Journal of Economic Theory **105**, 188–225 (2002)
doi:10.1006/jeth.2001.2886

Ramsey Equilibrium in a Two-Sector Model with Heterogeneous Households[1]

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Received October 24, 2000; final version received August 14, 2001;
published online April 23, 2002

We prove an existence theorem for a stationary perfect foresight equilibrium under borrowing constraints in a two-sector model with infinitely lived heterogeneous agents. The most patient agent holds all the capital in this solution. We also show that if the capital goods sector is capital intensive and capital income is increasing in the aggregate capital stock, then the aggregate capital stock eventually is monotonic and converges to the steady state stock. If the consumption goods sector is more capital intensive and capital income is increasing in aggregate capital we prove convergence to the steady state under more restrictive conditions. Periodic equilibria are shown to exist under weaker hypotheses. *Journal of Economic Literature* Classification Numbers: D52, D90, E13.

INTRODUCTION

There are important differences in representative agent dynamic equilibrium models and models with heterogeneous agents. Becker [2] introduced a one-sector model with a finite number of infinitely lived households based on Ramsey's [13] seminal paper. Agents were allowed to differ with respect to their initial endowments and intertemporal preferences. In particular, households were assumed to have different fixed discount factors

[1] We thank Professor Ciprian Foias for introducing us to the subject and commenting on our manuscript. We are also grateful to an anonymous referee and an associate editor for helpful and constructive comments.

0022-0531/02 $35.00

for future utility. Households were also assumed to face a borrowing constraint in the form of a nonnegativity restriction on their capital holdings. Using a perfect foresight equilibrium structure, Becker [2] showed the most patient agent would hold all of the economy's capital stock in the steady state or long run equilibrium. Subsequently Becker and Foias [4] showed that the aggregate capital stock was eventually monotonic and converged to the long run capital stock provided capital income was an increasing function of the capital stock. They also showed the economy exhibited the *turnpike property*—all but the most patient agent's capital became zero in finite time and remained at that level thereafter. They went on to show that if this capital income monotonicity condition failed to hold two-period equilibrium cycles could emerge in equilibrium.[2] By way of contrast, the representative agent one-sector model's equilibrium capital stock sequence is always monotonic.

Two-sector models have long been studied in growth theory. The representative agent version is known to exhibit non-monotonic equilibrium capital stock sequences, unlike the one-sector model. Indeed, two-sector representative agent models are known to admit specifications exhibiting cyclic or chaotic equilibrium capital sequences.[3] A natural question is to ask how a heterogeneous household two-sector model's equilibrium capital sequences contrast to its one-sector counterpart as well as its two-sector representative agent solution.

The purpose of our paper is to develop a heterogeneous agent, two-sector model along the lines of Becker's one-sector model. We prove there is a unique steady state with only the most patient household owning capital. Sufficient conditions on preferences and technology are given to prove the equilibrium has the turnpike property and the aggregate capital stock converges eventually monotonically to the stationary state capital stock. These sufficient conditions for convergence include a capital income monotonicity hypothesis analogous to the one-sector case as well as a restriction on the capital intensity of the consumption goods sector in relation to the capital goods sector.[4] Examples based on Cobb–Douglas production functions show two-period cycles are possible equilibria even though the capital income monotonicity hypothesis is satisfied. Our

[2] Becker and Foias [5] amplified this cycle example by a local bifurcation analysis. Sorger [15, 16] went on to show that multiple equilibria and chaotic equilibria could occur when capital income monotonicity failed to hold. Bewley [8] seems to be the first to suggest borrowing constraints as a basis for fluctuations in perfect foresight models.

[3] See Boldrin [7], Boldrin and Deneckere [9], Boldrin and Woodford [10], Nishimura and Yano [11, 12], and Baierl *et al.* [1].

[4] Our asymptotic result for the case where the consumption goods sector is the more capital intensive sector also utilizes a strong restriction on preferences. This condition is discussed at length in Section 4.

convergence theorems maintained assumptions limit the difference in sectoral factor intensities. The need for a supplementary capital intensity assumption in addition to capital income monotonicity contrasts to the one-sector case where only the latter condition is needed to prove all equilibria converge to the long run steady state solution.

The two-sector economy and the representation of its technology as a transformation function are presented in the first two sections. The existence and uniqueness of the steady state equilibrium is established next. The eventual distribution of capital and sufficient conditions for an eventually monotonic equilibrium capital stock are then demonstrated. Examples of cyclic equilibria are also given that illustrate the nature of the restrictions made in our convergence theorems. Proofs of certain technical results appear in the appendices.

1. THE TWO-SECTOR ECONOMY

The basic model is a many agent version of Uzawa's [18] two-sector model. We adapt it to a borrowing constraint framework. We assume discrete time t and two commodities, one serves as capital and the other one as the consumption good. Households are labeled by the index $h = 1, ..., H$. The utility function of h is U_h and it is defined over alternative consumption profiles consisting of sequences $C = \{c_t\}_{t=1}^{\infty}$ of nonnegative real numbers. At time zero household h is endowed with positive capital k^h. The technology is stationary and represented by a two-sector model with consumption goods output function F^0 and capital goods output function F^1. The labor force consists of H households each inelastically supplying one unit of labor which is split between the consumption and the production goods sectors. We assume that capital available at the beginning of period t is equal to the capital stock at the end of period $t-1$. Labor inputs at time t are used to produce goods during that period.

DEFINITION 1.1. A $(2H+2)$-tuple $(U_1, ..., U_H, k^1, ..., k^H, F^0, F^1)$ is called a two-sector economy.

1.1. *The Household Sector*

At each time t, households receive capital and labor income. The capital income is $r_t k^h_{t-1}$, where r_t represents rental payment in units of consumption earned at t by renting one unit of capital at the beginning of time t to the production sector. The wage income of each household at time t is denoted by w_t and is measured in units of consumption. Each household is subject to a borrowing constraint because capital markets are incomplete:

households may not borrow against anticipated wage income. Thus, each household's budget constraints can be written as follows,

$$c_t^h + q_t[k_t^h - (1-\mu)\, k_{t-1}^h] = r_t k_{t-1}^h + w_t,$$
$$k_0^h = k^h, \qquad c_t^h, k_t^h \geqslant 0 \quad (t = 1, 2, ...), \tag{1.1}$$

where c_t^h is consumption at time t of household h, q_t is the current price of capital in units of consumption at time t, μ is the depreciation rate, $0 \leqslant \mu \leqslant 1$. Households are competitive agents and perfectly anticipate the sequence of factor returns $\{r_t, w_t\}$ and the sequence of capital goods prices $\{q_t\}$. Given an expected sequence of factor returns, household h solves the problem of maximizing $U_h = U_h(c_1, c_2, ...)$ over all sequences of pairs of nonnegative numbers $\{c_t^h, k_{t-1}^h\}_{t=1}^{\infty}$ subject to budget constraints (1.1). Let $\mathbb{C}^p(X)$ denote the space of p-times continuously differentiable real-valued functions on the set X; $\mathbb{C}(X)$ refers to the set of continuous functions on X.

We will make the following assumptions:

(U1) $U_h(C) = \sum_{t=1}^{\infty} \delta_h^{t-1} u_h(c_t^h);$

(U2) $u_h \colon \Re_+ \to \Re_+, \quad u \in \mathbb{C}^2(\Re_{++}) \cap \mathbb{C}(\Re_+);$

$u_h' > 0, \quad u_h'' < 0, \quad u_h(0) = 0, \quad u_h'(+\infty) = 0;$

(U3) $1 > \delta_1 > \delta_2 \geqslant \cdots \geqslant \delta_H > 0.$

There are two necessary and sufficient conditions for $\{c_t^h, k_{t-1}^h\}_{t=1}^{\infty}$ to be an optimum for U_h subject to (1.1):

Euler inequality (No arbitrage inequality).

$$u_h'(c_t^h) \geqslant \delta_h u_h'(c_{t+1}^h)\, \frac{1}{q_t}\, [r_{t+1} + q_{t+1}(1-\mu)], \tag{1.2}$$

with equality if $k_t^h > 0$.

Transversality condition.

$$\lim_{t \to \infty} \delta_h^{t-1} u_h'(c_t^h)\, q_t k_t^h = 0. \tag{1.3}$$

1.2. *Production Sector*

Production takes place in two sectors. Consumption goods are produced in one sector and capital goods are produced in the other sector.

Consumption goods sector. Consumption goods output y_t^0 at time t depends on the level of capital input x_{t-1}^0 at time t and labor input l_t^0 utilized during period t. Thus, $y_t^0 = \mathscr{F}^0(x_{t-1}^0, l_t^0)$. We will assume that $\mathscr{F}^0 \colon \mathfrak{R}_+ \times \mathfrak{R}_+ \to \mathfrak{R}_+$ satisfies the following properties:

(F1) $\mathscr{F}^0(0,0) = 0, \quad \mathscr{F}^0(0,l) = 0, \quad \mathscr{F}^0(x,0) = 0;$

(F2) $\mathscr{F}^0 \in \mathbb{C}^2(\mathfrak{R}_{++}^2) \cap \mathbb{C}(\mathfrak{R}_+^2)$, concave, $\dfrac{\partial \mathscr{F}^0}{\partial x} > 0, \quad \dfrac{\partial \mathscr{F}^0}{\partial l} > 0;$

(F3) $\dfrac{\partial^2 \mathscr{F}^0}{\partial x^2} < 0, \quad \dfrac{\partial^2 \mathscr{F}^0}{\partial l^2} < 0;$

(F4) $\mathscr{F}^0$ is positively homogeneous of degree 1

$\forall \lambda > 0 \quad \mathscr{F}^0(\lambda x, \lambda l) = \lambda \mathscr{F}^0(x,l);$

(F5) $\mathscr{F}^0$ satisfies the Inada condition

$\dfrac{\partial \mathscr{F}^0(x,l)}{\partial x} \to 0 \quad \text{as} \quad x \to +\infty;$

(F6) $\dfrac{\partial \mathscr{F}^0}{\partial x} \to +\infty \quad \text{as} \quad x \to 0+ \quad \text{if} \quad l > 0.$

Remark 1. If $\mathscr{F}^0$ satisfies (F3) and (F4) then $\mathscr{F}_{xl}^0 = \mathscr{F}_{lx}^0 \geqslant 0$.

Let π_t^0 be the consumption goods profit function. Then the profit maximizing problem is specified for each time t as the following program,

$$\pi_t^0 = \max(y_t^0 - r_t x_{t-1}^0 - w_t l_t^0),$$
$$\text{s.t.}\, y_t^0 \leqslant \mathscr{F}^0(x_{t-1}^0, l_t^0), \quad x_{t-1}^0, l_t^0 \geqslant 0;$$

the pair (r_t, w_t) is taken as given.

Capital goods sector. Let y_t^1 be the output of new capital goods at time t. We assume that $y_t^1 = \mathscr{F}^1(x_{t-1}^1, l_t^1)$, where x_{t-1}^1 is capital input at time $t-1$ and l_t^1 is labor input at time t. Let $\mathscr{F}^1$ satisfy the same conditions as $\mathscr{F}^0$. If π_t^1 is the profit function then this sector's profit maximizing problem takes the following form,

$$\pi_t^1 = \max(y_t^1 - r_t x_{t-1}^1 - w_t l_t^1),$$
$$\text{s.t.}\, y_t^1 \leqslant \mathscr{F}^1(x_{t-1}^1, l_t^1), \quad x_{t-1}^1, l_t^1 \geqslant 0,$$

with (r_t, w_t) considered as given.

Notice that there is a maximum sustainable capital stock, k^M, where $\mathscr{F}^1(k^M, H) = \mu k^M$.

1.3. *Perfect Foresight Equilibrium*

DEFINITION 1.2. A sequence $\{w_t, r_t, q_t, c_t^h, k_{t-1}^h, x_{t-1}^0, x_{t-1}^1, l_t^0, l_t^1\}_{t=1}^{\infty}$ is a perfect foresight competitive equilibrium if:

(1) x_{t-1}^0, l_t^0, x_{t-1}^1, $l_t^1 > 0$ maximize profits in each sector given (q_t, r_t, w_t);

(2) the output market clears at each t:

$$\sum_{h=1}^{H} c_t^h = \mathscr{F}^0(x_{t-1}^0, l_t^0), \qquad \sum_{h=1}^{H} (k_t^h - (1-\mu)\, k_{t-1}^h) = \mathscr{F}^1(x_{t-1}^1, l_t^1);$$

(3) the input market clears at each time t:

$$x_{t-1}^0 + x_{t-1}^1 = \sum_{h=1}^{H} k_{t-1}^h, \qquad l_t^0 + l_t^1 = H;$$

(4) for each h, the sequences $\{c_t^h, k_{t-1}^h\}_{t=1}^{\infty}$ maximize U_h subject to (1.1).

We observe that with constant return to scale economic profits are zero in an equilibrium. Formally we state without proof

LEMMA 1.1. *$\pi_t^0 = 0$ and $\pi_t^1 = 0$ in equilibrium.*

2. THE TRANSFORMATION FUNCTION

The transformation function $\mathscr{T}(k, y)$ is given by:

$$\mathscr{T}(k, y) = \max \mathscr{F}^0(x^0, l^0) \text{ such that}$$
$$\mathscr{F}^1(x^1, l^1) \geqslant y, \quad x^0 + x^1 \leqslant k, \quad l^0 + l^1 \leqslant H, \quad x^0, x^1, l^0, l^1 \geqslant 0. \tag{2.1}$$

The transformation function inherits many of the properties of the underlying sectoral production functions:

(T1) Let $\mathscr{D} \subseteq \mathfrak{R}_+ \times \mathfrak{R}_+$ be the domain of $\mathscr{T}$. Then
$\mathscr{D} = \{(k, y) \in \mathfrak{R}_+ \times \mathfrak{R}_+ : y \leqslant \mathscr{F}^1(k, H)\}$;

(T2) $\mathscr{D}$ is a closed convex set with $(0, 0) \in \mathscr{D}$;

(T3) $\mathscr{T}(k, y) = 0$ if $y = \mathscr{F}^1(k, H)$;

(T4) $\mathscr{T}$ is concave in (k, y), increasing in k
for each y and decreasing in y for each k.

It can also be shown (see Appendix A) that:

(T5) $\mathcal{T} \in \mathbb{C}^1(\mathcal{D}/(0,0)) \cap \mathbb{C}^2(\mathring{\mathcal{D}})$;

(T6) $\dfrac{\partial \mathcal{T}}{\partial k} > 0, \quad \dfrac{\partial \mathcal{T}}{\partial y} < 0, \quad \dfrac{\partial^2 \mathcal{T}}{\partial k^2} < 0$;

(T7) $\dfrac{\partial \mathcal{T}}{\partial k}(k, y) \to +\infty \quad \text{as} \quad k, y \to 0$;

(T8) $-\dfrac{\dfrac{\partial \mathcal{T}}{\partial k}(k, y)}{\dfrac{\partial \mathcal{T}}{\partial y}(k, y)} \to +\infty \quad \text{as} \quad k, y \to 0$;

(T9) $\dfrac{\partial}{\partial k}\left(-\dfrac{\dfrac{\partial \mathcal{T}}{\partial k}(k, y)}{\dfrac{\partial \mathcal{T}}{\partial y}(k, y)}\right) < 0.$

The proofs of Propositions 2.1–2.3 can be found in Appendix A.

PROPOSITION 2.1. *$\mathcal{T}(k, y) = f(k) - y$ is a transformation function, where*

$$f \in \mathbb{C}^2(\mathfrak{R}_{++}), \quad f(0) = 0, \quad f' > 0, \quad f - kf' > 0, \quad f'' < 0,$$
$$f'(+0) = \infty, \quad f'(+\infty) = 0.$$

PROPOSITION 2.2. *If $\mathcal{T}(k, y) = \phi(k) - \psi(y)$ is a transformation function of $\mathcal{F}^0$, $\mathcal{F}^1$ which satisfies properties* (F1)–(F6) *then $\psi(y) \equiv by$, where b is a constant.*

PROPOSITION 2.3. *Let $\mathcal{E} = -\mathcal{T}_k/\mathcal{T}_y$. Then $\mathcal{E}_y + \gamma(\mathcal{E})\,\mathcal{E}_k = 0$, where $\gamma \in \mathbb{C}^1(\mathfrak{R}_+)$.*

The profit maximizing conditions can be rewritten in terms of the transformation function as follows (see Becker and Boyd [3, pp. 232–233]);

$$\mathcal{W}_t = \sup_{(k_{t-1},\, y_t) \in \mathcal{D}} \mathcal{P}_t, \qquad \text{where}$$
$$\mathcal{P}_t = \mathcal{P}_t(k_{t-1}, y_t) = \mathcal{T}(k_{t-1}, y_t) + q_t y_t - r_t k_{t-1}$$

is the profit function. Now we can rewrite the definition of a perfect foresight competitive equilibrium using $\mathcal{T}$ as follows:

DEFINITION 2.1. We will call a sequence $\{q_t, r_t, w_t, c_t^h, k_{t-1}^h, y_t, k_{t-1}\}_{t=1}^{\infty}$ a Ramsey equilibrium in the two-sector economy if:

$$q_t = -\left.\frac{\partial \mathscr{T}}{\partial y}\right|_{\substack{k=k_{t-1},\\ y=y_t}}, \qquad r_t = \left.\frac{\partial \mathscr{T}}{\partial k}\right|_{\substack{k=k_{t-1},\\ y=y_t}}, \qquad k_{t-1}, y_t > 0; \tag{2.2}$$

$$w_t = \frac{\mathscr{W}_t}{H} = \frac{1}{H}(\mathscr{T}(k_{t-1}, y_t) + q_t y_t - r_t k_{t-1}); \tag{2.3}$$

$\{c_t^h, k_{t-1}^h\}_{t=1}^{\infty}$ maximizes U_h subject to the budget constraints

$$\begin{aligned} &c_t^h + q_t[k_t^h - (1-\mu)\, k_{t-1}^h] = w_t + r_t k_{t-1}^h \qquad (t = 1, 2, ...) \\ &k_0^h = k^h, \quad c_t^h, k_t^h \geqslant 0, \qquad \text{with } (q_t, r_t, w_t) \text{ considered as given;} \end{aligned} \tag{2.4}$$

the output market clears

$$\sum_{h=1}^{H} c_t^h = \mathscr{T}(k_{t-1}, y_t) \qquad (t = 1, 2, ...); \tag{2.5}$$

the input market clears

$$\sum_{h=1}^{H} k_{t-1}^h = k_{t-1} \qquad (t = 1, 2, ...). \tag{2.6}$$

Conditions (2.2)–(2.6) imply

$$y_t = \sum_{h=1}^{H} (k_t^h - (1-\mu)\, k_{t-1}^h), \tag{2.7}$$

which is a form of Walras law.

Condition (2.2) implies that (k_{t-1}, y_t) maximizes the profit because $\mathscr{P}_t$ is concave and $\partial \mathscr{P}_t / \partial k = \partial \mathscr{P}_t / \partial y = 0$ evaluated at (k_{t-1}, y_t).

LEMMA 2.1. $w_t > 0$ *in equilibrium.*

Proof. See Appendix A. ▮

3. EXISTENCE AND UNIQUENESS OF A STATIONARY RAMSEY EQUILIBRIUM IN A TWO-SECTOR MODEL

The two-sector model's steady state theory is similar to the one-sector model's construction. Only the most patient household holds capital in the long run.

DEFINITION 3.1. A Ramsey equilibrium is stationary if $\{q_t, r_t, w_t, c_t^h, k_{t-1}^h, y_t, k_{t-1}\}$ is independent of t.

Set

$$y = y(k) = \mu k, \qquad q = q(k) = -\frac{\partial \mathcal{T}}{\partial y}\bigg|_{y=y(k)}, \qquad r = r(k) = \frac{\partial \mathcal{T}}{\partial k}\bigg|_{y=y(k)};$$

$$w = w(k) = \left[\frac{\mathcal{T}(k, y)+qy-rk}{H}\right]\Bigg|_{\substack{y=y(k)\\ q=q(k)\\ r=r(k)}};$$

$$k^h = 0, \qquad c^h = c^h(k) = w(k) \qquad \text{if} \quad h \geqslant 2;$$

$$k^1 = k, \qquad c^1 = c^1(k) = [w+k(r-q\mu)]\Big|_{\substack{w=w(k),\\ q=q(k),\\ r=r(k)}}.$$

LEMMA 3.1. *There exists at least one positive solution of*

$$\left[\delta_1\left(\frac{r}{q}+(1-\mu)\right)\right](k) = 1. \tag{3.1}$$

Proof. Equation $[\delta_1(\frac{r}{q}+(1-\mu))](k) = 1$ is equivalent to

$$-\frac{\mathcal{T}_k}{\mathcal{T}_y} = \frac{1}{\delta_1} - 1 + \mu. \tag{3.2}$$

The left-hand side of (3.2) goes to $+\infty$ as $k \to 0$. Multiplying both sides of (3.2) by a positive number $-\frac{\partial \mathcal{T}}{\partial y}$ we obtain

$$\frac{d}{dk}(\mathcal{T}(k, \mu k)) = \left(\frac{1}{\delta_1} - 1\right)\left(-\frac{\partial \mathcal{T}}{\partial y}\right)\bigg|_{y=\mu k}, \tag{3.3}$$

since $\frac{d}{dk}(\mathcal{T}(k, \mu k)) = \mathcal{T}_k + \mathcal{T}_y \mu$. The properties of $\mathcal{D}$ imply that there exists exactly one positive value k^* such that $\mathcal{T}(k^*, \mu k^*) = \mathcal{T}(0, 0) = 0$. Then there is $k^{**} \in (0, k^*)$ such that $\frac{d}{dk}\mathcal{T}(k, \mu k)|_{k=k^{**}} = 0$. That means that at $k = k^{**}$ the right-hand side of (3.3) is greater than the left-hand side. Hence there exists $\bar{k} \in (0, k^{**}]$ such that

$$\frac{d}{dk}(\mathcal{T}(k, \mu k))\bigg|_{k=\bar{k}} = \left(\frac{1}{\delta_1} - 1\right)\left(-\frac{\partial \mathcal{T}}{\partial y}\right)\bigg|_{\substack{k=\bar{k},\\ y=\mu\bar{k}}}.$$

Thus $[\delta_1(\frac{r}{q}+(1-\mu))](\bar{k}) = 1$. ∎

LEMMA 3.2. *Equation* (3.1) *has at most one solution.*

Proof. See Appendix A. ∎

Let $\bar{k}$ be the stationary capital stock found in Lemma 3.1. Set

$$\bar{y}=y(\bar{k}), \quad \bar{q}=q(\bar{k}), \quad \bar{r}=r(\bar{k}), \quad \bar{w}=w(\bar{k});$$

$$\bar{k}^h=0, \quad \bar{c}^h=\bar{w} \qquad \text{if} \quad h \geqslant 2; \qquad \bar{k}^1=\bar{k}, \quad \bar{c}^1=\bar{w}+(\bar{r}-\bar{q}\mu)\,\bar{k}.$$

It follows from Lemma 2.1 that $\bar{w}, \bar{c}^h>0$ for all h.

LEMMA 3.3. *The sequence* $\{\bar{c}^h, \bar{k}^h\}$ *maximizes* U_h *subject to* (2.4).

Proof. First we can see that $\{\bar{c}^h, \bar{k}^h\}$ satisfies the budget constraints. Second, let $\{c_t^h, k_t^h\}$ be any sequence that satisfies (2.4) and $c_t^h \neq \bar{c}^h$ for some t. It follows from strict concavity of u_h that

$$u_h(\bar{c}^h)-u_h'(\bar{c}^h)\,\bar{c}^h \geqslant u_h(c_t^h)-u_h'(\bar{c}^h)\,c_t^h;$$

the inequality becomes strict if $c_t^h \neq \bar{c}^h$. Rearranging, we obtain

$$u_h(\bar{c}^h)-u_h(c_t^h) \geqslant u_h'(\bar{c}^h)(\bar{c}^h-c_t^h).$$

Hence

$$\sum_{t=1}^{\infty} \delta_h^{t-1}[u_h(\bar{c}^h)-u_h(c_t^h)] > \sum_{t=1}^{\infty} \delta_h^{t-1} u_h'(\bar{c}^h)(\bar{c}^h-c_t^h). \tag{3.4}$$

Let us consider two cases:

(1) $h \geqslant 2$, then $\bar{c}^h=\bar{w}$ and from (2.4)

$$\bar{c}_t^h-c_t^h=\bar{q}[k_t^h-(1-\mu)\,k_{t-1}^h]-\bar{r}k_{t-1}^h.$$

Thus by (3.1) $\bar{c}_t^h-c_t^h \geqslant \bar{q}[k_t^h-k_{t-1}^h/\delta_h]$ and

$$\sum_{t=1}^{\infty} \delta_h^{t-1}[u_h(\bar{c}^h)-u_h(c_t^h)] > \bar{q}u_h'(\bar{c}^h) \sum_{t=1}^{\infty} \delta_h^{t-1}\left(k_t^h-\frac{k_{t-1}^h}{\delta_h}\right).$$

Now evaluate $\sum_{t=1}^{N} \delta_h^{t-1}[k_t^h-k_{t-1}^h/\delta_h]$:

$$\sum_{t=1}^{N} \delta_h^{t-1}\left[k_t^h-\frac{k_{t-1}^h}{\delta_h}\right]=\delta_h^{N-1}k_N^h-\frac{k_0^h}{\delta_h}=\delta_h^{N-1}k_N^h \geqslant 0$$

because $k_0^h = 0$. So

$$\bar{q}u_h'(\bar{w}) \sum_{t=1}^{\infty} \delta_h^{t-1} \left[k_t^h - \frac{k_{t-1}^h}{\delta_h} \right] \geqslant 0, \textit{ and}$$

$$\sum_{t=1}^{\infty} \delta_h^{t-1} [u_h(\bar{c}^h) - u_h(c_t^h)] > 0.$$

(2) $h = 1$, then $\bar{c}^1 = \bar{w} + (\bar{r} - \bar{q}\mu)\,\bar{k}$. If $\{c_t^1, k_{t-1}^1\}$ satisfies (2.4) then

$$\bar{c}^1 - c_t^1 = \bar{q}\left[k_t^1 - \frac{1}{\delta_1} k_{t-1}^1 \right] - \bar{q}\left[\frac{1}{\delta_1} - 1 \right] \bar{k}.$$

If $c_t^1 \neq \bar{c}^1$ for some t then using strict concavity of u_1 we obtain

$$\sum_{t=1}^{\infty} \delta_1^{t-1} [u_1(\bar{c}^1) - u_1(c_t^1)] > \bar{q}u_1'(\bar{c}^1) \sum_{t=1}^{\infty} \delta_h^{t-1} \left[\left(k_t^h - \frac{k_{t-1}^h}{\delta_h} \right) - \left(\frac{1}{\delta_1} - 1 \right) \bar{k} \right].$$

Also

$$\sum_{t=1}^{N} \delta_h^{t-1} \left[\left(k_t^h - \frac{k_{t-1}^h}{\delta_h} \right) - \left(\frac{1}{\delta_1} - 1 \right) \bar{k} \right] = \delta_1^{N-1} k_N^1 - \frac{k_0^1}{\delta_1} - \delta_1^{N-1} \bar{k} + \frac{\bar{k}}{\delta_1}$$

$$= \delta_1^{N-1} k_N^1 - \delta_1^{N-1} \bar{k}$$

because $k_0^1 = \bar{k}$. Hence

$$\bar{q}u_1'(\bar{c}^1) \liminf_{N \to \infty} \sum_{t=1}^{N} \delta_h^{t-1} \left[\left(k_t^h - \frac{k_{t-1}^h}{\delta_h} \right) - \left(\frac{1}{\delta_1} - 1 \right) \bar{k} \right] \geqslant 0$$

and

$$\sum_{t=1}^{\infty} [u_1(\bar{c}^1) - u_1(c_t^1)] > 0. \quad \blacksquare$$

Our main steady state result is

THEOREM 3.1. *A two-sector economy has a unique stationary Ramsey equilibrium.*

Proof. Existence follows from Lemmas 3.1, 3.3, and 3.4. If $(q, r, w, c^h, k^h, y, k)$ is a stationary Ramsey equilibrium then $c^h > 0$ for all h because $\{(c^h, k^h)\}$ maximizes U_h subject to (2.4) with $w > 0$. In addition, there exists at least one index i such that $k^i > 0$. Thus by the Euler inequality $1 = \delta_i[r/q + (1-\mu)]$. Then $1 < \delta_1[r/q + (1-\mu)]$ if $i > 1$. Therefore $i = 1$ and $k^h = 0$ for any $h > 1$. Finally, r/q solves (3.1) and by Lemma 3.2 $k = \bar{k}$ as well as $w = \bar{w}$. All the other equalities follow from (2.2) and (2.4). $\blacksquare$

4. DYNAMIC EQUILIBRIUM PROPERTIES

The analysis of dynamic equilibrium paths is undertaken with additional restrictions on preferences and the two-sector technology. We propose sufficient conditions for every equilibrium capital stock sequence to converge to the stationary capital stock. The dynamic properties of equilibria depend on the sign of $\mathscr{T}_{ky}$. Assume $\mu = 1$ hereafter.

We begin with the case $\mathscr{T}_{ky} \leqslant 0$. This corresponds to the specification where the consumption goods sector is more capital intensive than the capital goods sector. It is well known that this capital intensity condition can yield periodic or chaotic equilibria in the representative agent case.[5] Our convergence theorem limits the degree to which the consumption goods sector is the more capital intensive sector. Weaker results showing the ways in which capital is eventually distributed are demonstrated first.

We start our convergence analysis by proving under our basic preference and technology assumptions that the no capital state is *recurrent* for $h \geqslant 2$. That is, for $h \geqslant 2$ the corresponding equilibrium capital stock sequence has zero capital holdings infinitely often. If the most patient household eventually owns all the economy's capital, then we say that the *turnpike property* obtains. It is known from the one-sector model that additional assumptions are required to prove the turnpike property holds. We introduce a strong restriction on preferences to establish the turnpike property given the recurrence property. We assume that the relatively more impatient households are sufficiently myopic in relation to the most patient household. This is expressed by writing $\delta_h \lll \delta_1$ for $h \geqslant 2$. This strong discounting condition implies that once one of the relatively impatient households achieves the no capital state it remains in that state. Its myopia is sufficiently strong that it never reaccumulates capital even when it correctly anticipates fluctuations in capital's rental rate and price.

The introduction of this strong discounting assumption requires comment as it has no parallel in the one-sector theory developed by Becker and Foias [4]. The main difficulty in using the weaker one-sector approach when the consumption goods sector is more capital intensive than the capital goods sector is that we cannot prove, using only our basic preference and technology assumptions, that every equilibrium capital stock

[5] Benhabib and Nishimura [6] discuss a representative agent two-sector model in which the consumption goods sector is capital intensive and show a two-cycle can be an equilibrium. See their discussion (pp. 298–300) for a detailed interpretation of this capital intensity condition. Nishimura and Yano [12] study a two-sector model with Cobb–Douglas production functions and work out sufficient conditions for the existence of an equilibrium capital sequence to form a two-period cycle. Baierl *et al.* [1] extend those examples to admit incomplete capital depreciation within the period. These results provide concrete examples of economies satisfying Benhabib and Nishimura's sufficient conditions for a cyclic equilibrium.

sequence is always smaller than the stationary capital stock provided the economy's initial aggregate stock is. Our examples in Section 4.2 show periodic cycles that oscillate around the steady state are possible equilibrium sequences. This would rule out the Becker–Foias approach. Hence, we have taken the alternative route of restricting preferences. However, Becker and Foias's approach can be adapted to the two-sector case when the capital goods sector is the more capital intensive. This is worked out in Section 4.4.

The final step in our convergence result when $\mathcal{T}_{ky} \leqslant 0$ is based on an assumption that capital income is increasing in the aggregate capital stock (see below). With this additional hypothesis we are able to prove that the sequence of equilibrium capital stocks is eventually monotonic and converges to the steady state. We construct an example of an equilibrium two-cycle showing that the turnpike property, by itself, is insufficient for a convergence result. A similar example shows that assuming households are sufficiently myopic (that is, $\delta_h \ll \delta_1$ for $h \geqslant 2$) is, by itself, also insufficient for all equilibria to converge to the steady state.

The case $\mathcal{T}_{ky} \geqslant 0$ has a parallel, but simpler treatment than the case $\mathcal{T}_{ky} \leqslant 0$. This is not surprising given the monotonicity of optimal capital sequences found in the corresponding representative agent model.[6]

We say that *capital income monotonicity* holds provided $(\mathcal{T}_k(k, y)\, k)_k > 0$ for all k. This is the two-sector version of the capital income monotonicity condition utilized by Becker and Foias [4] to prove their convergence theorem. This is introduced as an additional assumption in order to prove our convergence results when either $\mathcal{T}_{ky} \leqslant 0$ or $\mathcal{T}_{ky} \geqslant 0$ hold. This condition is sufficient for all equilibria to converge to the steady state in the one-sector theory, but not so in our two-sector setup as the examples given in Section 4.2 show. Both of our cyclic solutions arise in models satisfying the capital income monotonicity assumption. This indicates that capital income monotonicity enters the two-sector model in a more subtle manner than the one-sector case. We show in Appendix B that when each sector's production function is Cobb–Douglas, then capital income monotonicity always holds when the consumption goods sector is the more capital intensive technology (see Lemma B.2). However, if the capital goods sector is more capital intensive than the consumption goods sector, then capital income monotonicity occurs with some restriction on the choice of the consumption good's capital share parameter, α (see Lemma B.4). Thus, a unitary elasticity of substitution for each sector's production function does not necessarily yield capital income monotonicity, unlike the one-sector case.[7]

[6] See Benhabib and Nishimura [6] for a detailed development of the implications of $\mathcal{T}_{ky} \geqslant 0$ when there is a representative agent.

[7] A sufficient condition for capital income monotonicity in a one-sector model is the production function's elasticity of substitution be greater than or equal to one.

4.1. *The Eventual Distribution of Capital when $\mathcal{T}_{ky} \leqslant 0$*

In this section we will discuss the eventual distribution of capital in an arbitrary Ramsey equilibrium. First, we show that a zero capital stock is achieved infinitely often for $h \geqslant 2$ if $\mathcal{T}_{ky} \leqslant 0$; this is the recurrence property. Second, we demonstrate that every $h \geqslant 2$ eventually reaches a no capital position and maintains that position thereafter under the additional assumption $\delta_h \ll \delta_1$ for $h \geqslant 2$.

The case $\mathcal{T}_{ky} \leqslant 0$ includes two examples. If the transformation function is given by Proposition 2.1, then $\mathcal{T}_{ky} = 0$. In particular, one-sector model transformation functions are admissible under the $\mathcal{T}_{ky} \leqslant 0$ hypothesis. The two-sector model specified by $\mathcal{F}^0(x, l) = x^\alpha l^{1-\alpha}$, $\mathcal{F}^1(x, l) = x^{1/2} l^{1/2}$ can also be shown to satisfy the condition $\mathcal{T}_{ky} \leqslant 0$ if $\alpha \geqslant 1/2$ (see Lemma B.1 in Appendix B).

We begin with

LEMMA 4.1. *If $\{r_t, q_t, k_{t-1}, y_t, c_t^h, k_{t-1}^h\}$ is a Ramsey equilibrium and $\mathcal{T}_{ky} \leqslant 0$, then there exists $\varepsilon > 0$ such that $k_{t-1} > \varepsilon$ for all $t \geqslant 1$.*

Proof. If $\delta_1 r_{t+1}/q_t \geqslant 1$ for all t then by the Euler inequality

$$u_1'(c_{t+1}^1) \leqslant \delta_1 \frac{r_{t+1}}{q_t} u_1'(c_{t+1}^1) \leqslant u_1'(c_t^1).$$

Hence $c_t^1 \leqslant c_{t+1}^1$ for all $t \geqslant 1$. We also have $c_t^1 > 0$ if $t > 1$. Thus $c_t^1 \geqslant \varepsilon^* > 0$, $t > 1$. In addition, the equality $\sum_{h=1}^H c_t^h = \mathcal{T}(k_{t-1}, k_t)$ implies that $\mathcal{T}(k_{t-1}, k_t) \geqslant \varepsilon^*$. Therefore $k_t \geqslant \varepsilon > 0$. Thus we can assume that there exists t_0 such that $\delta_1 r_{t_0+1}/q_{t_0} < 1$. It follows from the assumption $\mathcal{T}_{ky} \leqslant 0$ that the function of three variables $-\delta_1 \mathcal{T}_k(y, z)/\mathcal{T}_y(k, y)$ is decreasing in y and nonincreasing in k, z. Set $k_t^* = \max\{k_{t-1}, k_t, k_{t+1}\}$. Then

$$\delta_1 \frac{\mathcal{T}_k(k_{t_0}^*, k_{t_0}^*)}{-\mathcal{T}_y(k_{t_0}^*, k_{t_0}^*)} \leqslant \delta_1 \frac{\mathcal{T}_k(k_{t_0}, k_{t_0+1})}{-\mathcal{T}_y(k_{t_0-1}, k_{t_0})} < 1,$$

so $k_{t_0}^* \geqslant \bar{k} > 0$, where $\bar{k}$ is the capital stock in the stationary Ramsey equilibrium. Let us consider two special cases:

(1) if $k_{t_0}^* = k_{t_0-1}$, then $|\mathcal{T}_y(k_{t_0-1}, k_{t_0})| \leqslant M$ (any positive constant that depends only on $\mathcal{T}$ and δ_1 will be denoted either by M or by M^{-1}). For any transformation function $\mathcal{T}$, the partial derivative $\mathcal{T}_k(k, y) \to +\infty$ as $k, y \to 0$. Hence $k_{t_0} \geqslant M^{-1} > 0$,

(2) if $k_{t_0}^* = k_{t_0+1}$, then $k_{t_0} \geqslant M^{-1}$ because $(k_{t_0}, k_{t_0+1}) \in \mathcal{D}$. Thus in either case $k_{t_0} \geqslant M^{-1}$. If $\delta_1 r_{t+1}/q_t < 1$ for $\forall t \geqslant t_0$ then $k_{t-1} \geqslant \varepsilon > 0$ for all t.

Therefore, we can assume that there exists $t_1 > t_0$ such that $\delta_1 r_{t_1+1}/q_{t_1} \geqslant 1$ and $\delta_1 r_{t_1}/q_{t_1-1} < 1$. Then

$$\begin{aligned} \mathscr{W}_{t_1} &= \mathscr{W}_{t_1}(k_{t_1-1}, k_{t_1}) \\ &= \mathscr{T}(k_{t_1-1}, k_{t_1}) - \mathscr{T}_y(k_{t_1-1}, k_{t_1})\, k_{t_1} - \mathscr{T}_k(k_{t_1-1}, k_{t_1})\, k_{t_1-1} > M^{-1} \end{aligned}$$

because $\mathscr{T}_{ky} \leqslant 0$ implies that $\mathscr{W} = \mathscr{W}(k, y)$ is nondecreasing in both arguments. The budget constraint for household 1 at $t = t_1$

$$c^1_{t_1} + q_{t_1} k^1_{t_1} = w_{t_1} + r_{t_1} k^1_{t_1-1}$$

implies that either $q_{t_1} k^1_{t_1} > M^{-1}$ or $c^1_{t_1} > M^{-1}$. In the first case, since $q_{t_1} k^1_{t_1} > M^{-1}$, then the inequality $k_{t_1-1} > M^{-1}$ implies that $q_{t_1} < M$ and $k^1_{t_1} > M^{-1}$. In the second case, as $c^1_{t_1} > M^{-1}$, then by the Euler inequality $c^1_{t_1+1} \geqslant c^1_{t_1}$. Hence $\mathscr{T}(k_{t_1}, k_{t_1+1}) \geqslant M^{-1}$ and therefore $k_{t_1} \geqslant M^{-1}$. If we repeat the whole argument again we finally obtain $k_t \geqslant \varepsilon > 0$, $t \geqslant 0$, where $\varepsilon = \varepsilon(k_0, k_1, c^1_1, M^{-1})$. ▮

Lemma 4.1 implies that the sequence $\{w_t\}$ is bounded from below. The proof of the next result is immediate.

COROLLARY 4.1. *Under the hypotheses of Lemma* 4.1,

$$\varliminf_{t \to \infty} c^h_t > 0 \qquad (h = 1, ..., H). \tag{4.1}$$

LEMMA 4.2. *If* $\{r_t, q_t, w_t, k_{t-1}, y_t, c^h_t, k^h_{t-1}\}$ *is a Ramsey equilibrium, then*

$$\varlimsup_{T \to \infty} \prod_{t=1}^{T} \delta_1 \frac{r_{t+1}}{q_t} < \infty. \tag{4.2}$$

Proof. Iterate the first agent's Euler inequality T times. We obtain

$$\prod_{t=1}^{T} \delta_1 \frac{r_{t+1}}{q_t} \leqslant \frac{u_1'(c^1_1)}{u_1'(c^1_{T+1})} \leqslant \frac{u_1'(c^1_1)}{u_1'(k^M)}$$

because $c^1_t < k^M$. Therefore (4.2) holds. ▮

DEFINITION 4.1. We say that k is a *recurrent state* for the sequence $\{k_{t-1}\}$ if there exists $\{t_n\}_{n=1}^{\infty}$, $t_n < t_{n+1}$, and $k = k_{t_n-1} (n = 1, 2, ...)$. We call $k = 0$ the no capital state.

LEMMA 4.3. *The no capital state is recurrent for each* $h \geqslant 2$.

Proof. Suppose the no capital state is not recurrent for some $h \geqslant 2$. Then there exists t_0 such that $k_t^h > 0$ for all $t > t_0$. Iterating this household's Euler inequality we obtain

$$\prod_{t=t_0}^{T} \delta_h \frac{r_{t+1}}{q_t} = \frac{u'(c_{t_0}^h)}{u'(c_{T+1})},$$

where we drop the index h from the utility function u_h. In addition,

$$\lim_{T \to \infty} \prod_{t=t_0}^{T} \delta_h \frac{r_{t+1}}{q_t} = \lim_{T \to \infty} \left(\frac{\delta_h}{\delta_1}\right)^{T-t_0} \prod_{t=t_0}^{T} \delta_1 \frac{r_{t+1}}{q_t} = 0$$

by (4.2). Thus $c_t^h \to 0$ which contradicts Corollary 4.1. ▮

LEMMA 4.4. *Under the assumptions of Lemma* 4.1, *if* $\delta_h \ll \delta_1$ *then* $k_t^h = 0$ *eventually for all* $h \geqslant 2$.

Proof. Lemma 4.1 implies that r_{t+1}/q_t is bounded from above and w_t is bounded from below. If $\delta_h \ll \delta_1$ then we have

$$\delta_h \frac{r_{t+1}}{q_t} u_h'(w_{t+1}) < u_h'(w_t) \tag{4.3}$$

for all $h \geqslant 2$. Let us fix such an h and drop it from the notation. By Lemma 4.2 there exists t_1 such that $k_{t_1} = 0$. If k_t is not identically zero for $t \geqslant t_0$ then there exists $t_2 > t_1 \geqslant t_0$ such that $k_{t_2} = k_{t_1} = 0$ and $k_t > 0$ for $t_1 < t < t_2$. We have that $w_{t_1+1} > c_{t_1+1}$ and also if for some $t_1 < t < t_2 - 1$ we have $w_t > c_t$, then by (4.3)

$$\delta \frac{r_{t+1}}{q_t} u'(c_{t+1}) = u'(c_t) > u'(w_t) > \delta \frac{r_{t+1}}{q_t} u'(w_{t+1}).$$

Hence $c_{t+1} < w_{t+1}$. It follows that $c_t < w_t$ for all $t_1 < t \leqslant t_2$. Therefore, the alternative program $\{c_t'\}$, defined by $c_t' = w_t$ for $t_1 < t \leqslant t_2$ and $c_t' = c_t$ otherwise, will be preferred to (c_t, x_{t-1}). Thus eventually $k_t^h = 0$ for $\forall h \geqslant 2$. ▮

DEFINITION 4.2. We say that a Ramsey equilibrium has the turnpike property when the capital sequences of households $h \geqslant 2$ are eventually zero.

Throughout the rest of the paper we will assume that the turnpike property holds for $t \geqslant t_0$. Sufficient conditions supporting this assumption are given by Lemma 4.4.

4.2. *A Two-Cycle Equilibrium Examples*

The following example shows that the turnpike property by itself does not imply convergence of an equilibrium to the steady state. An equilibrium two-cycle can exist. We consider the two-sector model specified by $\mathcal{F}^0(x, l) = x^\alpha l^{1-\alpha}$ and $\mathcal{F}^1(x, l) = x^{1/2} l^{1/2}$. It satisfies the condition $\mathcal{T}_{ky} \leqslant 0$ if $\alpha \geqslant 1/2$ (see Lemma B.1 in Appendix B). We also assume that capital fully depreciates within a period so $\mu = 1$.

Let $\alpha = 0.76$ in the example given in Appendix B.[8] We set $k_1 = 0.25$, $k_2 = 0.35$. Using Maple we find that the first household's consumption levels are $c_2^1 < 0.038$ and $c_1^1 > 0.19$. Also note that $1 < \mathcal{T}_k(k_1, k_2)/\mathcal{T}_y(k_2, k_1) < 1.171$ and $\mathcal{T}_k(k_2, k_1)/\mathcal{T}_y(k_1, k_2) > 1.171$. Then there exist positive $\delta_1 < 1$ and concave u that solve the following two equations

$$\delta_1 \frac{\mathcal{T}_k(k_1, k_2)}{\mathcal{T}_y(k_2, k_1)} u'(c_2^1) = u'(c_1^1), \quad \delta_1 \frac{\mathcal{T}_k(k_2, k_1)}{\mathcal{T}_y(k_1, k_2)} u'(c_1^1) = u'(c_2^1).$$

If δ_2 is small enough then the Euler inequality for household 2 is satisfied. Indeed, for $\delta_1 = 0.853959999$ the steady state capital is $\bar{k} = 0.2998878415$. We can find a u such that $k_1 < \bar{k} < k_2$ and (k_1, k_2) constitutes an equilibrium two-cycle oscillating around the steady state stock.

This example shows a bit more. The capital income monotonicity condition holds since $\alpha \geqslant 1/2$ (Lemma B.2 in Appendix B). Therefore, this condition is also insufficient, by itself, to prove a convergence theorem in the two-sector model in contrast to the one-sector case.

We note that the choice $\delta_2 \ll \delta_1$ is also, by itself, not sufficient to yield the convergence property.

If we set $\alpha = 0.751$, $k_1 = 0.33$ and $k_2 = 0.35$, then $1 < \mathcal{T}_k(k_1, k_2)/\mathcal{T}_y(k_2, k_1) < 1.068161$ and $\mathcal{T}_k(k_2, k_1)/\mathcal{T}_y(k_1, k_2) > 1.068161$. Let $k_1^2 = 0.000003$ and $k_2^2 = 0$. Then $0.32 < k_1^1 < 0.33$, $k_2^1 = 0.35$, $c_1^1 > 0.12$, $0.093 < c_2^1 < 0.1$, $0.0912 < c_1^2 < 0.0913$, and $0.0911 < c_2^2 < 0.0912$. Note that $c_2^2 < c_1^2 < c_2^1 < c_1^1$. The following equations can be solved for δ_1, δ_2, u':

$$\delta_1^2 \frac{\mathcal{T}_k(k_1, k_2)}{\mathcal{T}_y(k_2, k_1)} \frac{\mathcal{T}_k(k_2, k_1)}{\mathcal{T}_y(k_1, k_2)} = 1,$$

$$\delta_1 \frac{\mathcal{T}_k(k_1, k_2)}{\mathcal{T}_y(k_2, k_1)} u'(c_2^1) = u'(c_1^1), \qquad \delta_2 \frac{\mathcal{T}_k(k_1, k_2)}{\mathcal{T}_y(k_2, k_1)} u'(c_2^2) = u'(c_1^2).$$

[8] This choice of $\alpha > 3/4$ in our examples is related to Nishimura and Yano's [12] cycle example. They constructed a two-cycle equilibrium for a representative agent economy with the same Cobb–Douglas technology as we study. They showed a cycle existed whenever $\alpha > 3/4$, the representative agent's discount factor was sufficiently small, and that agent's one-period return function was linear in consumption.

The solution will be consistent with concavity of u and $\delta_1 > \delta_2$. The inequality $\delta_2 \mathcal{T}_k(k_2, k_1)/\mathcal{T}_y(k_2, k_1)\, u'(c_1^2) < u'(c_2^2)$ will hold automatically. The steady state solution is $\bar{k} = 0.3399970803$ when $\delta_1 = 0.936187$. As before we can find u and δ_2 so that $k_1 < \bar{k} < k_2$ forms an equilibrium two-cycle.

Both of our examples implicitly constrain the choice of δ_1. It is known from the literature on optimal growth theory that the steady state is stable provided the planner's discount factor is sufficiently close to 1.[9] Our cycle examples state that given the technological parameters there are choices of the first household's discount factor that induce the two-period solution. It would be interesting to know if these examples could be turned into convergent cases when the first agent's discount factor is sufficiently close to unity.[10]

4.3. *Convergence of the Equilibrium Capital Stock when $\mathcal{T}_{ky} \leqslant 0$*

It follows from the Euler inequality that

$$\delta_1 \frac{r_{t+1}}{q_t} u_1'(c_{t+1}^1) = u_1'(c_t^1) \tag{4.4}$$

for all $t \geqslant t_0$, where $c_t^1 = (1/H)\, \mathcal{T}(k_{t-1}, k_t) + (1 - 1/H)(\mathcal{T}_k(k_{t-1}, k_t)\, k_{t-1} + \mathcal{T}_y(k_{t-1}, k_t)\, k_t)$.

LEMMA 4.5. *Under the conditions of Lemma* 4.1, *if* $\lim_{t \to \infty} k_t = k$ *exists, then* $k = \bar{k}$.

Proof. Existence of $\lim_{t \to \infty} k_t = k$ implies existence of $\lim_{t \to \infty} c_t^1 = c^1$, where $c^1 > 0$ by Corollary 4.1. Then $k = \bar{k}$ by (4.4). ▮

We are interested in finding sufficient conditions for the aggregate capital state, k_t, to be convergent. To that end, we state extra hypotheses in the following result. Our assumptions are designed to rule out cyclic equilibria that could otherwise arise, as shown by our examples. These hypotheses also include the capital income monotonicity condition as well as additional assumptions which constrain the difference between the consumption goods and capital goods sectors' relative capital intensities.

[9] See Scheinkman [14] for example.

[10] Nishimura and Yano [12] offer a more precise restriction on the choice of their representative agent's discount factor in terms of the technology parameter α in order to produce a periodic equilibrium. Their result depends on the assumed linearity of the agent's one-period reward function in consumption.

Intuitively speaking, conditions (2)–(4) show how close the two-sector economy is to a one-sector model. See Remark 2 for a more precise statement. Condition (5) is valid for $u(c) = c^{1-\sigma}/(1-\sigma)$ if $\sigma \neq 1$ and $u(c) = \ln c$ if $\sigma = 1$. These one-period reward functions place sufficient curvature on the most patient household's utility function to promote consumption smoothing behavior and thereby dampen oscillations that might persist with less curvature (as shown by Nishimura and Yano [12] in the representative agent case).

LEMMA 4.6. *Assume the previous conditions and that*

(1) $(\mathscr{T}_k(k, y)\, k)_k > 0$,

(2) $\|\mathscr{T}_y + 1\| < v$,

(3) $\|\mathscr{T}_{ky}\| < v$,

(4) $\|(\mathscr{T}_k(k, y)\, k - \mathscr{T}_k(k, 0)\, k)_k\| < v$, *where* $\|\phi\| = |\phi|_{L^\infty(\mathscr{D} \cap ((k, y) : k \geqslant \varepsilon))}$, ε *is the lower bound for* k, *and* $v > 0$ *is small enough,*

(5) $\lim_{c \to 0} u_1''(c)/u_1'(c) = -\infty$.

Then

(i) $c_t^1 = c_t^1(k_{t-1}, k_t)$ *is increasing in* k_{t-1} *and decreasing in* k_t;

(ii) $k_{t+1} = G(k_{t-1}, k_t)$ *is decreasing in* k_{t-1} *and increasing in* k_t.

Proof. (i) The inequality $(c_t^1)_{k_t} < 0$ follows from $\mathscr{T}_{ky} \leqslant 0$; $(c_t^1)_{k_{t-1}} > 0$ follows from (1), (3).

(ii) Differentiating the left-hand side of (4.4) with respect to k_{t+1}, we obtain

$$\delta_1 \frac{1}{-\mathscr{T}_y(k_{t-1}, k_t)} (\mathscr{T}_{ky}(k_t, k_{t+1})\, u_1'(c_{t+1}^1) + \mathscr{T}_k(k_t, k_{t+1})\, u_1''(c_{t+1}^1)\, \partial c_{t+1}^1/\partial k_{t+1})$$

which is positive by (3) and (i). Thus (4.4) defines k_{t+1} as a function of (k_{t-1}, k_t); i.e., $k_{t+1} = G(k_{t-1}, k_t)$. The partial derivative of the left-hand side of (4.4) with respect to k_t

$$\delta_1 \frac{1}{-\mathscr{T}_y(k_{t-1}, k_t)} (\mathscr{T}_{kk}(k_t, k_{t+1})\, u_1'(c_{t+1}^1) + \mathscr{T}_k(k_t, k_{t+1})\, u_1''(c_{t+1}^1)\, \partial c_{t+1}^1/\partial k_t)$$

$$-\delta_1 \frac{\mathscr{T}_k(k_t, k_{t+1})\, u_1'(c_{t+1}^1)\, \mathscr{T}_{ky}(k_t, k_{t+1})}{\mathscr{T}_y^2(k_{t-1}, k_t)}$$

is negative due to (2), (3), and (i). That provides $G_{k_t} > 0$. To prove that $(G(k_{t-1}, k_t))_{k_{t-1}} < 0$ we multiply both sides of (4.4) by $-\mathcal{T}_y(k_{t-1}, k_t)$. The derivative of $-\mathcal{T}_y(k_{t-1}, k_t)\, u_1'(c_t^1)$ with respect to k_{t-1}

$$-\mathcal{T}_{ky}(k_{t-1}, k_t)\, u_1'(c_t^1) - \mathcal{T}_y(k_{t-1}, k_t)\, u_1''(c_t^1)\, \partial c_t^1/\partial k_{t-1}$$

is negative due to (3), (4), (5), and (i). ∎

Hypotheses (1)–(5) enable us to obtain more information about the behavior of $\{k_t\}$ which is stated in Propositions 4.1–4.3.

PROPOSITION 4.1. *There is no $t_1 \geqslant t_0$ such that $k_{t_1+1} > k_{t_1} \geqslant \bar{k}$.*

Proof. Suppose that there exists $t_1 \geqslant t_0$ such that $k_{t_1+1} > k_{t_1} \geqslant \bar{k}$. The partial derivative

$$\frac{\partial}{\partial k_t}\left(\frac{r_{t+1}}{q_t}\right) = \frac{\partial}{\partial k_t}\left(\frac{\mathcal{T}_k(k_t, G(k_{t-1}, k_t))}{-\mathcal{T}_y(k_{t-1}, k_t)}\right)$$

is negative by $\mathcal{T}_{ky} \leqslant 0$ and (ii). It follows that

$$\frac{\mathcal{T}_k(k_{t_1+1}, G(k_{t_1}, k_{t_1+1}))}{-\mathcal{T}_y(k_{t_1}, k_{t_1+1})} < \frac{\mathcal{T}_k(k_{t_1}, G(k_{t_1}, k_{t_1}))}{-\mathcal{T}_y(k_{t_1}, k_{t_1})}.$$

From the inequality $k_{t_1} \geqslant \bar{k}$ we obtain $\delta_1 \frac{\mathcal{T}_k(k_{t_1}, k_{t_1})}{-\mathcal{T}_y(k_{t_1}, k_{t_1})} \leqslant 1$. Thus $G(k_{t_1}, k_{t_1}) \geqslant k_1$ because the left-hand side of (4.4) is increasing in k_{t+1}. Therefore,

$$\delta_1 \frac{\mathcal{T}_k(k_{t_1+1}, G(k_{t_1}, k_{t_1+1}))}{-\mathcal{T}_y(k_{t_1}, k_{t_1+1})} < \delta_1 \frac{\mathcal{T}_k(k_{t_1}, G(k_{t_1}, k_{t_1}))}{-\mathcal{T}_y(k_{t_1}, k_{t_1})} \leqslant 1.$$

If $G(k_{t_1}, k_{t_1+1}) < k_{t_1+1}$ then $c_{t_1+2}^1 > c_{t_1+1}^1$ and $\delta_1 r_{t_1+2}/q_{t_1+1} > 1$; then we have a contradiction. Hence $k_{t_1+2} = G(k_{t_1}, k_{t_1+1}) \geqslant k_{t_1+1}$. By induction we conclude that $k_{t+1} \geqslant k_t$ for $\forall t \geqslant t_1$. Thus the sequence $\{k_t\}$ is increasing and bounded by k^M. Therefore, $\lim_{t\to\infty} k_t = k > \bar{k}$. That contradicts Lemma 4.5. ∎

The same idea allows us to prove the following

PROPOSITION 4.2. *There is no $t_1 \geqslant t_0$ such that $\bar{k} \geqslant k_{t_1} > k_{t_1+1}$.*

PROPOSITION 4.3. *There is no $t_1 \geqslant t_0$ such that $k_{t_1-1} > \bar{k} > k_{t_1+1} \geqslant k_{t_1}$.*

Proof. Suppose that this is not true. It follows from (i) that $c^1_{t_1} > c^1_{t_1+1}$. Thus $\delta_1 r_{t_1+1}/q_{t_1} < 1$. Solve (4.4) for k_{t-1}; $k_{t-1} = N(k_t, k_{t+1})$, where $N_{k_t}, N_{k_{t+1}}$ are negative. Consider the following partial derivative

$$\frac{\partial}{\partial k_t}\left(\frac{\mathscr{T}_k(k_t, k_{t_1+1})}{-\mathscr{T}_y(N(k_t, k_{t_1+1}), k_t)}\right)$$
$$= \frac{-\mathscr{T}_{kk}(k_t, k_{t_1+1})\,\mathscr{T}_y(N(k_t, k_{t_1+1}), k_{t_1+1})}{\mathscr{T}^2_y(N(k_t, k_{t_1+1}))}$$
$$+ \frac{\left(\begin{array}{c}\mathscr{T}_k(k_t, k_{t_1+1})((\mathscr{T}_{ky}(N(k_t, k_{t_1+1}), k_t)\,N_{k_t}(k_t, k_{t_1+1})\\ +\mathscr{T}_{yy}(N(k_t, k_{t_1+1}), k_t))\end{array}\right)}{\mathscr{T}^2_y(N(k_t, k_{t_1+1}))}.$$

Let $k_t \in [k_{t_1}, k_{t_1+1}]$. There are two cases to consider:

(1) $N(k_t, k_{t_1+1}) \geqslant \bar{k}$. Then $c^1_t(N(k_t, k_{t_1+1}), k_t) > \bar{c}^1 > 0$. The lower bound for c^1_t gives us the upper bound for N_{k_t}. Thus the partial derivative $\partial/\partial k_t(-\mathscr{T}_k(k_t, k_{t_1+1})/\mathscr{T}_y(N(k_t, k_{t_1+1}), k_t))$ is negative if v is small enough. Therefore,

$$1 > \delta_1 \frac{\mathscr{T}_k(k_{t_1}, k_{t_1+1})}{-\mathscr{T}_y(k_{t_1-1}, k_{t_1})} \geqslant \delta_1 \frac{\mathscr{T}_k(k_{t_1+1}, k_{t_1+1})}{-\mathscr{T}_y(N(k_{t_1+1}, k_{t_1+1}), k_{t_1+1})}.$$

It follows from (4.4) and Lemma 4.6 that $N(k_{t_1+1}, k_{t_1+1}) \leqslant k_{t_1+1}$. Hence $-\delta_1 \mathscr{T}_k(k_{t_1}, k_{t_1+1})/\mathscr{T}_y(\bar{k}, k^*) < 1$, which is a contradiction, or

(2) there exists the smallest $k^* \in (k_{t_1}, k_{t_1+1})$ such that $N(k^*, k_{t_1+1}) = \bar{k}$. Then $\partial/\partial k_t(-\mathscr{T}_k(k_t, k_{t_1+1})/\mathscr{T}_y(N(k_t, k_{t_1+1}), k_t))$ is negative if $k_t \in [k_{t_1}, k^*]$. Thus

$$\delta_1 \frac{\mathscr{T}_k(k^*, k_{t_1+1})}{-\mathscr{T}_y(\bar{k}, k^*)} < 1,$$

which is a contradiction. ▮

Definition 4.3. A sequence $\{k_t\}$ is eventually monotonic if there exists t^* such that $\{k_t\}$ is increasing or decreasing if $t > t^*$.

Now we can prove our main convergence result when $\mathscr{T}_{ky} \leqslant 0$.

Theorem 4.1. *If the turnpike property holds as well as the assumptions of Lemma* 4.6, *then the sequence* $\{k_t\}$ *is eventually monotonic and* $\lim_{t\to\infty} k_t = \bar{k}$.

Proof. Assume that there exists $t_1 > t_0$ such that $k_{t_1-1} \geqslant k_{t_1} \leqslant k_{t_1+1}$ and at least one inequality is strict. It follows from (4.4) that $\delta_1 r_{t_1+1}/q_{t_1} < 1$. We need to consider two cases:

(1) $k_{t_1+1} \geqslant \bar{k}$. Then $k_{t_1+2} = G(k_{t_1}, k_{t_1+1}) > G(k_{t_1-1}, k_{t_1}) = k_{t_1+1} \geqslant \bar{k}$ and by Proposition 4.1 that is impossible;

(2) $k_{t_1+1} < \bar{k}$. Then $k_{t_1-1} > \bar{k}$ and by Proposition 4.3 that is impossible. Thus $\{k_t\}$ is eventually monotonic and by Lemma 4.5 $\lim_{t\to\infty} k_t = \bar{k}$. ∎

This theorem does not restrict the magnitude of the first agent's discount factor other than to place it in the interval (0, 1). Instead, we take as given the agents' preferences and their degrees of discounting of future utilities. Then we find sufficient conditions on technology for equilibria to be convergent sequences.

Remark 2. Hypotheses (2)–(5) of Lemma 4.6 hold if $|f-g|_{L^\infty} + |f'-g'|_{L^\infty}$ is small, where $f(x) = \mathscr{F}^0(x, H)$ and $g(x) = \mathscr{F}^1(x, H)$.

We note that Theorem 4.1 can be applied to models with a representative agent. The equilibrium path is a social optimum and vice versa.[11] We identify the most patient household in our results with the representative agent. Benhabib and Nishimura [6] and Nishimura and Yano [12] have shown that when u is linear in consumption and $\mathscr{T}_{ky} \leqslant 0$, then the resulting equilibrium (and thus optimal) capital sequence is oscillatory. Our result says something different—the equilibrium capital sequence may initially oscillate, but it is eventually monotonic. The nonlinearity of u is playing a crucial role in dampening any initial oscillations and eventually promoting monotonicity of the equilibrium capital sequence given that the two sectors' technologies are close to a one-sector technology. Nishimura and Yano [12, p. 243] note that whether or not the equilibrium oscillates depends on the relative strengths of two effects working in opposite directions. An increase in capital raises the marginal rate of transformation between consumption and investment goods as the consumption goods sector is more capital intensive than the capital goods sector. However, the resulting increase in the consumption goods output necessarily lowers the representative household's marginal utility of consumption. If the decline in marginal utility is sufficiently large, the agent will respond by accumulating capital. This is a consumption smoothing effect. If the decline is small, or zero (as in the case when u is linear), then the capital supply schedule can be shown to shift upward and the resulting path is oscillatory as the next period's capital will be smaller. When both sectoral production

[11] See, for example, the presentation of equivalence principles in Becker and Boyd [3].

functions are close to the one-sector case, then we expect the increased marginal rate of transformation to be small (it is a constant in the one-sector case) and the consumption smoothing effect dominates. Hence, u's strict concavity promotes the (eventual) monotonicity of the optimal capital sequence.

4.4. *Convergence of the Equilibrium Capital Stock when* $\mathcal{T}_{ky} \geqslant 0$

Suppose that the capital goods sector is the more capital intensive sector. Then $\mathcal{T}_{ky} \geqslant 0$ holds. We prove an asymptotic result in three steps. First, by making some additional assumptions on the household's capital income function we show the sequence of equilibrium capital stocks converges to the steady state. Second, we demonstrate that the turnpike property holds under no additional assumptions. In particular, the strong discounting condition used to prove convergence when $\mathcal{T}_{ky} \leqslant 0$ is not required here. Third, we show that the convergence is eventually monotonic. Our arguments are the natural extension of Becker and Foias's [4] one-sector methods to the two-sector case.

LEMMA 4.7. *Assume that*

(1) $\mathcal{T}_{ky} \geqslant 0$,

(2) $(\mathcal{T}_k(k, y)\, k)_k \geqslant 0$,

(3) $(\mathcal{T}_k(k, y)\, k + \mathcal{T}_y(k, y)\, y)_y \leqslant 0$.

Then there is no t_0 *such that* $k_{t_0} > \bar{k}$ *and* $k_{t_0} \geqslant k_{t_0-1}$.

Proof. Assume that there exists t_0 such that $k_{t_0} > \bar{k}$ and $k_{t_0} \geqslant k_{t_0-1}$. Suppose that $k_{t_0+1} < k_{t_0}$. Then $\mathcal{T}_k(k_{t_0}, k_{t_0+1})/\mathcal{T}_y(k_{t_0-1}, k_{t_0}) \leqslant \mathcal{T}_k(k_{t_0}, k_{t_0})/\mathcal{T}_y(k_{t_0}, k_{t_0})$ by (1). Thus $\delta_1 \mathcal{T}_k(k_{t_0}, k_{t_0+1})/\mathcal{T}_y(k_{t_0-1}, k_{t_0}) < 1$. Let $S_t = \{h \mid k_t^h > 0\}$, $X_t = \sum_{h \in S_t} k_t^h$, $\Gamma_t = \sum_{h \in S_t} c_t^h$, $H_t = \#S_t$. For $h \in S_{t_0}$ we have

$$u'(c^h_{t_0+1}) \geqslant \delta_1 \frac{\mathcal{T}_k(k_{t_0}, k_{t_0+1})}{\mathcal{T}_y(k_{t_0-1}, k_{t_0})} u'(c^h_{t_0+1}) = u'(c^h_{t_0});$$

it follows that $c^h_{t_0+1} \leqslant c^h_{t_0}$. Therefore $\Gamma_{t_0+1} \leqslant \Gamma_{t_0}$. We use budget constraints for households in S_{t_0} to obtain

$$\Gamma_{t_0} + q_{t_0} X_{t_0} = r_{t_0} X_{t_0-1} + H_{t_0} w_{t_0},$$
$$\Gamma_{t_0+1} + q_{t_0+1} X_{t_0+1} = r_{t_0+1} X_{t_0} + H_{t_0} w_{t_0+1}.$$

Note that $X_{t_0+1} \leqslant k_{t_0+1}$, $X_{t_0} = k_{t_0}$, $X_{t_0-1} \leqslant k_{t_0-1}$. Therefore

$$\begin{aligned} q_{t_0+1}k_{t_0+1} - q_{t_0}k_{t_0} &= q_{t_0+1}k_{t_0+1} - q_{t_0}X_{t_0} \\ &\geqslant q_{t_0+1}X_{t_0+1} - q_{t_0}X_{t_0} \\ &\geqslant r_{t_0+1}X_{t_0} - r_{t_0}X_{t_0-1} + H_{t_0}w_{t_0+1} - H_{t_0}w_{t_0} \\ &\geqslant r_{t_0+1}k_{t_0} - r_{t_0}k_{t_0-1} + H_{t_0}w_{t_0+1} - H_{t_0}w_{t_0}. \end{aligned}$$

Thus

$$q_{t_0+1}k_{t_0+1} - r_{t_0+1}k_{t_0} - H_{t_0}w_{t_0+1} \geqslant q_{t_0}k_{t_0} - r_{t_0}k_{t_0-1} - H_{t_0}w_{t_0}. \tag{4.5}$$

We consider the following function

$$Y(k, y) = -\mathscr{T}_y(k, y)\, y - \mathscr{T}_k(k, y)\, k - n(\mathscr{T}(k, y) - \mathscr{T}_y(k, y)\, y - \mathscr{T}_k(k, y)\, k),$$

where $0 < n \leqslant 1$. It follows from (2), (3) that $Y_k < 0$ and $Y_y > 0$. We assumed that $k_{t_0} \geqslant k_{t_0-1}$. Hence $k_{t_0+1} \geqslant k_{t_0}$ by (4.5). This contradicts the assumption we made in the beginning of the proof. Therefore $\{k_t\}$ is increasing for any $t \geqslant t_0$ and $\lim_{t\to\infty} k_t = k$. Then $\lim_{t\to\infty} w_t = w$, where $w = (\mathscr{T}(k, k) - \mathscr{T}_y(k, k)\, k - \mathscr{T}_k(k, k)\, k)/H$ and $k > \bar{k}$. The inequality $w > 0$ follows from Lemma 2.1. Hence

$$\overline{\lim_{t\to\infty}}\; c_t^h > 0 \qquad (h = 1, ..., H)$$

and $\lim_{t\to\infty} c_t^1 = c^1 > 0$. Then it follows from (4.4) that $k = \bar{k}$—a contradiction because $k_{t_0} > \bar{k}$. ∎

LEMMA 4.8. *Under the assumptions of Lemma* 4.7, *if there exists* t_0 *such that* $k_t > \bar{k}$ *for all* $t \geqslant t_0$, *then* $\{k_t\}$ *converges monotonically to the steady state* $\bar{k}$.

Proof. The inequality $k_{t+1} \leqslant k_t$ follows from Lemma 4.7. Thus $\lim_{t\to\infty} k_t = k$. Now we apply the same argument as in the proof of Lemma 4.7 to show that $k = \bar{k}$. ∎

LEMMA 4.9. *Under the conditions of Lemma* 4.7, *if* $k_t \leqslant \bar{k}$ *for all* t *large enough, then*

$$\lim_{t\to\infty} k_t = \bar{k}.$$

Proof. Assume that there exists t_0 such that $k_t \leqslant \bar{k}$ for all $t \geqslant t_0$. Then $1 \leqslant \delta_1 \mathcal{T}_k(k_t, k_{t+1}) / \mathcal{T}_y(k_{t-1}, k_t)$ for all $t > t_0$. Therefore $c_t^1 \leqslant c_{t+1}^1$,

$$1 \leqslant \delta_1 \frac{\mathcal{T}_k(k_t, k_{t+1})}{\mathcal{T}_y(k_{t-1}, k_t)} \leqslant \frac{u_1'(c_t^1)}{u_1'(c_{t+1}^1)} \to 1$$

and thus $\lim_{t\to\infty} k_t = \bar{k}$. ▮

LEMMA 4.10. *Under the assumptions of Lemma* 4.7 $k_t^h = 0$ *eventually holds for all* $h \geqslant 2$.

Proof. We proved that $\lim_{t\to\infty} k_t = \bar{k}$. Then (4.3) holds for all t large enough and we repeat the argument used in the proof of Lemma 4.4. ▮

THEOREM 4.2. *Under the assumptions of Lemma* 4.7 *the sequence of equilibrium capital stocks converges eventually monotonically to the steady state and the turnpike property holds.*

Proof. The turnpike property follows from Lemma 4.10. If $k_t \geqslant \bar{k}$ then by Lemma 4.8 $\{k_t\}$ converges to $\bar{k}$ monotonically. It follows from Lemma 4.7 that the only alternative left to consider is $k_t \leqslant \bar{k}$ for all large t. We assume that $k_t^h = 0, h \geqslant 2$, and $k_t \leqslant \bar{k}$ for all $t \geqslant t_0$. Then the budget constraints for household 1 are

$$c_{t+1} + q_{t+1} k_{t+1} = r_{t+1} k_t + w_{t+1},$$
$$c_t + q_t k_t = r_t k_{t-1} + w_t$$

and $c_{t+1} \geqslant c_t$, $t \geqslant t_0$. Subtracting one equation from the other we obtain

$$q_{t+1} k_{t+1} - r_{t+1} k_t - w_{t+1} \leqslant q_t k_t - r_t k_{t-1} - w_t.$$

If we assume that $k_t < k_{t-1}$ then the properties of Y, introduced in the proof of Lemma 4.7, imply that $k_{t+1} \leqslant k_t$. Thus $\lim k_t = k < \bar{k}$—a contradiction. ▮

CONCLUSION

The proofs that in every equilibrium the aggregate capital converges eventually monotonically to the steady state capital relied on some restrictions on the economy's transformation function. Our results included the one-sector model as a special case. One important structure governing the two-sector model's equilibrium dynamics with distinct consumption and capital goods turned out to be the economy's relative factor intensities. We explored models without factor intensity reversals and found cases where equilibria did not converge as well as conditions sufficient for convergence to the long run steady state solution. One problem for future research is to understand the emergence of cycles better. It would also be natural to conjecture that chaotic equilibria might be found.

It is possible to criticize the model's steady state analysis by questioning whether it makes sense for the most patient household to act as a price-taker in that solution. Sorger [17] addresses this in the one-sector context and it is reasonable to conjecture similar results should be available in the two-sector model. However, verifying that is a matter left for a future paper.

APPENDIX A

We begin by proving several properties of the transformation function.

LEMMA A.1. *Let* $x = x(k, y)$ *and* $l = l(k, y)$ *be such that* $\mathcal{T}(k, y) = \mathcal{F}^0(x, l)$. *Then* $x, l, \mathcal{T} \in \mathbb{C}^1(\mathring{\mathcal{D}})$.

Proof. First let us make a change of variables in (2.1). Set $\mathcal{T}_1(k, y) = \mathcal{T}(Hk, Hy)/H$. Then

$$\mathcal{T}_1(k, y) = \max(1/H)\, \mathcal{F}^0(x^0, y^0) \text{ such that}$$
$$\mathcal{F}^1(x^1, y^1) \geqslant Hy, \qquad x^0 + x^1 \leqslant Hk, \qquad l^0 + l^1 \leqslant H, \qquad x^0, l^0, x^1, l^1 \geqslant 0.$$

If we define new variables $l_1^0 = l^0/H, l_1^1 = l^1/H, x_1^0 = x^0/H, x_1^1 = x^1/H$ then

$$\mathcal{T}_1(k, y) = \max \mathcal{F}^0(x_1^0, y_1^0) \text{ such that}$$
$$\mathcal{F}^1(x_1^1, y_1^1) \geqslant y, \quad x_1^0 + x_1^1 \leqslant k, \quad l_1^0 + l_1^1 \leqslant 1, \quad x_1^0, l_1^0, x_1^1, l_1^1 \geqslant 0. \tag{A.1}$$

So without loss of generality we can assume that $H = 1$. The function $\mathcal{F}^0$ achieves its maximum value only when

$$\mathcal{F}^1(x^1, l^1) = y, \qquad x^0 + x^1 = k, \qquad l^0 + l^1 = 1 \tag{A.2}$$

because $\mathcal{F}^0, \mathcal{F}^1$ are increasing in both arguments. We solve Eqs. (A.2) for x^0 in terms of l^0. Set $f(x) = \mathcal{F}^0(x, 1)$ and $g(x) = \mathcal{F}^1(x, 1)$. Then the equalities $\mathcal{F}^0(x, l) = lf(x/l)$, $\mathcal{F}^1(x, l) = lg(x/l)$ follow from the homogeneity of $\mathcal{F}^0$ and $\mathcal{F}^1$. The functions f, g have the following properties:

(i) $f, g \in \mathbb{C}^2(\Re_{++})$, $f(0) = g(0) = 0$;
(ii) $f', g' > 0$, $f'', g'' < 0$;
(iii) $f'(0+) = +\infty$, $f'(+\infty) = 0$, $g'(0+) = +\infty$, $g'(+\infty) = 0$;
(iv) $f(x) - xf'(x) > 0$, $g(x) - xg'(x) > 0$.

If $(k, y) \in \mathring{\mathcal{D}}$ then it follows from (A.2) that $x^0, x^1, l^0, l^1 > 0$ and $\mathcal{F}^0$ achieves its maximum inside $(0, k) \times (0, 1)$. Solving the equations in (A.2), we obtain

$$l^1 = 1 - l^0, \qquad x^1 = l^1 g^{-1}\left(\frac{y}{l^1}\right), \qquad x^0 = k - (1 - l^0)\, g^{-1}\left(\frac{y}{1 - l^0}\right).$$

The definition of the transformation function can be written as the following maximization problem:

$$\mathcal{T}(k, y) = \max_{l \in (0, 1)} lf\left(\frac{1}{l}\left(k - (1 - l)\, g^{-1}\left(\frac{y}{1 - l}\right)\right)\right). \tag{A.3}$$

At a point of maximum

$$\frac{d}{dl}\left[lf\left(\frac{1}{l}\left(k - (1 - l)\, g^{-1}\left(\frac{y}{1 - l}\right)\right)\right)\right] = 0. \tag{A.4}$$

Equation (A.4) gives us

$$f\left(\frac{x}{l}\right) - \frac{x}{l} f'\left(\frac{x}{l}\right) + f'\left(\frac{x}{l}\right)\left[g^{-1}\left(\frac{y}{1 - l}\right) - \frac{y}{1 - l} g^{-1\prime}\left(\frac{y}{1 - l}\right)\right] = 0,$$

where $x = k - (1 - l)\, g^{-1}(y/(1 - l))$. Therefore

$$\left(\frac{f\left(\frac{x}{l}\right)}{f'\left(\frac{x}{l}\right)} - \frac{x}{l}\right) + \left(g^{-1}\left(\frac{y}{1 - l}\right) - \frac{y}{1 - l} g^{-1\prime}\left(\frac{y}{1 - l}\right)\right) = 0. \tag{A.5}$$

If we set $f_1(x) = f(x)/f'(x) - x$, $g_1(x) = xg^{-1\prime}(x) - g^{-1}(x)$ then (A.5) takes the following form

$$f_1\left(\frac{x}{l}\right) - g_1\left(\frac{y}{1 - l}\right) = 0. \tag{A.6}$$

We find that $f_1'(x) = -(f(x)\, f''(x))/(f'(x))^2 > 0$ and $g_1'(x) = xg^{-1''}(x) > 0$. In addition, $g_1(0) = 0$. Hence $g_1(x) > 0$ for all $x > 0$. The derivative of the function appearing in (A.6) with respect to l is equal to

$$f_1'\left(\frac{x}{l}\right)\left(-\frac{x}{l^2} - \frac{1}{l}\, g_1\left(\frac{y}{1-l}\right)\right) - g_1'\left(\frac{y}{1-l}\right)\frac{y}{(1-l)^2}$$

which is negative. Thus by the implicit function theorem (A.6) can be solved for l and for a given (k, y); l is unique. Finally $l = l(k, y)$, $x = x(k, y, l(k, y))$, $\mathcal{T} = \mathcal{T}(x(k, y, l(k, y)), l(k, y)) \in \mathbb{C}^1(\mathring{\mathcal{D}})$. ▮

LEMMA A.2. $\mathcal{T} \in \mathbb{C}^1(\mathcal{D}/(0, 0)) \cap \mathbb{C}^2(\mathring{\mathcal{D}})$, $\partial\mathcal{T}/\partial k > 0$, $\partial\mathcal{T}/\partial y < 0$, $\partial^2\mathcal{T}/\partial k^2 < 0$.

Proof. We have the following sequences of equalities

$$\frac{\partial \mathcal{T}}{\partial k}(k, y) = \frac{\partial \mathcal{F}^0}{\partial k}(x(k, y, l(k, y)), l(k, y))$$

$$= \frac{\partial \mathcal{F}^0}{\partial l}\frac{\partial l}{\partial k} + \frac{\partial \mathcal{F}^0}{\partial x}\frac{\partial x}{\partial k} + \frac{\partial \mathcal{F}^0}{\partial x}\frac{\partial x}{\partial l}\frac{\partial l}{\partial k} = \frac{\partial \mathcal{F}^0}{\partial x}\frac{\partial x}{\partial k},$$

$$\frac{\partial \mathcal{T}}{\partial y}(k, y) = \frac{\partial \mathcal{F}^0}{\partial y}(x(k, y, l(k, y)), l(k, y))$$

$$= \frac{\partial \mathcal{F}^0}{\partial l}\frac{\partial l}{\partial y} + \frac{\partial \mathcal{F}^0}{\partial x}\frac{\partial x}{\partial y} + \frac{\partial \mathcal{F}^0}{\partial x}\frac{\partial x}{\partial l}\frac{\partial l}{\partial y} = \frac{\partial \mathcal{F}^0}{\partial x}\frac{\partial x}{\partial y}.$$

In addition,

$$\frac{\partial x}{\partial k} = 1, \qquad \frac{\partial x}{\partial y} = -g^{-1\prime}\left(\frac{y}{1-l}\right).$$

Finally

$$\frac{\partial \mathcal{T}}{\partial k} = \frac{\partial \mathcal{F}^0}{\partial x} = f'\left(\frac{x}{l}\right), \tag{A.7}$$

$$\frac{\partial \mathcal{T}}{\partial y} = -\frac{\partial \mathcal{F}^0}{\partial x}\, g^{-1\prime}\left(\frac{y}{1-l}\right) = -f'\left(\frac{x}{l}\right) g^{-1\prime}\left(\frac{y}{1-l}\right), \tag{A8}$$

and thus

$$\frac{\partial \mathcal{T}}{\partial k} > 0, \frac{\partial \mathcal{T}}{\partial y} < 0, \qquad (k, y) \in \mathring{\mathcal{D}}.$$

It follows from equalities (A.7) and (A.8) that $\mathcal{T}_k, \mathcal{T}_y \in \mathbb{C}^1(\mathring{\mathcal{D}})$; therefore $\mathcal{T} \in \mathbb{C}^2(\mathring{\mathcal{D}})$. Differentiating (A.7) with respect to k we obtain the following equality

$$\frac{\partial^2 \mathcal{T}}{\partial k^2} = f''\left(\frac{x}{l}\right)\left[\frac{1}{l}\left(1 + l_k g^{-1}\left(\frac{y}{1-l}\right) - \frac{yl_k}{1-l} g^{-1\prime}\left(\frac{y}{1-l}\right)\right) - \frac{xl_k}{l^2}\right].$$

Now let us take the derivative of (A.6) with respect to k

$$f_1'\left(\frac{x}{l}\right)\left[\frac{1}{l}\left(1 + l_k g^{-1}\left(\frac{y}{1-l}\right) - \frac{yl_k}{1-l} g^{-1\prime}\left(\frac{y}{1-l}\right)\right) - \frac{xl_k}{l^2}\right]$$
$$- \frac{yl_k}{(1-l)^2} g_1'\left(\frac{y}{1-l}\right) = 0.$$

Combining these two equations, we finally obtain

$$\frac{\partial^2 \mathcal{T}}{\partial k^2} = \frac{f''(x/l)}{f'(x/l)} \frac{yl_k}{(1-l)^2} g_1'\left(\frac{y}{1-l}\right). \tag{A.9}$$

We can see that the sign of $\partial^2 \mathcal{T}/\partial k^2$ depends only on the sign of l_k. It follows from the definition that $\mathcal{T}$ is concave in (k, y); therefore $\partial^2 \mathcal{T}/\partial k^2, \partial^2 \mathcal{T}/\partial y^2 \leqslant 0$ in $\mathring{\mathcal{D}}$. If $l_k = 0$ in $\mathring{\mathcal{D}}$ then $f_1'(x/l) = 0$, which is impossible. Thus $l_k > 0$ everywhere in $\mathring{\mathcal{D}}$.

Next we prove that $\mathcal{T}_k, \mathcal{T}_y$ are continuous up to the boundary $\partial \mathcal{D}/(0, 0)$. We need to consider two cases:

(i) $(k, 0) \in \partial \mathcal{D}, k > 0$. Take $\{(k_n, y_n)\} \subset \mathring{\mathcal{D}}$ such that $\{k_n, y_n\} \to (k, 0)$. Then $(x_n^1, l_n^1) \to (0, 0)$ because $\mathcal{F}^1(x_n^1, l_n^1) = y_n \to 0$ and (x_n^0, l_n^0) maximizes $\mathcal{F}^0$. Hence $\{(x_n^0, y_n^0)\} \to (k, 1)$ and $x_n/l_n \to k$. In addition, $y_n/(1-l_n) \to g_1^{-1}(f_1(k))$ by (A.6). Finally,

$$\frac{\partial \mathcal{T}}{\partial k}(k_n, y_n) \to f'(k), \qquad \frac{\partial \mathcal{T}}{\partial y}(k_n, y_n) \to f'(k)\, g^{-1\prime}(g_1^{-1}(f_1(k))) \tag{A.10}$$

as $\{(k_n, y_n)\} \to (k, 0)$.

(ii) $(k, \mathscr{F}^0(k,1)) \in \partial\mathscr{D}, k>0$. Take $\{k_n, y_n)\} \subset \mathring{\mathscr{D}}$ such that $\{(k_n, y_n)\} \to (k, \mathscr{F}^0(k,1))$. Then $\{(x_n^1, l_n^1)\} \to (k,1); y_n/(1-l_n) \to g(k), x_n/l_n \to f_1^{-1}(g_1(g(k)))$ by (A.6). Therefore

$$\begin{aligned} \frac{\partial \mathscr{T}}{\partial k}(k_n, y_n) &\to f'(f_1^{-1}(g_1(g(k)))), \\ \frac{\partial \mathscr{T}}{\partial y}(k_n, y_n) &\to f'(f_1^{-1}(g_1(g(k))))\, g^{-1\prime}(g(k))) \end{aligned} \tag{A.11}$$

as $\{(k_n, y_n)\} \to (k, \mathscr{F}^0(k,1))$. ▌

LEMMA 5.3. *$\partial\mathscr{T}/\partial k \to +\infty$, $\mathscr{E} = -\mathscr{T}_k/\mathscr{T}_y \to +\infty$ as $k, y \to 0$.*

Proof. Consider a sequence $\{(k_n, y_n)\} \cap \mathring{\mathscr{D}}$ such that $\{(k_n, y_n)\} \to (0,0)$. If we assume that there exists a subsequence $\{(k_{n_m}, y_{n_m})\}$ such that $\{x_{n_m}/l_{n_m}\} = \{x(k_{n_m}, y_{n_m})/l(k_{n_m}, y_{n_m})\} \to a > 0$, then $\{(x_{n_m}, l_{n_m})\} \to (0,0)$ and $\{y_{n_m}/(1-l_{n_m})\} \to 0$. It follows from (A.6) and properties of f_1, g_1 that $\{x_{n_m}/l_{n_m}\} \to 0$ which contradicts the assumption. Hence $\{x_n/l_n\} \to 0$ as $\{(k_n, y_n)\} \to (0,0)$. Finally, by (A.7), (A.8)

$$\frac{\partial\mathscr{T}}{\partial k}(k_n, y_n) = f'\left(\frac{x_n}{l_n}\right) \to +\infty, \quad \mathscr{E} = -\frac{\mathscr{T}_k}{\mathscr{T}_y}(k_n, y_n) = \frac{1}{g^{-1\prime}\left(\dfrac{y_n}{1-l_n}\right)} \to +\infty$$

as $(k_n, y_n) \to (0,0)$. ▌

Proof of Proposition 2.1. Let us define technology functions as follows: $\mathscr{F}^0(x,l) = \mathscr{F}^1(x,l) = lf(x/l)$. Then $f_1(x) = f(x)/f'(x) - x$, $g_1(x) = xf^{-1\prime}(x) - f^{-1}(x)$. Set

$$l_0 = 1 - \frac{y}{f(k)}, \qquad l_1 = \frac{y}{f(k)}, \qquad x_0 = k - \frac{yk}{f(k)}, \qquad x_1 = \frac{yk}{f(k)}.$$

Then $l_0, l_1, x_0, x_1 \geqslant 0$, $l_0 + l_1 = 1$, $x_0 + x_1 = k$. In addition,

$$f_1\left(\frac{x}{l}\right) = \frac{f(k)}{f'(k)} - k, \qquad g_1\left(\frac{y}{1-l}\right) = f(k)\, f^{-1\prime}(f(k)) - f^{-1}(f(k)).$$

Therefore, $g_1(y/(1-l)) = f(k)/f'(k) - k$ and (A.6) holds. It was proved that for any positive k, y there is a unique l that solves (A.6). Thus x_0, l_0 maximize $\mathscr{F}^0$ subject to (2.1). Finally, $\mathscr{T} = l_0 f(x_0/l_0) = f(k) - y$. ▌

Proof of Proposition 2.2. If $\mathscr{T}(k, y) = \phi(k) - \psi(y)$ then $\mathscr{T}_{ky} \equiv 0$ and $\phi(0) = \psi(0) = 0$. Also $f(k) = \mathscr{T}(k, 0) = \phi(k)$. Equality (A.7) gives us $\mathscr{T}_{ky} = f''(x/l)(x/l)_y$. Thus $(x/l)_y \equiv 0$. Differentiating (A.6) with respect to y, we obtain

$$f_1'\left(\frac{x}{l}\right)\left(\frac{x}{l}\right)_y - g_1'\left(\frac{y}{1-l}\right)\frac{1-l+yl_y}{(1-l)^2} = 0.$$

Therefore, $1-l+yl_y \equiv 0$. All solutions of that differential equation are given by the formula $l = \alpha(k)\, y + 1$ for some function $\alpha(k)$. In addition, $\mathscr{T}(k, y) = lf(x/l)$. Thus we come to the following identity

$$(\alpha(k)\, y+1)\, f(\beta(k)) \equiv \phi(k) - \psi(y),$$

where $\beta(k)$ is found by solving (A.6) for x/l. Now we can conclude that $\psi(y) \equiv by$, $b = const$. ▮

Proof of Lemma 2.1. Suppose that $w_t = 0$. Then $\mathscr{P}_t \equiv 0$ along the line segment that connects (k_{t-1}, y_t) and $(0, 0)$. Then $\mathscr{T}$ is linear over the line segment; i.e., $\mathscr{T}(\lambda k_{t-1}, \lambda y_t) = \lambda \mathscr{T}(k_{t-1}, y_t)$, $\lambda \in (0, 1)$. In addition, there exist x^0, l^0, x^1, l^1 such that $\mathscr{T}(k_{t-1}, y_t) = \mathscr{F}^0(x^0, l^0)$, $l^0 + l^1 = 1$, $\mathscr{F}^1(x^1, l^1) = y_t$. We have the following inequality

$$\mathscr{T}(\lambda k_{t-1}, \lambda y_t) \geqslant \mathscr{F}^0(\lambda x^0, \lambda l^0) = \lambda \mathscr{T}(k_{t-1}, y_t)$$

with $\lambda l^0 + \lambda l^1 < 1$, $\mathscr{F}^1(\lambda x^1, \lambda l^1) = \lambda y_t$. That means that $\lambda x^0, \lambda l^0$ maximize $\mathscr{F}^0$ subject to (A.1). That contradicts (A.2). ▮

Proof of Lemma 3.2. According to (A.7), (A.8)

$$\mathscr{E}(k, y) = \frac{1}{g^{-1\prime}\left(\dfrac{y}{1-l(k, y)}\right)}. \tag{A.12}$$

Let us consider the following equation

$$\mathscr{E}(k, \lambda k) = a, \qquad \text{where } \lambda, a > 0 \text{ are given}, k \in (0, +\infty).$$

That equation is equivalent to

$$\frac{\lambda k}{1 - l(k, \lambda k)} = b, \qquad \lambda, b > 0 \text{ are given}, k \in (0, +\infty) \tag{A.13}$$

due to (A.12). If k solves (A.13) then $x/l = f_1^{-1}(g_1(b))$. Hence

$$l_1 = 1 - \frac{\lambda k}{b}, \qquad x_1 = \left(1 - \frac{\lambda k}{b}\right) f_1^{-1}(g_1(b)), \qquad x_2 = \frac{\lambda k}{b} g^{-1}(b).$$

Substitute these expressions for x_1, x_2 into the second equation of (A.2). Therefore k solves

$$\left(1 - \frac{\lambda k}{b}\right) f_1^{-1}(g_1(b)) + \frac{\lambda k}{b} g^{-1}(b) = k. \tag{A.14}$$

We notice that Eq. (A.14) is linear in k with a non-zero free term. So it can have at most one solution. Thus (A.13) and consequently (3.1) have at most one solution. ▌

Proof of Proposition 2.3. We will find a function γ such that $\mathscr{E}_y + \gamma(\mathscr{E})\, \mathscr{E}_k = 0$. We have

$$\mathscr{E}_k = \frac{g^{-1\prime\prime}\left(\frac{y}{1-l}\right)}{(g^{-1\prime}(1-l))^2} \frac{y l_k}{(1-l)^2}, \qquad \mathscr{E}_y = \frac{g^{-1\prime\prime}\left(\frac{y}{1-l}\right)}{(g^{-1\prime}(1-l))^2} \frac{1-l+y l_y}{(1-l)^2}.$$

Thus γ solves $y l_k \gamma(\mathscr{E}) + 1 - l + y l_y = 0$. We use (A.6) to find l_k, l_y. There are two equalities

$$A l_k + f_1'\left(\frac{x}{l}\right)\frac{1}{l} = 0,$$

$$A l_y - f_1'\left(\frac{x}{l}\right) g^{-1\prime}\left(\frac{y}{1-l}\right)\frac{1}{1-l} - g_1'\left(\frac{y}{1-l}\right)\frac{1}{1-l} = 0,$$

where

$$A = f_1'\left(\frac{x}{l}\right)\left[-\frac{x}{l^2} + \frac{1}{l}\left(g^{-1}\left(\frac{y}{1-l}\right) - g^{-1\prime}\left(\frac{y}{1-l}\right)\frac{y}{1-l}\right)\right]$$
$$- g_1'\left(\frac{y}{1-l}\right)\frac{y}{(1-l)^2}.$$

It follows that $A < 0$ in $\mathscr{D}$. Thus these equations can be solved for l_k, l_y and the equation for γ becomes

$$-\gamma(\mathscr{E})\, f_1'\left(\frac{x}{l}\right)\frac{1}{l} + y\left(f_1'\left(\frac{x}{l}\right) g^{-1\prime}\left(\frac{y}{1-l}\right)\frac{1}{1-l} + g_1'\left(\frac{y}{1-l}\right)\frac{1}{1-l}\right)$$
$$+ A(1-l) = 0.$$

Finally,

$$\gamma(\mathscr{E})+\frac{1-l}{y}\left(\frac{x}{l}-g^{-1}\left(\frac{y}{1-l}\right)\right)=0. \tag{A.15}$$

Equation (A.15) together with (A.6), (A.12) proves that γ is a function of $\mathscr{E}$ only.

Remark 3. Any $\mathbb{C}^1$ solution of $\mathscr{E}_y+\gamma(\mathscr{E})\,\mathscr{E}_k=0$ has a remarkable property—all level sets of $\mathscr{E}$ are straight lines.

2. APPENDIX B

LEMMA B.1. *The two-sector model specified by* $\mathscr{F}^0(x, l)=x^\alpha l^{1-\alpha}$, $\mathscr{F}^1(x, l)=x^{1/2}l^{1/2}$, $\alpha\geqslant 1/2$ *satisfies the condition* $\mathscr{T}_{ky}\leqslant 0$.

Proof. We will use the notations given in the proof of Lemma A.1. Then

$$f(x)=\mathscr{F}^0(x, 1)=x^\alpha, \qquad g(x)=\mathscr{F}^1(x, 1)=x^{1/2},$$

$$f_1(x)=f(x)/f'(x)-x=x(1/\alpha-1), \qquad g_1(x)=x(g^{-1})'\,(x)-g^{-1}(x)=x^2.$$

From (A.6)

$$\frac{x}{l}\left(\frac{1}{\alpha}-1\right)=\left(\frac{y}{1-l}\right)^2.$$

Also

$$x=k-(1-l)\,g^{-1}\left(\frac{y}{1-l}\right)=k-\frac{y^2}{1-l}.$$

Thus we get the following equation

$$\left(k-\frac{y^2}{1-l}\right)\left(\frac{1}{\alpha}-1\right)=\left(\frac{y}{1-l}\right)^2 l.$$

It is equivalent to the following quadratic equation

$$kl^2+l(-2k+y^2(1-\beta))-y^2+k=0, \tag{B.1}$$

where $\beta=\alpha/(1-\alpha)$. Solving (B.1) for l, we obtain

$$l=\frac{2k-y^2(1-\beta)-\sqrt{y^4(1-\beta)^2+4ky^2\beta}}{2k}. \tag{B.2}$$

It can be shown that $0 \leqslant l \leqslant 1$ if l is given by (B.2). According to (A.7)

$$\mathcal{T}_k = f'\left(\frac{x}{l}\right) = \alpha\left(\frac{x}{l}\right)^{\alpha-1} = \alpha\beta^{\alpha-1}\left(\frac{y}{1-l}\right)^{2\alpha-2}.$$

Now we can compute $\mathcal{T}_{ky}$.

$$\mathcal{T}_{ky} = \alpha(2\alpha-2)\,\beta^{\alpha-1}\left(\frac{y}{1-l}\right)^{2\alpha-3}\frac{1-l+yl_y}{(1-l)^2}.$$

We claim that $1-l+yl_y \geqslant 0$.

$$1-l+yl_y = \frac{y^2(1-\beta)+y\sqrt{y^2(1-\beta)^2+4k\beta}}{2k} - y\left(\frac{2y^2(1-\beta)^2+4k\beta}{2k\sqrt{y^2(1-\beta)^2+4k\beta}} + \frac{2y(1-\beta)}{2k}\right).$$

To prove $1-l+yl_y \geqslant 0$ we need to show that

$$y(1-\beta)+\sqrt{y^2(1-\beta)^2+4k\beta} - \frac{2y^2(1-\beta)^2+4k\beta}{\sqrt{y^2(1-\beta)^2+4k\beta}} - 2y(1-\beta) \geqslant 0,$$

or equivalently

$$\sqrt{y^2(1-\beta)^2+4k\beta} \geqslant \frac{2y^2(1-\beta)^2+4k\beta}{\sqrt{y^2(1-\beta)^2+4k\beta}} + y(1-\beta).$$

Having multiplied the last inequality by $\sqrt{y^2(1-\beta)^2+4k\beta}$ we obtain

$$y^2(1-\beta)^2 + y(1-\beta)\sqrt{y^2(1-\beta)^2+4k\beta} \leqslant 0$$

and $y^2(1-\beta)^2 \leqslant y^2(1-\beta)^2+4k\beta$ which is obviously true. ∎

LEMMA B.2. *The two-sector model specified by* $\mathcal{F}^0(x, l) = x^\alpha l^{1-\alpha}$, $\mathcal{F}^1(x, l) = x^{1/2} l^{1/2}$, $\alpha \geqslant 1/2$ *satisfies the condition* $(\mathcal{T}_k(k, y)\,k)_k > 0$.

Proof.

$$\mathcal{T}_k(k, y)\,k = \alpha\beta^{\alpha-1}\left(\frac{y}{1-l}\right)^{2\alpha-2} k,$$

so it suffices to prove

$$\left(\left(\frac{y}{1-l}\right)^{2\alpha-2} k\right)_k > 0$$

or

$$\left(\frac{y}{1-l}\right)^{2\alpha-2} + (2\alpha-2)\left(\frac{y}{1-l}\right)^{2\alpha-3} \frac{kyl_k}{(1-l)^2} > 0$$

which is equivalent to $1-l+(2\alpha-2)\,kl_k > 0$. Differentiating (A.2) with respect to k we then obtain

$$\frac{y^2(1-\beta)+y\sqrt{y^2(1-\beta)^2+4k\beta}}{2k} + (2\alpha-2)\,k\left(\frac{y^2(1-\beta)}{2k^2}+\frac{y\sqrt{y^2(1-\beta)^2+4k\beta}}{2k^2} - \frac{4y^2\beta}{4ky\sqrt{y^2(1-\beta)^2+4k\beta}}\right) > 0$$

which can be reduced to

$$(y(1-\beta)+\sqrt{y^2(1-\beta)^2+4k\beta})(2\alpha-1) - \frac{(2\alpha-2)\,2k\beta}{\sqrt{y^2(1-\beta)^2+4k\beta}} > 0.$$

Having simplified this inequality we obtain

$$y^2(1-\beta)^2\,(2\alpha-1)+(2\alpha-1)\,y(1-\beta)\sqrt{y^2(1-\beta)^2+4k\beta}+8k\beta\alpha > 0$$

which is equivalent to

$$(2\alpha-1)\,y^2(1-\beta)^2\,k\beta(8\alpha+4)+64k^2\beta^2\alpha^2 > 0$$

and the last inequality is true for any $k > 0$. ∎

LEMMA B.3. *If a two-sector model is defined by* $\mathcal{F}^0(x,l) = x^\alpha l^{1-\alpha}$, $\mathcal{F}^1(x,l) = x^{1/2} l^{1/2}$, $\alpha \leqslant 1/2$ *then*

(1) $\mathcal{T}_{ky} \geqslant 0$,

(2) $(\mathcal{T}_k k + \mathcal{T}_y y)_y \leqslant 0$.

Proof. (1) Follows from the proof of Lemma B.2.

(2) We will prove a slightly stronger inequality $\mathcal{T}_{ky}k+\mathcal{T}_y \leqslant 0$. We have

$$\mathcal{T}_k = \alpha\beta^{\alpha-1}\left(\frac{y}{1-l}\right)^{2\alpha-2}, \qquad \mathcal{T}_y = -\alpha\beta^{\alpha-1}\left(\frac{y}{1-l}\right)^{2\alpha-2} 2\left(\frac{y}{1-l}\right),$$

$$\mathcal{T}_{ky} = \alpha(2\alpha-2)\,\beta^{\alpha-1}\left(\frac{y}{1-l}\right)^{2\alpha-3}\frac{1-l+yl_y}{(1-l)^2}.$$

Then

$$\mathcal{T}_{ky}k+\mathcal{T}_y = \alpha\beta^{\alpha-1}\left(\frac{y}{1-l}\right)^{2\alpha-3}\frac{y^2}{(1-l)^2}\left((\alpha-1)\,k\,\frac{1-l+yl_y}{y^2}-1\right).$$

We need to show that

$$(\alpha-1)\,k\,\frac{1-l+yl_y}{y^2}-1 \leqslant 0.$$

There is a sequence of equalities

$$\begin{aligned}
&(\alpha-1)\,k\,\frac{1-l+yl_y}{y^2}-1\\
&\quad=\frac{\alpha-1}{2y}\left(y(1-\beta)+\sqrt{y^2(1-\beta)^2+4k\beta}-\frac{2y^2(1-\beta)^2+4k\beta}{\sqrt{y^2(1-\beta)^2+4k\beta}}-2y(1-\beta)\right)-1\\
&\quad=\frac{(1-\alpha)(1-\beta)}{2}-1+\frac{(1-\alpha)(1-\beta)^2}{2\sqrt{y^2(1-\beta)^2+4k\beta}}.
\end{aligned}$$

Therefore

$$(\alpha-1)\,k\,\frac{1-l+yl_y}{y^2}-1 \leqslant \frac{(1-\alpha)(1-\beta)}{2}-1+\frac{(1-\alpha)(1-\beta)}{2} \leqslant 0. \quad \blacksquare$$

LEMMA B.4. *If a two-sector model is defined by* $\mathcal{F}^0(x,l)=x^{\alpha}l^{1-\alpha}$, $\mathcal{F}^1(x,l)=x^{1/2}l^{1/2}$, *and* $1/4 \leqslant \alpha \leqslant 1/2$, *then* $(\mathcal{T}_k k)_k \geqslant 0$.

Proof. It follows from the proof of Lemma B.2 that we have to show that

$$y^2(1-\beta)^2\,(2\alpha-1)+(2\alpha-1)\,y(1-\beta)\,\sqrt{y^2(1-\beta)^2+4k\beta}+8k\beta\alpha \geqslant 0.$$

The function on the left-hand side of the inequality is decreasing in y. Therefore its minimum value is at $y=\sqrt{k}$. Then the following inequality must hold

$$(1-\beta)^2\,(2\alpha-1)+(2\alpha-1)(1-\beta)(1+\beta)+8\beta\alpha\geqslant 0,$$

or

$$2(1-\beta)(2\alpha-1)+8\beta\alpha\geqslant 0.$$

If we recall the definition of β then the last inequality can be reduced to $\alpha\geqslant 1/4$. ∎

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