Modification of Near Set Theory

M. E. Abd El-Monsef*, H. M. Abu-Donia** and E. A. Marei*

*Department of Mathematics, Faculty of Science, Tanta University, Egypt.
**Department of Mathematics, Faculty of Science, Zagazig University, Egypt.

Abstract

Most real life situations need some sort of approximation to fit mathematical models. The beauty of using topology in approximation is achieved via obtaining approximation for qualitative subsets without coding or using assumption. The aim of this paper is to introduce different approaches to near sets by using general relations and special neighbourhoods. Some fundamental properties and characterizations are given. We obtain a comparison between these new approximations and traditional approximations introduced by Peters [17-24].

Keywords: Information system, approximation space, topological space, rough set, feature, near set, β-open set.

1. Introduction

One of the most powerful notions in system analysis is the concept of topological structures [6] and their generalizations. Many works have appeared recently, for example in structural analysis [8], in chemistry [29], and physics [4].

The theory of rough set, proposed by Pawlak [14], is an effective tool for data analysis [3, 5, 7, 9, 10, 11, 12, 16, 25, 33]. It can be used in the attribute-value representation model to describe the dependencies among attributes and evaluate the significance of attributes and derive decision rules. Classical rough set philosophy is based on an assumption that every object in the universe of discourse is associated with some information. Objects characterized by the same information are indiscernible with the available information about them. The indiscernibility relation generated in this way is the mathematical basis for the rough set theory. Classical rough set theory has used successfully in the analysis of data in complete information systems. The indiscernibility relation is reflexive, symmetric and transitive. The set of all indiscernible objects is called an elementary set or equivalent class. Any set of objects, being a union of some elementary sets, is refereed to as crisp set, otherwise is called rough set. A rough set can be described by a pair of crisp sets, called the lower and upper approximations. By relaxing indiscernibility relation to more general
binary relation, classical rough set can be extended to a more general model. Skowron and Stepaniuk [26] and Yao and Wong [30] discussed generalized approximation space based on the reflexive and symmetry binary relation. Slowinski and Vanderpooten [28] proposed generalized definition of rough approximation based on reflexive binary relation and compared with other definitions. Lin and Yao [10, 31, 32] study the approximation operators defined by different neighborhood operators. Skowron and Stepaniuk [26, 27] defined generalized approximation space by using uncertain function and rough inclusion function and described its construction. Also they used the proposed techniques to investigate the problem of object selection. In practice, tolerance relation (reflexive, symmetric) and preference relation (reflexive, anti-symmetric, transitive) are important relations. Greco et al. [5] proposed rough approximations based on preference relation and applied it to multicriteria decision analysis; rough approximation based on tolerance relation has been used successfully to compute attribute reducts and derive decision rules in incomplete information systems [7, 9].

Near set theory introduced by J. F. Peters in [17-24] is as a generalization of rough set theory. In this theory Peter depends on the features of objects to define the nearness of objects, consequently the classification of our universal set with respect to the available information of the objects. It is clearly, the approximations of near set theory that can be considered as an important method used to get the approximations of any rough set.

Topology is a branch of mathematics, whose concepts exist not only in almost all branches of mathematics, but also in many real life applications. We believe that topological structure will be an important base for modification of knowledge extraction and processing.

We note that the near lower approximation defined by Peters is better than the rough lower approximation defined by Pawlak, but the near upper approximation is not good, so our considered problem in this paper is how to get modified upper approximations by using general relations, special neighbourhoods, topological structures and near open sets ($\beta$-open sets). And then test them to get the best approach to near sets, consequently we generalize some of Peter’s concepts.

2. Preliminaries

Rough set theory expresses vagueness, not by means of membership, but employing a boundary region of a set. If the boundary region of a set is empty, it means that the set is crisp, otherwise the set is rough (inexact). Nonempty boundary region of a set means that our knowledge about the set is not sufficient to define the set precisely.

Suppose we are given a set of objects $U$ called the universe and an indiscernibility relation $E \subseteq U \times U$, representing our lack of knowledge about elements of $U$. For the sake of simplicity we assume that $E$ is an equivalence relation and $X$ be a subset of $U$, we want
to characterize the set $X$ with respect to $E$. To this end we will need the basic concepts of rough set theory given below [14].

The equivalence class of $E$ determined by element $x$ is: 

$$[x]_E = \{ y \in X : E(x) = E(x') \}.$$ 

Hence $E$–lower, upper approximations and boundary region of a subset $X \subseteq U$ are:

$$E(X) = \bigcup \{ [x]_E : [x]_E \subseteq X \};$$
$$E(X) = \bigcup \{ [x]_E : [x]_E \cap X \neq \phi \};$$
$$BND_E(X) = E(X) - \overline{E(X)}.$$

It is easily seen that approximations are in fact interior and closure operations in a topology generated by the indiscernibility relation [2].

The rough membership function $\mu^E_X(x)$ is a degree that $x$ belongs to $X$ in view of information about expressed by $E$. It is defined as [15]:

$$\mu^E_X(x) : U \rightarrow [0, 1], \quad \mu^E_X(x) = \frac{|X \cap [x]_E|}{|[x]_E|},$$

$|\ast|$ denotes the cardinality of $\ast$.

A rough set can also be characterized numerically by the accuracy measure of an approximation [14] which is defined as:

$$\alpha^E(X) = \frac{|E(X)|}{|E(X)|}.$$

Obviously, $0 \leq \alpha^E(X) \leq 1$. If $\alpha^E(X) = 1$, then $X$ is crisp with respect to $E$ ($X$ is precise with respect to $E$), otherwise $X$ is rough with respect to $E$.

Underlying the study of near set theory is an interest in classifying sample objects by means of probe functions associated with object features. More recently, the term feature is defined as the form, fashion or shape (of an object).

Let $F$ denotes a set of features of objects in a subset $X$. For any feature $a \in F$, we associate a function $f_a$ that maps $X$ to some set $V_{f_a}$ (range of $f_a$).

The value of $f_a(x)$ is a measurement associated with feature $a$ of an object $x \in X$. The function $f_a$ is called a probe function [13, 20].

The following concepts introduced by J. F. Peters in [19, 22].

$N_r(B)$-lower (upper) approximation and boundary region of a set $X$ with respect to $r$ features from the probe functions $B$ are defined as:

$$N_r(B), X = \bigcup_{x : [x]_{B_r} \subseteq X} [x]_{B_r};$$

$$N_r(B), X = \bigcup_{x : [x]_{B_r} \supsetneq X} [x]_{B_r}.$$
\[ N_r(B)^*X = \bigcup_{x: [x]_{B^r} \cap X \neq \phi} [x]_{B^r}; \]

\[ BND_{N_r(B)}X = N_r(B)^*X - N_r(B)_rX. \]

\[ \text{GAS} = (U, F, N_r, \nu_{B^r}) \text{ is a generalized approximation space, where } U \text{ is a universe of objects, } F \text{ is a set of functions representing object features, } N_r \text{ is a neighbourhood family function defined as:} \]

\[ N_r(F) = \bigcup_{A \subseteq P_r(F)} [x]_A, \text{ where } P_r(F) = \{ A \subseteq F : |A| = r, 1 \leq r \leq |F| \}. \]

And \( \nu_{B^r} \) is an overlap function defined by:

\[ \nu_{B^r} : P(U) \times P(U) \to [0, 1], \nu_{B^r}(Y, N_r(B)_rX) = \frac{|Y \cap N_r(B)_rX|}{|N_r(B)_rX|}, \text{ where} \]

\( N_r(B)_rX \neq \phi, Y \) is a member of the family of neighbourhoods \( N_r(B) \) and if \( N_r(B)_rX = \phi, \) then \( \nu_{B^r}(Y, N_r(B)_rX) \) is equal to 1.

The overlap function \( \nu_{B^r} \) maps a pair of sets to a number in \([0, 1]\) representing the degree of overlap between the sets of objects with features \( B_r. \)

### 3. Generalized Near Set Theory

In the following, we use a general relation to deduce a new approach to near set theory, consequently we obtain a new general near lower (upper) approximation for any set. Also we introduce a modification of some concepts.

**Definition 3.1.** Let \( f \in B \) be a general relation on a nonempty set \( X. \) A general neighbourhood of an element \( x \in X \) is defined as:

\( (x)_{r_r} = \{ y \in X : |f(y) - f(x)| \leq r \}; \)

\( \xi(f_r) = \cup \{(x)_{f_r} : x \in X \}. \)

where \( r \) is the length of neighborhood with respect to the feature \( f \) and \( \xi(f_r) \) is the family of this general neighborhoods with respect to the feature \( f \) whose length \( r. \)

**Remark 3.1.** We will replace the equivalence class in the approximations defined by Peters by the general neighborhood defined in Definition 3.1.

**Definition 3.2.** Let \( B \subseteq F \) be a set of functions representing features of \( x, x' \in X. \) Objects \( x \) and \( x' \) are minimally near each other if \( \exists f \in B \) such that \( |f(x) - f(x')| \leq r, \) where \( r \) is the length of a general neighborhood defined in Definition 3.1, with respect to
the feature $f$, denoted by $xN_jx'$.

**Definition 3.3.** Let $Y, Y' \subseteq X$ and $B \subseteq F$. A set $Y$ is near to $Y'$ if $\exists x \in Y, x' \in Y'$ and $f \in B$ such that $xN_jx'$, denoted by $YN_jY'$.

**Remark 3.2.** We can determine the degree of nearness $K$ between two sets $Y$ and $Y'$ as the following:

$$K = \frac{|\phi_i \in B : YN_{\phi_i}Y'|}{|B|}.$$  

**Proposition 3.1.** Let $x, y \in X$ and $f \in B$. Then $x$ is near to $y$ if $x \in (y)_{fr}$ or $y \in (x)_{fr}$.

**Proof.** From Definitions 3.1 and 3.2, we get the proof obviously.

**Proposition 3.2.** Any subset of a set $A$ is near to $A$.

**Proof.** Obvious.

**Postulation 3.1.** Every set $A$ is called near set, near to itself, as every element $x \in A$ is near to itself.

**Definition 3.4.** Let $(X, \tau_{\phi_i})$ be topological spaces, where $\phi_i \in B, 1 \leq i \leq |B|$. Then the lower and upper approximations for any subset $A \subseteq X$ with respect to the feature $\phi_i$ are defined as:

$$N_{\phi_i}(A) = \text{int}_{\phi_i}(A) \quad \text{and} \quad N_{\phi_i}(A) = \text{cl}_{\phi_i}(A),$$

where

$$\text{int}_{\phi_i}(\text{cl}_{\phi_i})$$

is the interior (closure) with respect to the topology $\tau_{\phi_i}$, whose subbase is the family of general neighborhoods defined in Definition 3.1.

**Definition 3.5.** Let $(X, \tau_{\phi_i})$ be topological spaces, where $\phi_i \in B, 1 \leq i \leq |B|$. A new near lower and upper approximations for any subset $A \subseteq X$ with respect to one feature of the probe functions $B$ are defined as:

$$\text{apr}_1(A) = \bigcup_{\phi_i \in B} N_{\phi_i}(A) \quad \text{and} \quad \text{apr}_1(A) = \bigcap_{\phi_i \in B} N_{\phi_i}(A).$$

Consequently

$$b_{\text{apr}_1}(A) = \text{apr}_1(A) - \text{apr}_1(A).$$

**Remark 3.3.** The new near lower and upper approximations with respect to two features of the probe functions $B$ will be defined as:

$$\text{apr}_2(A) = \bigcup_{\phi_i, \phi_j \in B} N_{\phi_i, \phi_j}(A) \quad \text{and} \quad \text{apr}_2(A) = \bigcap_{\phi_i, \phi_j \in B} N_{\phi_i, \phi_j}(A),$$

where

$$N_{\phi_i, \phi_j}(A) = \text{int}_{\phi_i, \phi_j}(A) \quad \text{and} \quad N_{\phi_i, \phi_j}(A) = \text{cl}_{\phi_i, \phi_j}(A).$$
Consequently
\[
\text{apr}_{|\mathcal{B}|} (A) = \bigcup_{\phi_1, \phi_2, \ldots, \phi_{|\mathcal{B}|} \in \mathcal{B}} \overline{\mathcal{N}}_{\phi_1, \phi_2, \ldots, \phi_{|\mathcal{B}|}} (A);
\]
\[
\overline{\text{apr}}_{|\mathcal{B}|} (A) = \bigcap_{\phi_1, \phi_2, \ldots, \phi_{|\mathcal{B}|} \in \mathcal{B}} \mathcal{N}_{\phi_1, \phi_2, \ldots, \phi_{|\mathcal{B}|}} (A).
\]

**Definition 3.6.** Let \((X, \tau_{\phi_i})\) be topological spaces, where \(\phi_i \in \mathcal{B}, 1 \leq i \leq |\mathcal{B}|\). The accuracy measure of any subset \(A \subseteq X\) with respect to \(i\) features is defined as:
\[
\alpha_i' (A) = \frac{|\text{apr}_i (A)|}{|\overline{\text{apr}}_i (A)|}, A \neq \phi.
\]

**Proposition 3.3.** For any subset \(A \subseteq X\), \(\text{apr}_i (A)\) and \(\overline{\text{apr}}_i (A)\) are near to \(\overline{\text{apr}}_i (A)\).

**Proof.** Obvious.

**Remark 3.4.** A set \(A\) with a boundary \(|\overline{\text{apr}}_i (A)| \geq 0\) is a near set.

**Definition 3.7.** Let \((X, \tau_{\phi_i})\) be topological spaces, where \(\phi_i \in \mathcal{B}, 1 \leq i \leq |\mathcal{B}|\). The new generalized lower rough coverage of any subset \(Y\) of the family of neighborhoods with respect to the probe functions \(B\) is defined as:
\[
\nu_i' (Y, \overline{\text{apr}}_i (D)) = \frac{|Y \cap \overline{\text{apr}}_i (D)|}{|\overline{\text{apr}}_i (D)|}, \text{where}
\]
\[D\] is the decision class, means the acceptable objects [22], \(\text{apr}_i (D) \neq \phi\) and if \(\overline{\text{apr}}_i (D) = \phi\), then \(\nu_i' (Y, \overline{\text{apr}}_i (D)) = 1\).

**Remark 3.5.** \(0 \leq \nu_i' \leq 1\). It measures the degree that the subset \(Y\) covers the acceptable objects (sure region).

**Remark 3.6.** Let \((X, \tau_{\phi_i})\) be topological spaces, where \(\phi_i \in \mathcal{B}, 1 \leq i \leq |\mathcal{B}|\). Hence the following definition is equivalent to Definition 3.5.
\[
\mathcal{N}_i' (A) = \bigcup \{G : G \in \mathcal{N}_1(B), G \subseteq A\};
\]
\[
\mathcal{N}_1' (A) = \bigcap \{F : F \in [\mathcal{N}_1(B)]^c, A \subseteq F\}, \text{where}
\]
\(\mathcal{N}_1 (B) = \{G : G \in \bigcup_{\phi_i \in B} \tau_{\phi_i}\}\) and \(\tau_{\phi_i}\) is the topology generated from the family of general neighborhoods with respect to the probe function \(\phi_i \in \mathcal{B}\).
4. β—approach to Near Set Theory

In this section we introduce a new approach to near sets by using β-open sets. Also we obtain another β-modification of some concepts.

**Definition 4.1.** [1] A subset $A$ of a topological space $(X, \tau)$ is called:

$$\beta - \text{open set if } A \subset \text{cl}(\text{int}(\text{cl}(A))).$$

**Definition 4.2.** Let $(X, \tau_{\phi_i})$ be topological spaces, where $\phi_i \in B, 1 \leq i \leq |B|$. The β-near lower and upper approximations for any subset $A \subseteq X$ with respect to one feature of the probe functions $B$ are defined as:

$$\mathcal{N}_{\beta_1} (A) = \bigcup \{ G : G \in \mathcal{N}_{\beta_1} (B), G \subseteq A \};$$

$$\overline{\mathcal{N}}_{\beta_1} (A) = \bigcap \{ F : F \in [\mathcal{N}_{\beta_1} (B)]^c, A \subseteq F \},$$

where

$$\mathcal{N}_{\beta_1} (B) = \{ G : G \in \bigcup_{i=1,2,\ldots,|B|} \beta_i O(X) \}$$

and $\beta_i O(X)$ is the family of beta open sets with respect to the topologies $\tau_{\phi_i}$. Hence the boundary region of $A$ with respect to one feature is defined as:

$$b_{\mathcal{N}_{\beta_1}} = \overline{\mathcal{N}}_{\beta_1} (A) - \mathcal{N}_{\beta_1} (A).$$

**Remark 4.1.** The β-near lower and upper approximations with respect to two features take the form:

$$\mathcal{N}_{\beta_2} (A) = \bigcup \{ G : G \in \mathcal{N}_{\beta_2} (B), G \subseteq A \};$$

$$\overline{\mathcal{N}}_{\beta_2} (A) = \bigcap \{ F : F \in [\mathcal{N}_{\beta_2} (B)]^c, A \subseteq F \},$$

where

$$\mathcal{N}_{\beta_2} (B) = \{ G : G \in \bigcup_{i,j=1,2,\ldots,|B|,i \neq j} \beta_{i,j} O(X) \}$$

and $\beta_{i,j} O(X)$ is the family of beta open sets with respect to the topologies $\tau_{\phi_i, \phi_j}$. Consequently

$$\mathcal{N}_{\beta_{|B|}} (A) = \bigcup \{ G : G \in \mathcal{N}_{\beta_{|B|}} (B), G \subseteq A \};$$

$$\overline{\mathcal{N}}_{\beta_{|B|}} (A) = \bigcap \{ F : F \in [\mathcal{N}_{\beta_{|B|}} (B)]^c, A \subseteq F \}.$$

**Definition 4.3.** Let $(X, \tau_{\phi_i})$ be topological spaces, where $\phi_i \in B, 1 \leq i \leq |B|$, hence we can define the β-near accuracy measure of any subset $A \subseteq X$ with respect to $i$ features of the probe functions $B$ as:

$$\alpha_{\mathcal{N}_{\beta_i}} (A) = \frac{\mathcal{N}_{\beta_i} (A)}{\overline{\mathcal{N}}_{\beta_i} (A)}, A \neq \phi.$$

**Proposition 4.1.** For any subset $A \subseteq X$, we have $\mathcal{N}_{\beta_i} (A)$ and $b_{\mathcal{N}_{\beta_i}} (A)$ are near to $\overline{\mathcal{N}}_{\beta_i} (A)$, where $1 \leq i \leq |B|$.
Proof. From Definition 4.2 and Remark 4.1, we can deduce that $N_{\beta_i}(A) \subseteq \overline{N}_{\beta_i}(A)$. Hence from Proposition 3.2, we get the proof. The other part can be proved in the same way.

**Remark 4.3.** A set $A$ with a boundary $|b_{\nu_{\beta_i}}(A)| \geq 0$ is a near set.

**Definition 4.4.** Let $(X, \tau_x)$ be topological spaces, where $\phi_i \in B, 1 \leq i \leq |B|$. The generalized $\beta$-lower rough coverage of any subset $Y \subseteq X$ is defined as:

$$\nu_{\beta_i}(Y, N_{\beta_i}(D)) = \frac{|Y \cap N_{\beta_i}(D)|}{|N_{\beta_i}(D)|}, \text{ where } N_{\beta_i}(D) \neq \phi, \text{ otherwise it is 1.}$$

**Example 4.1.** Let $s, a, r$ be three features defined on a nonempty set $X = \{x_1, x_2, x_3, x_4\}$ as in Table 1.

<p>| | | | |</p>
<table>
<thead>
<tr>
<th></th>
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</tr>
</thead>
<tbody>
<tr>
<td>$x_1$</td>
<td>0.51</td>
<td>1.2</td>
<td>0.53</td>
</tr>
<tr>
<td>$x_2$</td>
<td>0.56</td>
<td>3.1</td>
<td>2.35</td>
</tr>
<tr>
<td>$x_3$</td>
<td>0.72</td>
<td>2.8</td>
<td>0.72</td>
</tr>
<tr>
<td>$x_4$</td>
<td>0.77</td>
<td>1.9</td>
<td>0.95</td>
</tr>
</tbody>
</table>

Table 1: The values of three measures.

If the length of the neighborhood of the feature $s$ (resp $a$ and $r$) equals to 0.2 (resp 0.9 and 0.5), then:

$N_1(B) = \{\xi(s_{0.2}), \xi(a_{0.9}), \xi(r_{0.5})\}$, where

$\xi(s_{0.2}) = \{\{x_1, x_2\}, \{x_1, x_2, x_3\}, \{x_2, x_3, x_4\}, \{x_3, x_4\}\};$

$\xi(a_{0.9}) = \{\{x_1, x_4\}, \{x_1, x_2, x_3\}, \{x_2, x_3, x_4\}, \{x_1, x_3, x_4\}\};$

$\xi(r_{0.5}) = \{\{x_1, x_3, x_4\}, \{x_2\}\}$. As

$(x_1)_{s_{0.2}} = \{x_1, x_2\}, (x_2)_{s_{0.2}} = \{x_1, x_2, x_3\}, (x_3)_{s_{0.2}} = \{x_2, x_3, x_4\}, (x_4)_{s_{0.2}} = \{x_3, x_4\}$.

$(x_1)_{r_{0.9}} = \{x_2\}, (x_2)_{r_{0.9}} = \{x_2, x_3\}, (x_3)_{r_{0.9}} = \{x_2, x_3, x_4\}, (x_4)_{r_{0.9}} = \{x_2, x_3, x_4\}$.

$(x_1)_{r_{0.5}} = \{x_2\}, (x_2)_{r_{0.5}} = \{x_1, x_3, x_4\}, (x_3)_{r_{0.5}} = \{x_1, x_3, x_4\}$.

So

$\tau_{s_{0.2}} = \{\{x_2\}, \{x_4\}, \{x_1, x_2\}, \{x_2, x_3\}, \{x_3, x_4\}, \{x_1, x_2, x_3\}, \{x_2, x_3, x_4\}, X, \phi\}$;
\[ \tau_{0.9} = \{\{x_3\}, \{x_4\}, \{x_3, x_4\}, \{x_2, x_3\}, \{x_1, x_4\}, \{x_2, x_3, x_4\}, X, \phi\}; \]
\[ \tau_{0.5} = \{\{x_2\}, \{x_1, x_3, x_4\}, X, \phi\}. \] Hence
\[ \mathcal{N}_1(B) = \{\{x_2\}, \{x_3\}, \{x_4\}, \{x_1, x_2\}, \{x_1, x_3\}, \{x_2, x_3\}, \{x_3, x_4\}, \{x_1, x_2, x_3\}, \{x_1, x_3, x_4\}, \{x_2, x_3, x_4\}, X, \phi\}. \] Also, we can get:
\[ \mathcal{N}_2(B) = \{\{s_{0.2}, a_{0.9}\}, \xi(S_{0.2}, r_{0.5}), \xi(a_{0.9}, r_{0.5})\}, \]
\[ \xi(s_{0.2}, a_{0.9}) = \{\{x_2\}, \{x_3, x_4\}\}; \]
\[ \xi(s_{0.2}, r_{0.5}) = \{\{x_2\}, \{x_3, x_4\}\}; \]
\[ \xi(a_{0.9}, r_{0.5}) = \{\{x_1, x_4\}, \{x_2\}, \{x_3, x_4\}, \{x_1, x_3, x_4\}\}. \]

As
\[ \tau_{0.2}^{s_{0.9}} = \{\{x_2\}, \{x_3\}, \{x_2, x_3\}, \{x_1, x_2\}, \{x_1, x_3\}, \{x_2, x_3, x_4\}, \{x_1, x_2, x_3\}, \{x_1, x_3, x_4\}, X, \phi\}; \]
\[ \tau_{0.2}^{r_{0.5}} = \{\{x_1\}, \{x_2\}, \{x_1, x_2\}, \{x_1, x_3\}, \{x_2, x_4\}, \{x_1, x_2, x_3, x_4\}, X, \phi\}; \]
\[ \tau_{0.5}^{s_{0.9}} = \{\{x_2\}, \{x_3\}, \{x_2, x_3\}, \{x_1, x_2\}, \{x_1, x_3\}, \{x_2, x_4\}, \{x_1, x_2, x_3, x_4\}, X, \phi\}. \]

So \[ N'_3(B) = \mathcal{N}_3(B) = \{\{x_2\}, \{x_3\}, \{x_4\}, \ldots, X, \phi\}. \] Also in the same way we can get that: \[ \tau_{s_{0.2}^{r_{0.5}}} = \tau_{s_{0.2}^{r_{0.5}}}^{s_{0.9}}. \]

That means the reduct of these features is \{s, r\}, so the feature \{a\} can be cancelled.

For \( \beta \)-approach to near sets, we get:
\[ \mathcal{N}_{\beta_1}(B) = \mathcal{N}_{\beta_2}(B) = \mathcal{N}_{\beta_3}(B) = \{\{x_1\}, \{x_2\}, \{x_3\}, \{x_4\}, \ldots, X, \phi\}. \]

## 5. Special Approach to Near Set Theory

We aim in this section to introduce a generalized approach to near sets by using a special neighborhood. Deduce a modification of some concepts.

**Definition 5.1.** Let \( \phi_i \in B \) be a general relation on a nonempty set \( X \). Hence we can deduce a special neighborhood of an element \( x \in X \) as:
\[ x_{[\phi_i]} = \bigcap \{ (y)_{\phi_i} : x \in (y)_{\phi_i} \}, \quad \text{where} \]

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(y)_{φ_y} is the general neighborhood of an element y defined in Definition 3.1.

**Remark 5.1.** Let φ_i ∈ B be general relations on a nonempty set X, where 1 ≤ i ≤ |B|.

The special neighborhood of an element x with respect to two features is defined as:

\[x_{φ_iφ_j} = x_{φ_i} \cap x_{φ_j}, \quad i \neq j.\]

Consequently, with respect to |B| features is defined as:

\[x_{φ_1...φ_{|B|}} = x_{φ_1} \cap x_{φ_2} \cap ... \cap x_{φ_{|B|}}.\]

**Definition 5.2.** Let B be probe functions defined on a nonempty set X. The family of special neighborhoods with respect to one feature is defined as:

\[N_{[1]}(B) = \bigcup \{x_{φ_i} : x \in X, \ φ_i \in B\}.\]

**Proposition 5.1.** Let B ⊆ F be probe functions representing features of x, y ∈ X. Then x is near to y if y ∈ x_{φ_i}, where φ_i ∈ B, 1 ≤ i ≤ |B|.

**Proof.** Obvious.

**Definition 5.3.** Let B be probe functions on a nonempty set X. The lower and upper approximations for any subset A ⊆ X by using the special neighborhood are defined as:

\[N_{[i]}(A) = \bigcup \{y \in N_{[i]}(B) : y \subseteq A\};\]

\[\overline{N}_{[i]}(A) = \bigcup \{y \in N_{[i]}(B) : y \cap A \neq \phi\}.\]

Consequently, the boundary region is:

\[b_{N_{[i]}} = \overline{N}_{[i]}(A) - N_{[i]}(A), \text{ where } 1 \leq i \leq |B|.\]

**Definition 5.4.** Let B be probe functions on a nonempty set X. The accuracy measure for any subset A ⊆ X by using the special neighborhood with respect to i features is:

\[α_{[i]}(A) = \frac{|N_{[i]}(A)|}{|\overline{N}_{[i]}(A)|}, \quad \overline{N}_{[i]}(A) \neq \phi.\]

**Definition 5.5.** Let B be probe functions on a nonempty set X. The special generalized lower rough coverage of any subset Y of the family of special neighborhoods takes the form:

\[ν_{[i]}(Y, N_{[i]}(D)) = \frac{|Y \cap N_{[i]}(D)|}{|N_{[i]}(D)|}, \quad \overline{N}_{[i]}(D) \neq \phi.\]

If N_{[i]}(D) = φ, then ν_{[i]}(Y, N_{[i]}(D)) = 1.
Definition 5.6. Let $B$ be probe functions on a nonempty set $X$. The modified near lower, upper and boundary approximations for any subset $A \subseteq X$ are defined as:

$$N'_i(A) = \bigcup \{ y \in N_i(B) : y \subseteq A \};$$

$$\overline{N}_i(A) = \overline{[N'_i(A)]};$$

$$b_{N'_i} = \overline{N}_i(A) - N'_i(A), \text{ where } 1 \leq i \leq |B|.$$  

Definition 5.7. Let $B$ be probe functions on a nonempty set $X$. The modified accuracy measure for any subset $A \subseteq X$ takes the form:

$$\alpha'_i(A) = \frac{|N'_i(A)|}{|N_i(A)|}, \quad N'_i(A) \neq \phi.$$  

Proposition 5.2. Let $A \subseteq X$, then:

1. $N'_i(A)$ is near to $\overline{N}_i(A)$ and $\overline{N}_i(A)$.

2. $b_{N'_i}(A)$ is near to $\overline{N}_i(A)$ and $\overline{N}_i(A)$.

3. $\overline{N}_i(A)$ is near to $N_i(A)$.

4. $b_{N'_i}(A)$ is near to $b_{N_i}(A)$.

Proof. Obvious.

Definition 5.8. Let $B$ be probe functions on a nonempty set $X$. The modified lower rough coverage of any subset $Y$ of the family of special neighborhoods defined in Definition 5.1, takes the form:

$$\nu'_i(Y, N'_i(D)) = \frac{|Y \cap N'_i(D)|}{|N'_i(D)|}, \quad N'_i(D) \neq \phi.$$  

If $N'_i(D) = \phi$, then $\nu'_i(Y, N'_i(D)) = 1$.

Now, we give an example to explain these definitions.

Example 5.1. From Examble 4.1, we can get the following results:

- $x_1 = \{x_1, x_2\}$, $x_2 = \{x_2\}$, $x_3 = \{x_3\}$, $x_4 = \{x_3, x_4\}$, $x_4 = \{x_1, x_4\}$, $x_2 = \{x_2, x_3\}$,
- $x_3 = \{x_2\}$, $x_4 = \{x_4\}$, $x_1 = \{x_3\}$, $x_4 = \{x_1, x_3, x_4\}$, $x_2 = \{x_2\}$.

So
\[ N_{[1]}(B) = \{ \{ x_2 \}, \{ x_3 \}, \{ x_4 \}, \{ x_1, x_2 \}, \{ x_1, x_4 \}, \{ x_2, x_3 \}, \{ x_3, x_4 \}, \{ x_1, x_3, x_4 \} \}. \] Also, we get:
\[ x_1_{[s,a]} = \{ x_1 \}, x_2_{[s,a]} = \{ x_2, x_3 \}, x_3_{[s,a]} = \{ x_3 \}, x_4_{[s,a]} = \{ x_3, x_4 \}, x_1_{[a,r]} = \{ x_1, x_4 \}, x_2_{[a,r]} = \{ x_2 \}, x_3_{[a,r]} = \{ x_3 \}, x_4_{[a,r]} = \{ x_3, x_4 \}, x_1_{[s,a,r]} = \{ x_1 \}, x_2_{[s,a,r]} = \{ x_2 \}, x_3_{[s,a,r]} = \{ x_3 \}, x_4_{[s,a,r]} = \{ x_3, x_4 \}, \]

**Proposition 5.3.** Every rough set \( A \subseteq X \) is a near set but not every near set is a rough set.

**Proof.** There are two cases to consider:
1. \( |b_{N_{[1]}(A)}| > 0 \). Given a set \( A \subseteq X \) that has been approximated with a nonempty boundary, this means \( A \) is a rough set as well as a near set.
2. \( |b_{N_{[1]}(A)}| = 0 \). Given a set \( A \subseteq X \) that has been approximated with an empty boundary, this means \( A \) is a near set but not a rough set.

The following example shows Proposition 5.3.

**Example 5.2.** From Example 4.1, if \( A = \{ x_3, x_4 \} \). Then \( N_3(B)^\ast A = N_3(B), A = A, N_{[2]}(A) = \overline{N}_{[2]}(A) = A \) and \( N_{[1]}(A) = \overline{N}_{[1]}(A) = A \). Hence \( A \) is a near set in each case, but is not rough set with respect to three features by using the approximations introduced by Peters, with respect to two features by using our approach defined in Definition 5.3 and with respect to only one feature by using \( \beta \)-near approach defined in Definition 4.2 and our modified approach defined in Definition 5.6.

Now the following example deduces a comparison between the traditional and new general near approaches by using the accuracy measures of them.

**Example 5.3.** From Example 4.1, we introduce Table 2, where \( Q(X) \) is a family of subsets of \( X \) and \( II = \alpha_{[2]} = \alpha_{[3]} = \alpha_{2} = \alpha_{3} = \alpha_{[2]} = \alpha_{[3]} = \alpha_{N_{[2]}} = \alpha_{N_{[2]}} = \alpha_{N_{[3]}} \).
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<tr>
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<th>$\alpha_3$</th>
<th>$\alpha_{[1]}$</th>
<th>$\alpha_1' = \alpha_{[1]}'$</th>
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Table 2: Comparison between the traditional and modified approximations.

From Table 2, we note that when we use our modified approximations of near sets defined in Definition 5.6, with respect to one feature many of subsets become exact sets. And with respect to two features, all subsets become completely exact. Also, when we use the $\beta$-near approximations with respect to only one feature all subsets of $X$ become completely exact.
6. Medical Application

If we consider that $B = \{a, s, r\}$ in Example 4.1, represent measurements for a kind of diseases and the objects $X = \{x_1, x_2, x_3, x_4\}$ be patients, then:

For any group of patients, we can determine the degree of this disease in it, by using the lower rough coverage based on the decision class $D$. As in the following examples.

**Example 6.1.** In Example 4.1, if the decision class $D = \{x_1, x_3\}$ and we consider the following groups of the patients: $\{x_1, x_3\}$, $\{x_2, x_3\}$, $\{x_3, x_4\}$, $\{x_1, x_2, x_3\}$ and $\{x_2, x_3, x_4\}$. Then, we get the following results: $N_1(B)_D = \phi$, $N_2(B)_D = N_3(B)_D = \{x_1\}$, $\overline{N}_1''(D) = \{x_3\}$ and $\overline{N}_2''(D) = \overline{N}_3''(D) = N_{\beta_1}(D) = N_{\beta_2}(D) = N_{\beta_3}(D) = \{x_1, x_3\}$.

So these sets cover the sure region by the following degrees, where

$$II = \nu''_{[2]} = \nu''_{[3]} = \nu_{\beta_1} = \nu_{\beta_2} = \nu_{\beta_3}.$$ 

<table>
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<tr>
<th>$Q(X)$</th>
<th>$\nu_1$</th>
<th>$\nu_2$</th>
<th>$\nu_3$</th>
<th>$\nu''_{[1]}$</th>
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</table>

Table 3: The degrees that some subsets $Q(X)$ cover the sure region.

**Remark 6.1.** If we want to determine the degree that, the sure region covers the set $Y$, then we use the following formulas:

$$\nu^*_y(Y, N_y(B)_D) = \frac{|Y \cap N_y(B)_D|}{|Y|}, \quad Y \neq \phi;$$

$$\nu''_{[i]}(Y, \overline{N}'_{[i]}(D)) = \frac{|Y \cap \overline{N}'_{[i]}(D)|}{|Y|}, \quad Y \neq \phi;$$

$$\nu_{\beta_i}(Y, N_{\beta_i}(D)) = \frac{|Y \cap N_{\beta_i}(D)|}{|Y|}, \quad Y \neq \phi.$$
**Example 6.2.** In Example 4.1, if we interest in the degree that the sure region (acceptable objects) covers these groups, we get Table 4, where

\[ II = \nu^{[2]} = \nu^{[3]} = \nu^{[1]} \nu^{[2]} = \nu^{[3]} \nu^{[1]} \]

<table>
<thead>
<tr>
<th>( Q(X) )</th>
<th>( \nu_1^{*} )</th>
<th>( \nu_2^{*} )</th>
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<td>0</td>
<td>( \frac{1}{3} )</td>
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</table>

Table 4: The degrees that the sure region covers some subsets \( Q(X) \).

From Table 4, we can say that our approaches defined in Definitions 4.2 and 5.6, are better than the traditional approximations of near set theory, as our lower approximations is increasing the acceptable objects.

For example, when we used traditional approximations, the group \( \{x_1, x_3\} \) with respect to one feature has no disease and with respect to two or three features has this disease with ratio 50%. Although this group is itself the decision class of this disease, so there is a contradiction.

In fact these results are not true as the result must be 100% (\( \{x_1, x_3\} \) is itself the decision class of this disease).

But when we used our modified approximations defined in Definition 5.6, with respect to two or three features, we find the fact of this disease, the degree of disease in this group is 100%.

In addition, when we use the \( \beta \)-near approximations, we get the accurated results by using only one feature (with the ratio 100%).

7. **Conclusion**

In this paper, we introduce different approaches to near sets and by studied comparisons, we deduce that the best of them is \( \beta \)-near approach. Then we consider that our \( \beta \)-near approach is a start point of real life applications in many fields of science such as data
reduction.
Our future work is how to use the bitopological space with $\beta$-near open sets to get a new approach to near sets.

References


