

SUITP-98-11
SUNY-NTG-98-54
September, 1998

A Simple Derivation of the Hard Thermal Loop Effective Action

Per Elmfors¹ and T. H. Hansson¹

Institute of Theoretical Physics
University of Stockholm
Box 6730, S-113 85 Stockholm, Sweden
elmfors@physto.se, hansson@physto.se

Ismail Zahed²

Department of Physics and Astronomy
SUNY at Stony Brook
Stony Brook, New York, 11794, USA
zahed@nuclear.physics.sunysb.edu

ABSTRACT

We use the background field method along with a special gauge condition, to derive the hard thermal loop effective action in a simple manner. The new point in the paper is to relate the effective action explicitly to the S -matrix from the onset.

¹Supported by the Swedish Natural Science Research Council.

²Supported by the US Department of Energy under Grant No. DE-FG02-88ER40388.

1 INTRODUCTION

It is by now well established that the resummation of the so called hard thermal loops (HTLs) is a necessary part of any perturbative scheme to finite temperature QCD. Since the original work by Pisarski[1], Braaten and Pisarski[2] and Frenkel and Taylor[3], there have been several papers deriving the HTL amplitudes using various techniques, including Chern-Simons eikonal[4], color transport theory[5], and Wong equations[6]. It has also been shown that all the HTL amplitudes can be derived from a simple gauge-invariant effective action that incorporates the Ward-identities originally derived by Braaten and Pisarski[2] and Frenkel and Taylor[3]. This effective action has been given in several forms first by Taylor and Wong[7] who gave an expression involving string operators, and by Braaten and Pisarski, who found the following particularly elegant expression ($SU(N)$ Yang-Mills theory)

$$\Gamma_{\text{HTL}}[A_\mu] = \frac{g^2 NT^2}{6} \int d^4x \int \frac{d^2\hat{q}}{4\pi} \text{tr} F^{\mu\sigma} \frac{\hat{Q}_\sigma \hat{Q}_\lambda}{[\hat{Q} \cdot D(A)]^2} F^\lambda{}_\mu \quad , \quad (1)$$

where $D(A)_\mu = \partial_\mu + gA_\mu$ with A in the adjoint representation, and \hat{Q} a 4-vector of the form $\hat{Q} = (1, \hat{q})$, so $\hat{Q}^2 = 0$.³ This form was guessed by Braaten and Pisarski based on general properties of perturbation theory, and indeed was shown to reproduce the HTL Ward identities in the latter. At the same time Frenkel and Taylor derived essentially the same result by proving that the action satisfies certain conditions that are restrictive enough to have a unique solution[8]. Again their analysis was based on an analysis of the explicit (one-loop) n -gluon amplitudes in the HTL approximation.

The purpose of this paper is to give a simple derivation of Eq. (1), using the background field method and the specific gauge choice

$$\hat{Q}_\mu A^\mu = 0 \quad . \quad (2)$$

It is crucial in the derivation to show that it is possible to use a \hat{Q} dependent gauge choice. Using this gauge we can solve for the gauge potential,

$$A^\mu(\hat{Q}) = \frac{1}{\hat{Q} \cdot D(A)} \hat{Q}_\nu F^{\nu\mu} \quad , \quad (3)$$

³We shall always use Minkowski metric, so an Euclidean 4-vector is of the form (iq_0, \vec{q}) .

where the parametric dependence on the gauge condition is shown explicitly. Substituting in Eq. (1), we obtain

$$\Gamma_{\text{HTL}}[A_\mu] = \frac{g^2 N T^2}{6} \int d^4x \int \frac{d^2\hat{q}}{4\pi} \text{tr} A(\hat{Q}; x)^2 \quad , \quad (4)$$

as was first shown by Frenkel and Taylor[8], who also stressed that nothing is gained in simplicity by rewriting Γ_{HTL} in this gauge, since the full non-local and non-linear structure is hidden in the complicated parametric \hat{Q} -dependence of $A^\mu(\hat{Q})$.

It is also not clear that the gauge in Eq. (2) would be useful in trying to derive the effective action, in spite of the simple form of Eq. (4), since it involves an integral over the parameter \hat{Q} that enters in the gauge condition. Although, using the background field method, the effective action by construction is invariant under background gauge transformations, it is by no means obvious that it can be expressed as an integral over \hat{Q} of a gauge invariant object, as in Eq. (1). The key ingredient in the derivation to follow in this paper is to find such a formulation by relating the effective action to a gauge invariant quantity, namely the S -matrix. The \hat{q} integral is naturally interpreted as an integral over the (on-shell) momenta Q of the particles in the heat bath.

Many of the results used and/or derived in this paper have already appeared in the literature. For example, the connection between Γ_{HTL} and forward scattering amplitudes was already stressed by Frenkel and Taylor, and the gauge condition Eq. (2) was discussed in [8] and used to simplify the derivation of the effective action in [9]. We believe, however, that our approach, where the starting point is a direct relation between the effective action and the S -matrix, is novel and provides a very simple and physical way to derive Eq. (1).

When facing the problem of how to express the HTL effective action in terms of a gauge invariant object, it is important to understand its physical significance beyond the formal definition of being the generating functional of the $O(T^2)$ parts of the proper n -point functions. For static fields, Γ_{HTL} is nothing but the $O(T^2)$ contribution to the (negative) pressure of thermal particles interacting with the field. For time-dependent configurations the current is the natural object to relate to the S -matrix, as was stressed in this context by Jackiw and Nair [10]. The action Γ_{HTL} is then obtained by integration with respect to the gauge field. In the next section we shall first consider the static case and then the time-dependent one. It is worth remembering that although the final formulae derived in these cases are the same, the physical interpretations are rather different.

The detailed derivations in the next section will be given for adjoint scalars rather than gauge particles. From the presentation it will be quite obvious that the only thing

that matters for the HTL effective action is the number of physical degrees of freedom, and their charge and statistics. For those readers who are convinced that this is the case, the result for a complex adjoint scalar can immediately be taken over to the case of YM theory since the gluon also has two physical degrees of freedom. For those who want a formal proof, this is provided in Appendix A using HTL power counting arguments. There, we also prove that the HTL effective action, as calculated using background field technique, is independent of the quantum gauge fixing parameter. The extension of our method to include fermions is straightforward and can be found in Appendix B.

2 THE FREE ENERGY AND THE EFFECTIVE ACTION

As mentioned above, we shall consider a heat bath of charged scalars in the background of a non-abelian field A_μ . By construction, the one-loop background field effective action for static field configurations is nothing but the free energy of a gas of scalar particles interacting with the background. The free energy may in turn be directly related to the S -matrix [11]. Hence,

$$\Gamma_{\text{HTL}}^{\text{stat}} = \frac{1}{\beta} \text{Tr}_\beta \ln[-\square] = F = F_0 - \frac{1}{\beta} \int_0^\infty dE e^{-\beta E} \frac{1}{4\pi i} \text{tr} \left(S^\dagger \frac{\partial S}{\partial E} - S \frac{\partial S^\dagger}{\partial E} \right)_C, \quad (5)$$

where the trace is over all connected diagrams in the notation of [11]. For particles that do not interact mutually, but only with the external field, the sum over multiparticle states can be performed and the free energy can be related to the one-particle density of states

$$F = \frac{1}{\beta} \int_0^\infty dE \ln(1 - e^{-\beta E}) (\rho_0 + \Delta\rho(E)), \quad (6)$$

where the shift of the density of states is related to the one-particle S -matrix by

$$\Delta\rho(E) = \frac{1}{4\pi i} \text{tr} \left(S^\dagger \frac{\partial S}{\partial E} - S \frac{\partial S^\dagger}{\partial E} \right) \Big|_{1\text{-part}}. \quad (7)$$

Since the S -matrix is gauge invariant for each physical momentum state we can use different gauge choices for different momenta and thus the choice in Eq. (2) is allowed. Then, only the first two terms in

$$S = \text{---} + \text{---} \begin{array}{c} \text{z} \\ \text{z} \\ \text{z} \end{array} + \text{---} \begin{array}{c} \text{z} \\ \text{z} \end{array} + \text{---} \begin{array}{c} \text{z} \\ \text{z} \end{array} \begin{array}{c} \text{z} \\ \text{z} \end{array} + \dots \quad (8)$$

contribute. All other diagrams are either zero because of the gauge choice or suppressed at high temperature. The counting here is very much the same as in Appendix A. A direct expansion of $S = \mathbb{1} + iT$ in a static background gives

$$\langle q|S(E)|q\rangle = 1 + \frac{ig^2N}{E_qV} 2\pi\delta(E_q - E) \int d^3x (A_\mu^a(\hat{Q}; x))^2 \quad , \quad (9)$$

using the normalization $\langle q|q'\rangle = \frac{(2\pi)^3}{V}\delta^{(3)}(q - q')$ and $\text{tr} = \frac{V}{(2\pi)^3} \int d^3q$. From Eq. (7) we obtain

$$\Delta\rho(E) = \frac{g^2N}{2\pi^2} \int d^3x \int \frac{d^2\hat{q}}{4\pi} (A_\mu^a(\hat{Q}; x))^2 \quad . \quad (10)$$

The energy integral in Eq. (6) is then trivial and we arrive at

$$F = F_0 - \frac{g^2NT^2}{12} \int d^3x \int \frac{d^2\hat{q}}{4\pi} (A_\mu^a(\hat{Q}; x))^2 \quad , \quad (11)$$

which agrees with Eq. (4) when we use the relation $\Gamma = -\int_0^{-i\beta} dx_0 F$.

The calculation above can be generalized to time dependent background fields but, as already mentioned, the physical interpretation is different. The free energy is an equilibrium concept and we shall instead start from the expectation value of the current in a background field to derive an effective action. The current is given by [10, 12]

$$j(t, x) = \frac{1}{Z} \text{tr} [e^{-\beta H} U^\dagger(t, -\infty) \hat{j}(-\infty, x) U(t, -\infty)] = \frac{1}{Z} \text{tr} [e^{-\beta H} S^\dagger \frac{-i\delta}{\delta A(t, x)} S] \quad , \quad (12)$$

with the S -matrix in the interaction picture. There are two pieces in the current when written in terms of the T -matrix $S = \mathbb{1} + iT$

$$j(t, x) = \frac{1}{Z} \text{tr} e^{-\beta H} \left[\frac{\delta T}{\delta A(t, x)} - iT^\dagger \frac{\delta T}{\delta A(t, x)} \right] \quad , \quad (13)$$

The first term is a total derivative of an action (which we call the effective action for time-ordered n -point functions). To one-loop order the second term is imaginary and has support only when the external field is on the light-cone. This is the term that makes up for the difference between time-ordered and retarded n -point functions.

Looking only for the real part we can integrate the first piece in Eq. (13) with respect to A . We obtain

$$e^{i\Gamma[A]} = \frac{1}{Z} \text{tr} [e^{-\beta H} S] = \frac{1}{Z} \exp \left[V \int \frac{d^3q}{(2\pi)^3} \ln \left(\sum_{n_q} e^{-\beta E_q n_q} \langle n_q | S | n_q \rangle \right) \right] \quad . \quad (14)$$

where $|n_q\rangle$ is a state with n_q particles of momentum q . Since the particles do not interact with each other the expectation value of the S -matrix factorizes $\langle n_q|S|n_q\rangle = (\langle q|S|q\rangle)^{n_q} = (1 + i\langle q|T|q\rangle)^{n_q}$. The sum over n_q and the internal group indices can be performed easily and we find

$$i\Gamma[A] = -\ln Z + VN \int \frac{d^3q}{(2\pi)^3} \ln\left(\frac{1}{1 - e^{-\beta E_q}(1 + i\langle q|T|q\rangle)}\right) . \quad (15)$$

Since the expectation value of the T -matrix is gauge invariant we can again choose the gauge $Q_\mu A^\mu = 0$. Only the $g^2 A^2$ piece in the interaction enters and in fact only a single such insertion since multiple insertions are again suppressed by powers of $1/T$. An expansion in $\langle q|T|q\rangle$ gives

$$i\Gamma[A] = iVN \int \frac{d^3q}{(2\pi)^3} \frac{1}{e^{\beta E_q} - 1} \langle q|T|q\rangle . \quad (16)$$

Using Eq. (9) for non-static background fields one finds

$$\langle q|T|q\rangle = \frac{1}{E_q V} \int d^4x A^2(\hat{Q}; x) , \quad (17)$$

After substituting into Eq. (16) and performing the q -integration, we have

$$\Gamma_{\text{HTL}}[A] = \frac{g^2 NT^2}{12} \int d^4x (A_\mu^a)^2(\hat{Q}; x) , \quad (18)$$

which again agrees with Eq. (4). This completes our derivation of the HTL effective action for general field configurations.⁴ Finally, we want to stress the simplicity of the arguments leading from Eq. (5) to Eq. (11) and from Eq. (12) to Eq. (18). Basically all steps are written out, and the only thing that requires some care is to get the various normalizations of the S -matrix elements right. This should be contrasted with the rather involved chain of arguments that have appeared in previous derivations of the HTL effective action. This paper was concerned with a new derivation of known results, but one could also try to use our methods to calculate subleading terms by including interactions between the fast thermal particles. This would amount to include contributions from the 2-body, 3-body etc S -matrix. Whether or not this would lead to physically interesting approximations remains to be seen.

Acknowledgement: We thank Rob Pisarski for commenting on an early version of this paper. I.Z. also thanks Rob Pisarski and Hidenaga Yamagishi for discussions.

⁴As usual, the form Eq. (1) is appropriate only off the light cone.

A APPENDIX

We shall now formally show that the leading high T contribution to the one-loop effective action for gluons is the same as for a single complex scalar field in the adjoint representation. Again we stress that this is rather straightforward from the calculation in Section 2. We start from the Euclidean formulation of finite temperature field theory, and, in the notation of [13], write the one-loop finite T , gauge invariant background field effective action as

$$i\Gamma[A, \alpha] = -\text{Tr}_\beta \ln [-\square] + \frac{1}{2} \text{Tr}_\beta \ln \left[-\square g_{\mu\nu} - 2gF_{\mu\nu} - \left(\frac{1}{\alpha} - 1 \right) D_\mu D_\nu \right] \quad , \quad (19)$$

where $\square = D_\mu D^\mu$, $gF_{\mu\nu} = [D_\mu, D_\nu]$, α is the (covariant background field) gauge-fixing parameter, and the trace is over color and Lorentz indices as well as over spacetime. The background field effective action is by construction gauge invariant, with respect to the background field A^μ , but to prove physical gauge invariance one must also establish α -independence. We first consider the background field Feynman gauge, $\alpha = 1$, expand the gluon trace in powers of $F^{\mu\nu}/\square$, and combine the leading term with that from the ghosts to get,

$$i\Gamma[A, 1] = \text{Tr}_\beta \ln [-\square] + \text{Tr}_\beta \left[\frac{1}{\square} 2F^{\mu\nu} \right] - \frac{1}{2} \text{Tr}_\beta \left[\frac{1}{\square} 2F^{\mu\rho} \frac{1}{\square} 2F^{\rho\nu} \right] + \dots \quad , \quad (20)$$

where $\square = \partial^2 + g\{\partial_\mu, A_\mu\} + g^2 A^2$. Now recall the rules for power counting in HTL. Each gluon propagator contributes a term $\sim 1/q^2$, so naively it would be expected to give a suppression $\sim T^{-2}$ to the amplitude, but as stressed by Braaten and Pisarski, this is not correct. The leading contribution arises when the loop momentum is large but the propagators also almost on shell. Performing the q_0 integration by closing the contour puts one propagator on shell, i.e. $q_\mu = q\hat{Q}_\mu$, contributing a factor $1/2q$ to the dq integration, while all the other propagators contribute with denominators $(\hat{Q} + p_i)^2 = 2q\hat{Q} \cdot p_i + p_i^2 \approx 2q\hat{Q} \cdot p_i$. In this approximation the dq integration factorizes and immediately gives the T behavior of the graph by power counting. The leading contribution to a diagram with m 3-gluon vertices, n 4-gluon vertices, and l insertions of $F^{\mu\nu}$ is $\sim T^4 T^m T^{-2-(m+n+l-1)} = T^{3-n-l}$ where the contributions are from the integration measure, the momentum dependence of the 3-gluon vertices and the propagators respectively. Note that the terms corresponding to $l = 1$ is $\sim F^{\mu\rho}$ and thus zero because of the Lorentz trace, and since the $l = 2$ term is already at most $\sim T$, the HTL action comes entirely from the graphs with no $F^{\mu\nu}$

insertions. This conclusion is independent of the gauge choice for the background field. Thus, the YM effective action $\text{Tr}_\beta \ln[-\square]$ can be calculated using charged scalar inside the loop and multiplying with the appropriate group factors.

Finally we show that the α -dependence in Eq. (19) is suppressed by powers of $1/T$. Following [14, 13] we write

$$\frac{\partial \Gamma}{\partial \alpha} = \frac{1}{2} \text{Tr}_\beta \left(\frac{1}{\square} D_\mu \mathcal{E}_\mu \frac{1}{\square} \right) - \frac{1}{2} \text{Tr}_\beta \left(\frac{1}{\square} \mathcal{E}_\mu G_{\mu\nu} \mathcal{E}_\nu \frac{1}{\square} \right) , \quad (21)$$

where $\mathcal{E}_\mu = [D_\nu, F_{\mu\nu}]$ and G the full covariant gluon propagator satisfying

$$\left(\square g_{\mu\lambda} + 2F_{\mu\lambda} + \frac{1-\alpha}{\alpha} D_\mu D_\lambda \right) G_{\lambda\nu}(x, y) = g_{\mu\nu} \delta^4(x - y) . \quad (22)$$

It is easy to see that, since \mathcal{E}_μ is independent of the loop momentum and the presence of a double pole in the first term in Eq. (21) gives an extra power of T compared to diagrams with only single poles, there is no T^2 contribution. In the second term we can expand G like

$$G = \sum_n \left(\frac{1}{\square} (2F_{\mu\nu} + \frac{1-\alpha}{\alpha} D_\mu D_\nu) \right)^n \frac{1}{\square} . \quad (23)$$

Now we have potentially dangerous terms with powers of $\frac{1}{\square} D_\mu D_\nu$ which naively go like T^n . However, the D factors can always be commuted around so that a contraction is possible. For example

$$\frac{1}{\square} D_\alpha D_\beta \frac{1}{\square} D_\beta D_\gamma = \frac{1}{\square} D_\alpha D_\gamma + \frac{1}{\square} D_\alpha D_\beta \frac{1}{\square} [D_\beta, \square] \frac{1}{\square} D_\gamma , \quad (24)$$

where the second term goes like T since $[D_\beta, \square] = g(D_\mu F_{\beta\mu} + F_{\beta\mu} D_\mu) \sim T$. Each contraction lowers the naive power by one factor of T and therefore the dangerous powers can be eliminated. Terms with factors of F are of course subleading according to the same power counting.

B APPENDIX

It is not hard to generalize the above argument to include fermions by adding a background of fermionic sources. In the Feynman gauge the total effective action is

$$i\Gamma[A, \bar{\Psi}, \Psi] = -\text{Tr}_\beta \ln[-\square] + \frac{1}{2} \text{STr}_\beta \ln \begin{bmatrix} -\square g_{\mu\nu} - 2gF_{\mu\nu} & ig\bar{\Psi}\gamma_\nu \\ ig\gamma_\mu\Psi & i\mathcal{D} \end{bmatrix} , \quad (25)$$

where the second trace now is a supertrace over both gauge bosons and fermions. This is related to the S -matrix in the same way as Eq. (5) and therefore we can use the same Q -dependent gauge as in Eq. (2). It can be rewritten in terms of an ordinary trace as [15]

$$i\Gamma[A, \bar{\Psi}, \Psi] = \frac{1}{2}\text{Tr}_\beta \ln [-\square g_{\mu\nu} - 2gF_{\mu\nu}] - \frac{1}{2}\text{Tr}_\beta \ln [i \not{D}] - \frac{1}{2}\text{Tr}_\beta \ln \left[1 - \frac{g^2}{i \not{D}} \gamma_\mu \Psi \left(\frac{1}{\square + 2gF} \right)_{\mu\nu} \bar{\Psi} \gamma_\nu \right]. \quad (26)$$

The second term is the contribution to the gauge boson effective action from dynamical fermions which is simply equal to $-\frac{1}{4}\text{Tr}_\beta \ln[-\square]$. The only difference with the term evaluated earlier is the statistics of the hard particles and that the group trace is in the fundamental representation and gives a factor $N_f/2$. The last term in Eq. (26) gives the effective action for the fermionic background fields. It can also be analysed with the methods described above. After expanding in powers of $\bar{\Psi}\Psi$ and in powers of F it is only the zeroth order term

$$N \text{tr}_\beta \left[\frac{g^2}{\square} \bar{\Psi} i \not{D} \Psi \right] \quad (27)$$

that can go like T^2 . The remaining factors of $1/\square$ are expanded in powers of g and with the gauge in Eq. (2) only the leading $1/\partial^2$ remain in the high T limit. The two poles correspond to forward scattering of gauge bosons and fermions, respectively. After performing the thermal trace over Lorentz indices we obtain

$$i\Gamma[\bar{\Psi}, \Psi] = \frac{g^2 C_f T^2}{8} \int \frac{d^2 \hat{q}}{4\pi} \bar{\Psi} \frac{\gamma_\mu \hat{Q}^\mu}{\hat{Q} \cdot \partial} \Psi. \quad (28)$$

Equation (28), just like Eq. (4), is gauge dependent and valid only in the gauge Eq. (2). It is, however, straightforward to write them in a explicitly gauge invariant by expressing A and $\hat{Q} \cdot \partial$ in terms of F and $\hat{Q} \cdot D$ and thereby recovering the standard HTL effective action.

References

- [1] R. D. Pisarski, *Scattering amplitudes in hot gauge theories*, *Phys. Rev. Lett.* **63** (1989) 1129.

- [2] E. Braaten and R. D. Pisarski, *Resummation and gauge invariance of the gluon damping rate in hot QCD*, *Phys. Rev. Lett.* **64** (1990) 1338; *Soft amplitudes in hot gauge theories: A general analysis*, *Nucl. Phys.* **B337** (1990) 569.
- [3] J. Frenkel and J. C. Taylor, *High temperature limit of thermal QCD*, *Nucl. Phys.* **B334** (1990) 199.
- [4] R. Efraty and V. P. Nair, *The secret Chern-Simons action for the hot gluon plasma*, *Phys. Rev. Lett.* **68** (1992) 2891; *Chern-Simons theory and the quark - gluon plasma*, *Phys. Rev.* **D47** (1993) 5601.
- [5] J. P. Blaizot and E. Iancu, *Kinetic equations for long wavelength excitations of the quark - gluon plasma*, *Phys. Rev. Lett.* **70** (1993) 3376; *Soft collective excitations in hot gauge theories*, *Nucl. Phys.* **B417** (1994) 608.
- [6] P. F. Kelly, Q. Liu, C. Lucchesi and C. Manuel, *Deriving the hard thermal loops of QCD from classical transport theory*, *Phys. Rev. Lett.* **72** (1994) 3461; *Classical transport theory and hard thermal loops in the quark - gluon plasma*, *Phys. Rev.* **D50** (1994) 4209.
- [7] J. C. Taylor and S. M. H. Wong, *The effective action of Hard Thermal Loops in QCD*, *Nucl. Phys.* **B346** (1990) 115.
- [8] J. Frenkel and J. C. Taylor, *Hard thermal QCD, forward scattering and effective actions*, *Nucl. Phys.* **B374** (1992) 156.
- [9] J. Frenkel, E. A. Gaffney and J. C. Taylor, *High temperature field theory in curved space-time*, *Nucl. Phys.* **B439** (1995) 131.
- [10] R. Jackiw and V. P. Nair, *High temperature response functions and the nonabelian Kubo formula*, *Phys. Rev.* **D48** (1993) 4991, hep-ph/9305241.
- [11] R. Dashen, S-k. Ma and H. J. Bernstein, *S-matrix formulation of statistical mechanics*, *Phys. Rev.* **187** (1969) 345.
- [12] H. Yamagishi and I. Zahed, *Chiral Perturbation Theory and Effective Lagrangians*, SUNY-Report 1988, unpublished; *A Master Formula for Chiral Symmetry Breaking*, *Ann. Phys.* **247** (1996) 292.

- [13] T. H. Hansson and I. Zahed, *Plasmons from a gauge invariant effective action for hot gluons*, *Nucl. Phys.* **B292** (1987) 715.
- [14] A. O. Barvinsky and G. A. Vilkovisky, *Beyond the Schwinger-DeWitt technique: Converting loops into trees and in-in currents*, *Nucl. Phys.* **B282** (1987) 163.
- [15] B. DeWitt, *Supermanifolds*, Cambridge University Press (1987).