from $\hat{\mu}$ by taking a regular conditional probability. Identical arguments to those used in the proof of Steps 4 and 5 of Proposition 1 in KRS show that $\hat{\pi}$ satisfies (IC), (MP), and (IR) for $(e, u, u_0)$.

But,

$$C(e, u_k, u_{0k}) = \int_X \pi(x, e, u_k, u_{0k}) dP(x|e)$$

$$= \int_{X \times [p_*, \infty)} p d\mu((x, p)|e, u_k, u_{0k}).$$

Since payments are bounded from below the portmanteau theorem (see Footnote) implies that

$$\liminf_{k \to \infty} C(e, u_k, u_{0k}) \geq \int_{X \times [p_*, \infty)} p d\hat{\mu}((x, p))$$

$$= \int \hat{\pi}(x) dP(x|e).$$

But, since $e$ is boundedly implementable, $C(e, u, u_0)$ is continuous in $(u, u_0)$ (by Lemma 5). Hence, $\int \hat{\pi}(x) dP(x|e) \leq C(e, u, u_0)$, and so, in particular, $\hat{\pi}$ is an optimal contract given $(e, u, u_0)$. However, by Corollary 2 in KRS, the optimal contract in our setting is unique up to events of $P(\cdot|e)$-measure zero, and we are done. □

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30 Tightness here means that for all $\varepsilon > 0$ there exists a compact set $D_\varepsilon \subset X \times [p_*, \infty)$ such that for all $k$, $\mu(D_\varepsilon|e, u_k, u_{0k}) \geq 1 - \varepsilon$. Intuitively, this follows since payments are uniformly bounded from below by $p_*$ and yet expected costs are uniformly bounded from above by $C(e, \tau, \tau)$. Thus, it must be that for all $k$, $\mu(\cdot|e, u_k, u_{0k})$ assigns a small probability to large payments, implying the existence of a compact set that uniformly carries most of the probability mass.

31 The assumption that $\xi(x, e', e)$ is lower semi-continuous is used to establish (IC). This is also where we use the assumption that $\lim_{w \to \infty} u'(w) = 0$. 
Appendix C. An example of a non-boundedly implementable effort level

Let $e_k = x_k = 1 - \frac{1}{3^k}, k \in \mathbb{N}$, and let $E = X = \{1 - \frac{1}{3^k}\}_{k \in \mathbb{N}} \cup \{1\}$. For $k \in \mathbb{N}$, let $c(e_k) = -\frac{1}{2^{3k}}$, with $c(1) = 0$. For $\hat{k}$ and $k$ in $\mathbb{N}$ with $k \neq \hat{k}$ let

$$p(x_k | e_{\hat{k}}) = \frac{1}{\frac{1}{3} - \frac{1}{3^k}} \sum_{k \in \mathbb{N}\setminus \hat{k}} \frac{1}{3^k} u_k - c(e_{\hat{k}}) \leq 2 \sum_{k \in \mathbb{N}} \frac{1}{3^k} u_k$$

for which a necessary condition is

$$2 \sum_{k \in \mathbb{N}\setminus \hat{k}} \frac{1}{3^k} u_k - c(e_{\hat{k}}) \leq 2 \sum_{k \in \mathbb{N}} \frac{1}{3^k} u_k$$

or, using that $c_{\hat{k}} = -\frac{1}{2^{3\hat{k}}}$,

$$u_{\hat{k}} \geq \frac{1}{2} \left(\frac{3}{2}\right)^{\hat{k}}$$

which diverges, and thus $e = 1$ is not boundedly implementable for any weakly concave $u$. But, taking, for example, $u = \sqrt{w}$, the cost to the principal of the contract given by $\sqrt{\pi(k)} = \frac{1}{2} \left(\frac{3}{2}\right)^k$ is

$$\sum_{k \in \mathbb{N}} 2 \frac{1}{3^k} \frac{1}{4} \left(\frac{3}{2}\right)^k = \frac{1}{2}.$$

So, while $e = 1$ cannot be boundedly implemented, it can be implemented at finite expected cost.

References


