Modified Boltzmann Transport Equation and Freeze Out

L.P. Csernai[1,2], V.K. Magas[3], E. Molnár[1], A. Nyiri[1] and K. Tamosiunas[1]
[1] *Section for Theoretical and Computational Physics, and Bergen Computational Physics Laboratory, BCCS-Unifob, University of Bergen, Allegaten 55, 5007 Bergen, Norway*
[2] *MTA-KFKI, Research Inst of Particle and Nuclear Physics, H-1525 Budapest 114, P.O.Box 49, Hungary*
[3] *Departamento de Física Teórica and IFIC Centro Mixto Universidad de Valencia-CSIC, Institutos de Investigación de Paterna Apdo. correos 22085, 46071, Valencia, Spain*

Abstract: We study Freeze Out process in high energy heavy ion reaction. The description of the process is based on the Boltzmann Transport Equation (BTE). We point out the basic limitations of the BTE approach and introduce Modified BTE. The Freeze Out dynamics is presented in the 4-dimensional space-time in a layer of finite thickness, and we employ Modified BTE for the realistic Freeze Out description.

I. INTRODUCTION

The Freeze Out (FO) is an important phase of dynamical reactions. It is of primary importance in case of rapid, dynamical processes where the originally strongly-interacting and locally equilibrated matter undergoes a rapid explosive process, in which matter properties change considerably, the interaction vanishes in a relatively small space-time layer, and local equilibrium disappears. The connection of the kinetic description of this process and the Boltzmann Transport Equation (BTE) raised considerable attention recently [1, 2].

The problem is to calculate the phase-space (PS) distribution of the post FO particles. Earlier such kinetic FO calculations were performed in one-dimensional models [3, 4, 5, 6], where the dynamics was governed by two constants: a re-thermalization parameter and a FO parameter. This latter one is governed by the phase-space FO probability, which was constructed recently in a fully covariant form [7].

The FO is a kinetic process and one would think it can be handled perfectly by using the Boltzmann Transport Equation, which may describe equilibrium and non-equilibrium processes equally well in a 4-dimensional space-time volume element, which is usually a FO layer. This work and ref. [2] follows this approach. This finite layer is frequently idealized as a 3-dimensional FO hypersurface. In ref. [1, 12] author analyzes the features of this idealized discontinuity [27].

The FO can also be simultaneous with a phase transition, especially when the phase transition reduces the number of degrees of freedom and contributes to the FO process this way. As an example let us describe a gradual hadronization and FO of the Quark-Gluon Plasma in a layer, where quasi-hadrons or hadrons are formed, the new particles gradually cease to interact, their PS distribution changes and the matter gradually freezes out.

Free hadrons, which are formed, do not interact with anything and propagate directly to the detector. Although, the formation of these fragments can be most suitably described in a coalescence or recombination model, most finally observed baryon abundances follow the statistical model predictions. The reason is simple: the formation cross sections are governed by the same statistical factors as the thermal equilibrium, because the radial part of the formation probability for s-wave hadrons is about the same. Exceptions are the excited states, e.g. the p-wave hadrons like $\Lambda(1520)$, which have a smaller radial form factor and consequently they are suppressed in comparison to the statistical equilibrium abundance, which is sensitive to their weight only.

In this work we do not discuss issues related to simultaneous hadronization and FO, for simplicity we consider one type of particles only and study their kinetic evolution. If hadronization happens simultaneously with kinetic FO the kinetic description presented here can and should be extended. For example, some features of the fast hadronization and FO of supercooled Quark-Gluon Plasma, which might be created in ultra-relativistic heavy ion collisions (for the first time such a scenario was proposed in refs. [8, 9]) are discussed in refs. [10, 11]. The simultaneous FO and hadronization can be described in an idealized way by using the 3-dimensional FO hypersurface approach as suggested in ref. [1]. Then this simplified approach makes it possible to solve some of the basic problems. The simultaneous hadronization and FO can also be handled by assuming idealized hadronization at the hypersurface, which is the inside boundary of the FO layer of finite thickness, L. In this way the present work is also relevant for the simultaneous hadronization and FO problem.

In the present work we analyze the situation, discuss the applicability of BTE, and point out the physical causes which, limit the applicability of the BTE for describing FO. And the aim is to show how can we overcome this obstacle. For this purpose we will modify the BTE and then will show how one can derive out of it a simple one dimensional kinetic model, similar to the one used

by some of the authors in earlier calculations.

II. PARTICLES EMERGING FROM FREEZE OUT HYPERSURFACE

Not only in heavy ion reactions, but in many dynamical processes particle creation (or condensation) happens mostly in a directed way: the phenomenon propagates into some direction, i.e. it happens in some layer or front (like detonations, deflagrations, shocks, condensation waves or FO across a layer with space-like normal). The reason is that neighboring regions in the front may interact to minimize the energy of the front by evening it out, providing energy to neighboring regions to exceed the threshold conditions. Even in those relativistic processes that are time-like (have time-like normal), and so the neighboring points of a front cannot be in causal connection, the dynamical processes may and frequently have a direction. See the example in ref. [13]. This can be a simple consequence of the initial and boundary conditions.

These fronts have a characteristic direction (or normal, $d\sigma^\mu$). Let us look at an example when particles in a domain of the space-time (ST) are characterized by a phase-space distribution, $f(x,p)$. Then the space-time current density of these particles, $N^\mu(x)$ can be described as

$$N^\mu = \int \frac{d^3p}{p^0}\, p^\mu\, f(x,p)\ . \tag{1}$$

The net number of particles crossing an arbitrary hypersurface element $d\sigma_\mu$ is

$$dS = N^\mu d\sigma_\mu = \int \frac{d^3p}{p^0}\, p^\mu\, d\sigma_\mu\, f(x,p)\ . \tag{2}$$

If we want to describe the FO, particles are allowed to cross the FO hypersurface "outwards" only, i.e., only in the direction of $d\sigma^\mu$. Thus,

$$\begin{aligned} S_{FO} &= \int N^\mu_{FO} d\sigma_\mu \qquad (3)\\ &= \int\int \frac{d^3p}{p^0}\, p^\mu\, d\sigma_\mu\, f_{FO}(x,p)\Theta(p^\mu\, d\sigma_\mu)\ , \end{aligned}$$

where either the phase space distribution, $f_{FO}(x,p)$, should have only particles with momenta pointing outwards (post FO distribution), and/or this is secured by the step function $\Theta(p^\mu\ d\sigma_\mu)$. Eq. (4) yields the modified Cooper-Frye FO formula, where $f_{FO}(x,p)$ should be determined in such a way that all conservation laws across the FO hypersurface are satisfied and overall entropy does not decrease! [3, 14, 15]

III. NON-ISOTROPIC PARTICLE SOURCES

The FO-fronts or FO-layers are not necessarily narrow, but they have a characteristic direction (or normal, $d\sigma^\mu$), and it is more realistic to assume a continuous, 4-volume FO in a layer (or domain) of the space-time. At the inside boundary of this layer no particles are frozen out yet, while at the outside boundary hypersurface all particles are frozen out and no interacting particles remain (see Figure 1). For the sake of simplicity let us also assume that the total particle number is conserved, even if simultaneous freeze out, hadronization and particle formation are frequently discussed.

Thus, while the total number of particles remains constant, in this domain, the number of interacting particles decreases and the number of frozen out or free particles increases:

$$\begin{aligned} N^\mu(x) &= N^\mu_i(x) + N^\mu_f(x)\ , &(4)\\ \partial_\mu N^\mu(x) &= 0 &(5)\\ \partial_\mu N^\mu_i(x) &= -\,\partial_\mu N^\mu_f(x)\ . &(6) \end{aligned}$$

Then the space-time (ST) volume element, d^4x, in the layer of interest can be converted into $d^4x \longrightarrow ds^\mu\ d\sigma_\mu$, where ds^μ is the length element in the direction of the 4-vector $d\sigma^\mu$, which can be space-like or time-like, i.e.: time-like, $d\sigma^\mu\ d\sigma_\mu = +1$, or space-like, $d\sigma^\mu\ d\sigma_\mu = -1$.

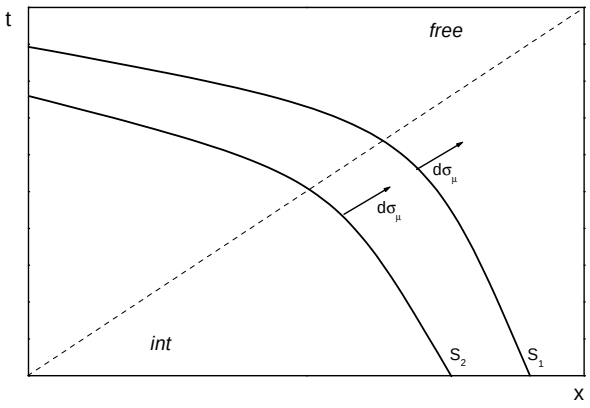

FIG. 1: Space-time picture of the FO process. At early times, centrally in the collision region we have intensively interacting matter, which is equilibrated and thermalized, this is the fluid-dynamical domain bordered by the S_2 hypersurface, which has a normal 4-vector $d\sigma_\mu$. The location of this surface is given by the fact that the interacting fluid is cooling and expanding, and reaches a point when interactions are not frequent enough to maintain full thermal and hydrodynamical equilibrium locally. Some particles will not interact any more beyond this hypersurface. Later on in the expansion and cooling we reach an other hypersurface, S_1. By reaching this surface on their way all particles become non-interacting, or free. Thus, when reaching this surface the FO process is completed. The momentum distribution of particles does not change any more. This is the (post) FO distribution

Let us also assume that the ST domain, where free particle formation happens, is a layer, which is relatively

narrow compared to the bulk of matter (see Figure 1). Assume also that the boundaries of this layer are parallel or approximately parallel, and the thickness of the layer does not vary much. Under these conditions one can describe the change of free particle number in the layer via the divergence of the 4-current of the particles by the expression:

$$\begin{aligned}\Delta N_i &= \int d^4x \;\; \partial_\mu N_i^\mu(x) \qquad (7)\\ &= \int ds^\mu \; d\sigma_\mu \int d^3p \; \frac{p^\mu}{p^0} \; \partial_\mu f_i(x,p)\,.\end{aligned}$$

According to the physical assumptions discussed above, the 4-divergence is maximal in the direction of $d\sigma_\nu$, and negligible in the other 3 orthogonal directions.

The emission or freeze out probability may depend on physical processes, cross sections, transition rates, and the actual PS distributions. Furthermore, f_i can be space-time dependent, and must be determined *self-consistently* during the detonation, deflagration or FO process [3, 4, 5, 6, 16].

We will return to realistic FO probabilities later in section VII.

IV. FREEZE OUT AND THE BOLTZMANN TRANSPORT EQUATION

One can derive the Boltzmann Transport Equation from the conservation of charges in a ST domain [17], Δ^4x, assuming the standard conditions: (i) only binary collisions are considered, (ii) we assume "molecular chaos", i.e. that the number of binary collisions at position x is proportional to $f(x,p_1)\times f(x,p_2)$, and that (iii) $f(x,p)$ is a smoothly varying function compared to the mean free path (m.f.p.). The conservation laws lead then to the requirement that the integral of the 4-divergence of conserved charges should vanish

$$\int_{\Delta^4x}\int_{\Delta^3p} d^4x\frac{d^3p}{p^0}p^\mu\partial_\mu f(x,p) = 0\;. \qquad (8)$$

As the choice of the ST 4-volume element is arbitrary we obtain the differential form of the conservation law, which describes the evolution of the PS distribution, $f(x,p)$, of a particle with momentum p. However, if we take into account that particles can scatter into this PS volume element around p, or can scatter out from this volume element, we have to add Gain- and Loss- collision terms to the conservation equation (see e.g. sect 3.2 of ref. [17]):

$$\begin{aligned}p^\mu\partial_\mu f(p) &= \frac{1}{2}\int {}_{12}\mathcal{D}_4 f(p_1)f(p_2)W^{pp_4}_{p_1p_2}\\ &- \frac{1}{2}\int {}_{2}\mathcal{D}_{34} f(p)f(p_2)W^{p_3p_4}_{pp_2}\;. \qquad (9)\end{aligned}$$

Here we assume elementary collisions where in the initial state two particles collide with momenta p_1 and p_2 into a final state of two particles with momenta p_3 and p_4. In case of the Gain term the particle described by the BTE, with momentum p (without an index), is one from the two final state particles, while in case of the Loss term this particle is one of the initial state particles. This is indicated by the indexes of the invariant transition rate [17]. We integrate over the momenta of the other three particles participating in this binary collision. We use the notation

$${}_{12}\mathcal{D}_3 \equiv \frac{d^3p_1}{p_1^0}\frac{d^3p_2}{p_2^0}\frac{d^3p_3}{p_3^0}\;.$$

We can shorten the notation further by suppressing the arguments of the PS distribution functions, and the indexes of the momenta in the argument will be carried by the distribution function f and the collision term $W^{pp_4}_{p_1p_2}\equiv W^{p4}_{12}$:

$$p^\mu\partial_\mu f = \frac{1}{2}\int {}_{12}\mathcal{D}_4 f_1^i f_2^i W^{p4}_{12} - \frac{1}{2}\int {}_{2}\mathcal{D}_{34} f^i f_2^i W^{34}_{p2} \qquad (10)$$

Now, aiming for the description of FO process let us split up the distribution function, f, into $f = f^i + f^f$, where f^f is the phase-space distribution function of the "free" or frozen out particles, which do not collide any more, while f^i is the interacting component [2, 3]. Thus, the FO process is represented here by gradually populating and building up the "free" component, while draining particles from the interacting component. As the particles belonging to the free component may not collide any more, they do not appear in the initial state components of collision integrals!

$$p^\mu\partial_\mu(f^i+f^f) = \frac{1}{2}\int {}_{12}\mathcal{D}_4 f_1^i f_2^i W^{p4}_{12} - \frac{1}{2}\int {}_{2}\mathcal{D}_{34} f^i f_2^i W^{34}_{p2}\;. \qquad (11)$$

The gain term, $f_1^i\; f_2^i\; W^{p4}_{12}$ populates both the interacting, f^i, and free, f^f, components, so we will introduce a FO probability, which 'feeds' the free component. The probability is phase-space dependent. In principle it may depend on the positions and momenta of both incoming particles, and it can weight the outgoing phase space for one (or both) outgoing particles. In the most simple case we have to assume that it depends at least on the momentum of the outgoing particle, which belongs to the component f^f: $\mathcal{P}^{FO}(x,p)\equiv\mathcal{P}_f$.

$$\begin{aligned}p^\mu\partial_\mu(f^i+f^f) &= \frac{1}{2}\int {}_{12}\mathcal{D}_4 f_1^i f_2^i\Big[\mathcal{P}_f W^{p4}_{12} + (1-\mathcal{P}_f)W^{p4}_{12}\Big]\\ &- \frac{1}{2}\int {}_{2}\mathcal{D}_{34} f^i f_2^i W^{34}_{p2}. \qquad (12)\end{aligned}$$

Now, we can separate the two components into two equations. The sum of these two equations returns the complete BTE above:

$$p^\mu\partial_\mu f^f = \frac{1}{2}\int {}_{12}\mathcal{D}_4 f_1^i f_2^i\; \mathcal{P}_f W^{p4}_{12} \qquad (13)$$

$$p^\mu \partial_\mu f^i = \frac{1}{2}\int {}_{12}\mathcal{D}_4 f_1^i f_2^i \,(1-\mathcal{P}_f) W_{12}^{p4} - \frac{1}{2}\int {}_2\mathcal{D}_{34} f^i f_2^i W_{p2}^{34} \tag{14}$$

The free component does not have a Loss term, because particles in the free component cannot collide, and so, the free component cannot loose particles due to collisions. Rewriting the second equation yields:

$$\begin{aligned} p^\mu \partial_\mu f^i \;&=\; -\frac{1}{2}\int {}_{12}\mathcal{D}_4 f_1^i f_2^i \mathcal{P}_f W_{12}^{p4} + \frac{1}{2}\int {}_{12}\mathcal{D}_4 f_1^i f_2^i W_{12}^{p4} \\ &-\; \frac{1}{2}\int {}_2\mathcal{D}_{34} f^i f_2^i W_{p2}^{34} \end{aligned} \tag{15}$$

The first term is a drain term, describing the "escape" or "freeze out" of particles from the interacting component. It is the inverse of the gain term (or source term) for the free component, f^f. The last two terms are influencing the interacting term by redistributing particles in the momentum space. These latter two terms do not include the FO probability factors! Thus, these two terms drive the interacting component towards re-thermalization. As a usual approximation these two terms can be approximated by the relaxation time approximation as in refs. [4, 5, 6]. Thus, the BTE describing FO in this situation reads as:

$$p^\mu \partial_\mu f^f = \frac{1}{2}\int {}_{12}\mathcal{D}_4 f_1^i f_2^i \mathcal{P}_f W_{12}^{p4} \tag{16}$$

$$p^\mu \partial_\mu f^i = -\frac{1}{2}\int {}_{12}\mathcal{D}_4 f_1^i f_2^i \mathcal{P}_f W_{12}^{p4} \;+\; p^0\,\frac{f_{eq}^i - f^i}{\tau_{rel}} \tag{17}$$

The first equation, eq. (16), describes the gain of the free component, i.e. that part of the earlier gain term, which will not collide any more. The first term in the second equation has the same value with opposite sign. This describes the part of f^i, which is leaving the interacting component and does not take part in the re-thermalization.

In fact the above described collision integrals can be highly simplified, by exploiting the symmetries and conservation laws in the invariant transition rate, W, so that only one phase-space integral remains to be executed (see section 3.3 and eq. (3.27) in ref. [17]).

V. MODIFIED BOLTZMANN TRANSPORT EQUATION

Now, the question arises: can the BTE handle realistically the FO process? We have seen that the structure of the kinetic equations, used earlier to describe FO [3, 4, 5, 6], and the separation of the "escape" and "re-thermalization" terms come out in a simple, straightforward way from the BTE.

However, the usual structure of the collision terms in the BTE are not adequate for describing rapid FO, in a layer which is comparable to the m.f.p. If we assume the existence of such a layer this immediately contradicts assumption (iii): the change is not negligible in the direction of $d\sigma^\nu$. The assumption (ii) of "molecular chaos" is also violated in a FO process because number of collisions is not proportional with $f(x,p_1)\times f(x,p_2)$, but it is delocalized in the normal direction with $f(x+\lambda,p_1)\times f(x-\lambda,p_2)$. (The fact that the FO is a delocalized kinetic process, was already used in ref. [2] when integrals along the path of propagating particles were introduced, but the consequences regarding the details of the collision terms and the validity of the molecular chaos assumption were not discussed.)

Based on the above considerations, one might conclude that the changes of the distribution function are mediated by the transfer of particles, and consequently only slowly propagating changes are possible. I.e., the front propagates slowly, and its normal, $d\sigma^\mu$, is always space-like. This was a common misconception, until recently, where all "superluminous" shock, detonation, deflagration fronts or discontinuities were considered unphysical based on early studies [18]. However, it was shown recently, that discontinuous changes may happen simultaneously in spatially neighboring points, i.e. the normal of the discontinuity-hypersurface can be time-like [13, 19]. This applies to the FO process also. Thus, the direction of characteristic or dominant change, $d\sigma^\mu$, may be both space-like and time-like in the FO process.

From all the processes mentioned above (i.e. shocks, detonations, deflagrations etc.) the FO is the most special one. Because the number of interacting particles is constantly decreasing as the FO proceeds and correspondingly the m.f.p. is increasing and, in fact, it reaches infinity when the complete FO is finished. This simply means that we strictly speaking can not make FO in finite layer of any thickness smooth enough to be modeled with BTE. It is also obvious that if FO has some characteristic length scale (thickness of the layer or even some characteristic parameter for infinitely long FO [4]), it is not proportional with the m.f.p., because m.f.p. increases as the density of interacting component becomes smaller, while FO becomes faster in this limit, so its characteristic scale should decrease.

Since, there is a strong gradient in the FO direction: the free component rapidly increases, while the interacting component decreases along the FO direction, we can conclude that the collision terms in their usual form are not adequate to describe the FO process, particularly not the "escape" probability or "escape" term. The appropriate equations to describe this system can be a Modified Boltzmann Transport Equation (MBTE) [20] :

$$\begin{aligned} p^\mu \partial_\mu f(p) \;&=\; \frac{1}{2}\int {}_{12}\mathcal{D}_4 \overline{f(x,p_1)}^{\,x}\,\overline{f(x,p_2)}^{\,x} W_{p_1p_2}^{pp_4} \\ &-\; \frac{1}{2}\int {}_2\mathcal{D}_{34} \overline{f(x,p)}^{\,x}\,\overline{f(x,p_2)}^{\,x} W_{pp_2}^{p_3p_4} \end{aligned} \tag{18}$$

where $\overline{f(x,p_i)}^{\,x}$ is an average over all possible origins of the particle in the backward lightcone of the ST point $x=(t,\vec{x})$:

$$\overline{f(x,p)}^{\,x} = \frac{\int_{t_0}^{t} dt_1 \int d^3x_1 \delta^3(\vec{x}-\vec{x}_1-\vec{v}(t-t_1)) f(x_1,p) e^{-\int_{t_1}^{t} dt_2 \int d^3x_2 \sigma n(x_2) v \delta^3(\vec{x}_2-\vec{x}_1-\vec{v}(t_2-t_1))}}{\int_{t_0}^{t} dt_1 \int d^3x_1 \delta^3(\vec{x}-\vec{x}_1-\vec{v}(t-t_1)) e^{-\int_{t_1}^{t} dt_2 \int d^3x_2 \sigma n(x_2) v \delta^3(\vec{x}_2-\vec{x}_1-\vec{v}(t_2-t_1))}}, \tag{19}$$

where $\delta^3(\vec{x}-\vec{x}_1-\vec{v}(t-t_1))$ fixes the ST trajectory, along which the particles with given momentum can reach the ST point x, time t_0 is given by the initial or boundary conditions, $\vec{v}=\vec{p}/p^0$ ($v=|\vec{v}|$), and the exponential factor accounts for the probability not to have any other collision from the origin x_1 till x. In the arguments of exponents $n(x)$ is the particle density in the calculational frame, $n(x)=N^0(x)$, and σ is the total scattering cross section. After performing integrations over d^3x with a help of δ-functions we can write the MBTE equation in the form:

$$\begin{aligned} & p^\mu \partial_\mu f(p) = \\ = \; & \frac{1}{2}\int {}_{12}\mathcal{D}_4^{t_1 t_2} f(t_1,p_1) G(t_1,p_1) f(t_2,p_2) G(t_2,p_2) W_{p_1 p_2}^{p p_4} \\ - \; & \frac{1}{2}\int {}_{2}\mathcal{D}_{34}^{t_1 t_2} f(t_1,p) G(t_1,p) f(t_2,p_2) G(t_2,p_2) W_{p p_2}^{p_3 p_4}\,, \end{aligned} \tag{20}$$

where

$${}_{12}\mathcal{D}_4^{t_1 t_2} = \frac{1}{2}\int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2 \int {}_{12}\mathcal{D}_4\,, \tag{21}$$

$$f(t_1,p) = f(t_1, \vec{x}-\vec{v}(t-t_1), p)\,, \tag{22}$$

$$G(t_1,p) = \frac{e^{-\int_{t_1}^{t} dt_2 \sigma n(t_2, \vec{x}-\vec{v}(t-t_2)) v}}{C(x,p)}\,, \tag{23}$$

$$C(x,p) = \int_{t_0}^{t} dt_1 e^{-\int_{t_1}^{t} dt_2 \sigma n(t_2, \vec{x}-\vec{v}(t-t_2)) v}\,. \tag{24}$$

Interestingly, Molecular Dynamics models do not use the local molecular chaos assumption, and follow the trajectories of the colliding particles instead. Thus such models do actually solve the MBTE, and not BTE, although this was not realized before.

The obvious limit in which MBTE is reduced to BTE is a completely homogeneous ST distribution function (i.e. no external forces, no boundaries). Another possibility is the hydrodynamic limit, $\lambda = 1/\sigma n \to 0$, when the exponential factors (23,24) will be reduced to $\sim \delta(t - t_{1,2})$, reproducing the BTE after t_1, t_2 integrations.

The symmetries and the assumption of local molecular chaos lead to the consequence that local conservation laws can be derived from the original BTE, i.e. $\partial_\mu T^{\mu\nu} = 0$ and $\partial_\mu N^\mu = 0$, where T and N are given as integrals over the single particle PS distribution and the momentum. Although now we have delocalized the equations, the local conservation laws can still be derived in the same way, as it was shown in Ref. [20]. The very essential property of the BTE is the Boltzmann H-theorem. Here the situation is more complicated and the behaviour of the entropy current in MBTE is a subject of future studies. Nevertheless, for adiabatic expansion ($S^\mu_{\ ,\mu} = 0$) a sufficient condition is the same as for BTE, namely $f(x,p_1)f(x,p_2) = f(x,p_3)f(x,p_4)$ [20].

The obtained MBTE is considerably more complicated, than the original BTE. In order to proceed let us make a further simplification assuming that all the particles arrive into the collision point x from one m.f.p. distance [21] (instead of allowing them to arrive from any distance with the corresponding probabilities, as it is done in eq. (19)). This then leads to the following simplified equation, which nevertheless is still adopted to the strongly non-homogeneous systems much better then the original BTE:

$$\begin{aligned} p^\mu \partial_\mu f(p) \; = \; & \frac{1}{2}\int {}_{12}\mathcal{D}_4 f(\tilde{x}_1,p_1) f(\tilde{x}_2,p_2) W_{p_1 p_2}^{p p_4} \\ - \; & \frac{1}{2}\int {}_{2}\mathcal{D}_{34} f(\tilde{x},p) f(\tilde{x}_2,p_2) W_{p p_2}^{p_3 p_4}\,, \end{aligned} \tag{25}$$

where x_k is the origin of colliding particles, i.e. the ST point where the colliding particles were colliding last, $\tilde{x}_k = x - u_k \tau(x, \vec{v}_k/v_k)$, $u_k^\mu = (\gamma_k, \gamma_k \vec{v}_k)$, $\gamma = 1/\sqrt{1-\vec{v}^2}$ and $\vec{v}_k = \vec{p}_k/p_k^0$. Here τ is the collision time, such that $|\vec{v}|\tau(x,\vec{v}/v) = \lambda(x,\vec{v}/v)$. Note that the m.f.p. depends not only on the position, but also on the direction of the particle motion. This is an essential modification if the PS distribution has a large gradient in the space-time. This gradient defines a ST 4-vector characterizing the direction of the process, $d\sigma^\mu$. In ref. [2] the direction $d\sigma^\mu$ is also introduced, however, it is not discussed why and it is not connected to the delocalization of the BTE.

For the FO modeling, repeating for the eq. (25) the same step as for BTE above, we then obtain:

$$p^\mu \partial_\mu f^f(x,p) = \frac{1}{2}\int {}_{12}\mathcal{D}_4 \mathcal{P}_f W_{12}^{p4}\; f^i(\tilde{x}_1,p_1) f^i(\tilde{x}_2,p_2)\,, \tag{26}$$

$$\begin{aligned} p^\mu \partial_\mu f^i(x,p) \; = \; & -\frac{1}{2}\int {}_{12}\mathcal{D}_4 \mathcal{P}_f W_{12}^{p4}\; f^i(\tilde{x}_1,p_1) f^i(\tilde{x}_2,p_2) \\ + \; & p^0\,\frac{f^i_{eq} - f^i}{\tau_{rel}}\,. \end{aligned} \tag{27}$$

A simple general solution of the MBTE (26,27) cannot be given but it serves as a basis for simplified, phenomenological kinetic models describing the FO process.

VI. APPROXIMATE KINETIC FREEZE OUT MODELS

In this section our goal is to present a schematic derivation of a simple kinetic FO model used by some of the authors earlier. This represents only one particular possibility and the general MBTE equation can be solved or approximated in other ways also. The approximation we present is one of the simplest possibilities, but not necessarily the most realistic one.

If the ST distribution is non-uniform and the direction of steepest gradient can be clearly identified, one may replace one (or more) of the integrals over d^3p_1 (or d^3p_2) by space-time integrals over the origins of the incoming particle(s), d^4x', requiring that the particle reaches the ST point, x, when needed. This requirement determines p^μ for a given x'^μ. It is reasonable to assume that after converting some of the integrals to ST integrals and performing them, we get an effective FO term reflecting the properties of the local PS distribution, transition rate, the ST configuration (e.g. gradient of density change, and its direction) and characteristics of the FO layer.

Let us return to the basic integral form of the kinetic theory, eqs. (26,27), and discuss the FO probability. We will study equation (26) without performing the integrals in a formal way, rather illustrating the procedure giving a better insight into the problem.

When we are in the FO layer, close to the boundary of complete FO we have to calculate here the collision rate. According to the MBTE this depends on the PS distribution of the incoming particles at their origins, $f^i(\tilde{x}_1,p_1)\ f^i(\tilde{x}_2,p_2)$. Assume that the FO direction points in the direction of $d\sigma^\mu$, as it is shown in Figure 2. On the right hand side of the collision point the density of interacting particles is low or zero, while on the left hand side it is larger, closer to the pre FO value (see Figure 2).

It is more probable, that particles arrive to the collision point x from the left side, because of the higher density of the interacting particles on the left. Consequently, most outgoing particles leave to the right. Thus, the collision rate at x depends on the conditions what we have around $\tilde{x}_1$ and $\tilde{x}_2$, i.e. deeper inside the interactive matter. Consequently, the collision rate is still higher than the conditions at x could secure! Then $\mathcal{P}_f$ determines what fraction of the outgoing particles will freeze out from those which collided. The collision rate does not go to zero even if we are at the outside boundary of the FO layer, because particles still can arrive from the left where we still have interacting particles. As there are no interacting particles on the right hand side, all of these particles should freeze out, i.e. $\mathcal{P}_f \to 1$, when $x \to L$ (see Figure 2).

Let us execute two of the phase space integrals for one

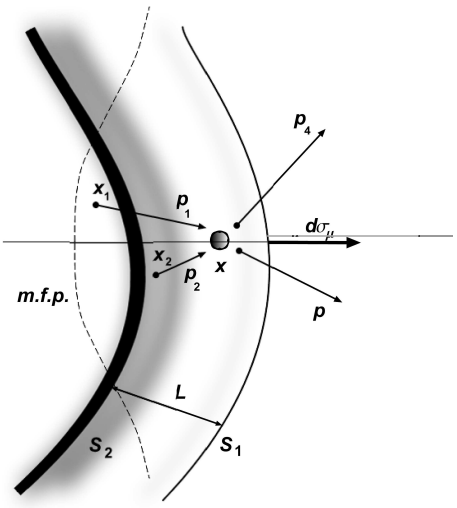

FIG. 2: The plot of one of the last collisions at x, shown in the spatial cross section of the FO layer. Particles arrive from positions $\tilde{x}_1$ and $\tilde{x}_2$ to point x with momenta p_1 and p_2. Within the FO layer of thickness, L, the density of interacting particles gradually decreases (indicated by shading) and disappears at the outside boundary, S_1 (thin line) of the layer. R.h.s. from this boundary there are no interacting particles. Particles can reach x from a region closer than the mean free path (m.f.p. indicated by the dashed line), but only from places where the interacting particle density is still not zero, i.e. mostly from the left. The inside boundary of the FO layer, S_2 (thick line) indicates the points where the FO starts. Left of this line there is only interacting matter and the FO probability is assumed to be zero for collisions happening in the interacting region.

incoming and one outgoing particle, $\int \frac{d^3p_2}{p_2^0}\ \frac{d^3p_4}{p_4^0}$

$$p^\mu\partial_\mu f^f = \frac{1}{2}\int {}_{12}\mathcal{D}_4 f_1^i f_2^i\ \ \mathcal{P}_f W_{12}^{p4} = \tag{28}$$

$$= \frac{1}{2}Q_2V_4\int \frac{d^3p_1}{p_1^0} f^i(\tilde{x}_1,p_1)\ \ \mathcal{P}_f W_{1\bar{2}}^{p\bar{4}}, \tag{29}$$

where $Q_2 = \int \frac{d^3p_2}{p_2^0}\ f^i(\tilde{x}_2,p_2)$ and V_4 are invariant scalars. Eq. (29) resembles eq. (3.27) in ref. [17], but one of the incoming particle distributions, $f^i(\tilde{x}_2,p_2)$ is integrated out, and leads to an integral quantity, Q_2. This can be approximated by the invariant scalar density at $\bar{x}_2$, i.e.

$$Q_2 \approx n_2(\bar{x}_2)\ .$$

Here V_4 is not known directly, but can in principle be calculated based on the distributions, $f^i(\tilde{x}_1,p_1)$ and $f^i(\tilde{x}_2,p_2)$, and the transition rate, W. The resulting transition rate will then be averaged over particles 2 and 4, $W_{1\bar{2}}^{p\bar{4}}$:

$$p^\mu\partial_\mu f^f(x,p) = \frac{1}{2}Q_2V_4\int \frac{d^3p_1}{p_1^0} f^i(\tilde{x}_1,p_1)\mathcal{P}_f W_{1\bar{2}}^{p\bar{4}}\ . \tag{30}$$

As we see this resulting equation is delocalized - in a rapid dynamical process the distributions at x and $\tilde{x}_1$ are not the same, as it was discussed above.

Now, eq. (30) can be integrated either in the $\tilde{x}_1$ -space, or in the p_1 -space, as the two are connected by the fact that a particle should travel from $\tilde{x}_1$ to x with momentum p_1. We should integrate over all $\tilde{x}_1$ points from where one can reach x in a collision time, $\tau(x, \vec{v}/v)$. This brings in information about the local spatial gradient of the ST distribution function, as we discussed it above. The collision terms in the original BTE contain only local information, which is assumed to be isotropic (or slowly changing), so it is neglected.

In addition the FO probability, $\mathcal{P}_f$, may include integrated information about the FO process, e.g. the probability not-to-collide with anything on the way out, reasonably should depend on the integral number of interacting particles on the way out.

For the sake of simplicity let us assume small angle scatterings, and the propagation of a single particle $W_{1\bar{2}}^{p\bar{4}} \approx w_{\bar{2}}^{\bar{4}}\ \delta(p - p_1)$, then

$$p^\mu \partial_\mu f^f(x,p) = \frac{1}{2} Q_2 V_4\ f^i(\tilde{x}_1, p)\ \mathcal{P}_f\ w_{\bar{2}}^{\bar{4}} . \qquad (31)$$

The cumulative effect of all particles which can reach the ST point x in a collision time, leads to a change directed into the direction given by $d\sigma^\mu$. The transition rate, $w_{\bar{2}}^{\bar{4}}$ can be estimated as $\langle \sigma v_{rel} \rangle \sim p^\mu/p^0$, what yields to:

$$p^\mu \partial_\mu f^f(x,p) = f^i(\tilde{x}_1, p)\ \left\{ \frac{1}{2} Q_2 V_4\ \mathcal{P}_f\ \ d\sigma^\mu p_\mu / p^0\ . \right\} \qquad (32)$$

As we mentioned, the spatial variation of the phase-space distribution cannot be neglected in rapid dynamical processes as the FO, and this brings in a direction of the dominant change, $d\sigma^\mu$.

Let us now consider the FO situation, where we have a directed process in a layer. The dominant change happens in the direction of the normal of the FO hypersurface, $d\sigma^\mu$ (where $d\sigma^\mu\ d\sigma_\mu = \pm 1$). We can decompose the 4-vector, p^ν on the l.h.s. of the above equations into four orthogonal directions:

$$\begin{aligned} p^\nu &= (p^\mu d\sigma_\mu) d\sigma^\nu + (p^\mu d\sigma_{1\mu}) d\sigma_1^\nu \qquad (33) \\ &+ (p^\mu d\sigma_{2\mu}) d\sigma_2^\nu + (p^\mu d\sigma_{3\mu}) d\sigma_3^\nu\ , \end{aligned}$$

where the 4-vectors, $d\sigma_1^\nu$, $d\sigma_2^\nu$, and $d\sigma_3^\nu$, are tangent to the hypersurface and orthogonal to the normal, $d\sigma^\nu$. This leads to:

$$\begin{aligned} p^\nu \partial_\nu f(x,p) &= \left[(p^\mu d\sigma_\mu) d\sigma^\nu + (p^\mu d\sigma_{1\mu}) d\sigma_1^\nu \right. \qquad (34) \\ &+ \left. (p^\mu d\sigma_{2\mu}) d\sigma_2^\nu + (p^\mu d\sigma_{3\mu}) d\sigma_3^\nu \right] \partial_\nu f(x,p)\ . \end{aligned}$$

Here we assumed that the change happens in the direction of the normal and negligible along the hypersurface of the front, thus the last three terms can be neglected:

$$p^\nu \partial_\nu f(x,p) \approx (p^\mu d\sigma_\mu)\ d\sigma^\nu \partial_\nu\ f(x,p)\ .$$

Inserting the above equation into (32) yields a kinetic equation describing the directional derivative of the distribution function in the direction of the dominant change, $d\sigma^\mu$ as

$$d\sigma^\mu \partial_\mu f^f(x,p) = f^i(\tilde{x}_1, p)\ P_{esc}^*\ , \qquad (35)$$

where the escape probability depends on the ST coordinates, on the interacting part of the PS distribution, on the transfer properties and the FO probability: $P_{esc}^*(x, p, f^i, d\sigma, w, \mathcal{P}_f)$. The $\tilde{x}_1$ in this case means $\tilde{x}_1 = x - d\sigma^\mu \lambda_\mu$, where λ_μ is a four-vector of m.f.p.

This $\tilde{x}_1$ in the argument of the distribution function on the r.h.s. of the eq. (35) is extremely important. Certanly, we can repeat all the steps from eq. (28) to eq. (35) based on the BTE amd the result will be the same except for the delocalization of f on the r.h.s. This $x - d\sigma^\mu \lambda_\mu$ dependence of the distribution function reflects the FO property which was discussed at the beginning of this section and illustrated in Fig. 2 - the collision rate at some point x, and correspondingly the number of particles, which will freeze out after this collision, feels the properties of the matter deeper (by about one m.f.p.) inside the interacting matter.

The derivation above did neglect several details and features, however, reflects the basic structure of ad hoc kinetic FO models [3, 4, 5, 6]. In these models the infinitely long FO was studied, and therefore the delocalization of eq. (35) was not so important. For the FO modeling in the finite layer [22] this effect will cause a substantial difference making FO faster.

VII. ESCAPE PROBABILITY

The escape probability in eq. (35) can be estimated based on fundamental physical principles, like it is done in the above mentioned works. The approach can, nevertheless, be improved if we take into consideration the origin of the above derivation, especially the requirement of full covariance of the model and the requirement that the FO process may point in any space-time direction. The first significant advances, where these principles were applied, are presented in [7]. It incorporates the achievements of recent years, by cutting negative contributions in the FO density [15] and making the FO direction dependent [3]. Here we just present briefly a direct estimate for the escape probability [25, 26].

The escape probability includes the FO probability, which separates from among the outgoing (gain) particles, which fraction of them is still colliding and which not. The probability not-to-collide with anything on the way out, reasonably should depend on the number of particles, which are in the way of a particle moving outwards in the direction $\vec{p}/p$, across a FO layer of estimated thickness L (representing the fact that we have finite number of particles on the way out to collide with [4]). If we are in this FO layer and progressed from the beginning of the layer to a position x^μ, there is still

$$\frac{L - x^\mu d\sigma_\mu}{\cos\Theta}$$

distance ahead of us, where Θ is an angle between the normal vector and $\vec{p}/p$. We assume then that the FO probability is inversely proportional to some power of this quantity [25, 26]. Thus

$$P^*_{esc} = \frac{1}{\lambda(\bar{x}_1)} \left(\frac{L}{L - x^\mu d\sigma_\mu} \right)^a (\cos\Theta)^a \, \Theta(p^\mu d\sigma_\mu) \,, \quad (36)$$

where the power a is influencing the FO profile across the front, and the cut factor is eliminating negative contributions to FO. In papers [3, 4, 5, 6] the authors have used $a = 1$, and modeled FO in an infinite layer, i.e. in $L \to \infty$ limit. Furthermore, they were using a constant characteristic length λ instead of $\lambda(\bar{x}_1)$:

$$P^*_{esc} = \frac{\cos\Theta}{\lambda} \, \Theta(p^\mu d\sigma_\mu) \,. \quad (37)$$

Comparing eqs. (36) and (37) one can see that now we replace the constant characteristic length λ, which was clearly oversimplifying the situation, with two factors. The first is the collision rate, which is proportional with $\frac{1}{\lambda(\bar{x}_1)} \approx \langle n(\bar{x}_1)\sigma \rangle$, and this does not tend to zero even if we reach the outside boundary of the FO layer, as this parameter is characteristic to the interior region at $\bar{x}_1$. The other is the generalized FO probability, which depends on the direction of the outgoing particle and on the number of interacting particles left in the way to collide with, i.e. $\propto \frac{L}{L-x}$, where we have fixed $d\sigma^\mu = (0, 1, 0, 0)$. So, we have generalized eq. (37) by replacing

$$\lambda \to \lambda'(x) = \lambda(\bar{x}_1) \frac{L - x}{L} \,. \quad (38)$$

Now the new characteristic length $\lambda'(x)$ gradually decreases as FO proceeds and the number of interacting particles becomes smaller and smaller, and goes to 0 when the FO is finished, as it was discussed in section V.

The simple angular factor, $\cos\Theta$, maximizes the FO probability of those particles, which propagate in the direction closest to the normal of the layer, $d\sigma_\mu$. The quantities, $\cos\Theta = p^x/|\vec{p}|$ for FO in x-direction and $\cos\Theta = 1$ for FO in t-direction, are not Lorentz invariant. Therefore, to make our description completely invariant we shall generalize it to $\frac{p^\mu d\sigma_\mu}{p^\mu u_\mu} \sim \cos\Theta$.

So, we write the invariant escape probability, within the FO layer covering both the time-like and space-like parts of the layer [26], as

$$P^*_{esc} = \frac{1}{\lambda(\bar{x}_1)} \left(\frac{L}{L - x^\mu d\sigma_\mu} \right)^a \left(\frac{p^\mu d\sigma_\mu}{p^\mu u_\mu} \right)^a \, \Theta(p^\mu d\sigma_\mu) \,. \quad (39)$$

If we take the four velocity equal to $u_\mu = (1, 0, 0, 0)$, in the Rest Frame of the Front (RFF), i.e. where $d\sigma_\mu = (1, 0, 0, 0)$, then the momentum dependent part of the Escape Probability, $P(p)$, is unity. Otherwise, in the Rest Frame of the Gas (RFG), where $u_\mu = (1, 0, 0, 0)$, the escape probability $P(p)$ is $P(p) = p^\mu d\sigma_\mu \; \Theta(p^\mu d\sigma_\mu)/p^0$. More detailed investigation about escape probability $P(p)$ for different $d\sigma_\mu$ can be found in [7].

To calculate the parameters of the normal vector $d\sigma_\mu$ for different cases listed above, we simply make use of the Lorentz transformation. The normal vector of the time-like part of the FO hypersurface may be defined as the local t'-axis, while the normal vector for the space-like part may be defined as the local x'-axis. As the $d\sigma_\mu$ normal vector is normalized to unity its components may be interpreted in terms of γ_σ and v_σ, as $d\sigma_\mu = \gamma_\sigma(1, v_\sigma, 0, 0)$, where $\gamma_\sigma = \frac{1}{\sqrt{1-v_\sigma^2}}$ for time-like normals and $\gamma_\sigma = \frac{1}{\sqrt{v_\sigma^2-1}}$ for space-like normals.

The detailed results of the application of this covariant escape probability will be presented elsewhere [22].

In refs. [4, 5] the post FO distribution was evaluated for space-like gradual FO in a kinetic model. Initially we had an equilibrated, interacting PS distribution, $f_{int}(p, x)$, and an escape probability, similar to eq. (39), but simplified one. It was dependent on the angle of the two vectors only. After some small fraction of particles were frozen out as the FO process progressed in the front, the interacting component were re-equilibrated with smaller particle number, smaller energy and momentum to account for the quantities carried away by the frozen out particles. This was then repeated many times in small steps along the FO front and the frozen out particles were accumulated in the post FO PS distribution, f_{free}. The resulting distribution was highly anisotropic and obviously non-equilibrated. The details of the post FO distribution depend on the details of the escape probability, and on the level of re-equilibration of the remaining, interacting component.

Bugaev assumed earlier [15] (see also ref. [23]), that the post FO distribution is a (sharply cut) "Cut-Jüttner" distribution, but the above mentioned model shows that this can only be obtained if re-equilibration is not taking place. The kinetic model provided an asymmetric but smooth PS distribution [4], while the escape probability (39) yields a somewhat different, but also smooth PS distribution [26]. These can be well approximated by the "Canceling Jüttner" distribution [24].

In ref. [6] the same infinite 1D model (as in refs. [4, 5]) was applied for the FO through the layer with time-like normal vector. The model in this case can be solved analytically (since $\cos\Theta \equiv 1$) and thus, the exact form of the post FO distribution for pions and protons was obtained. Although analytical expressions for these distribution are different from the thermal Jüttner distribution, the forms of the functions are very similar for intermediate and high momenta. Deviations at the low momenta seems to be due to infinite long FO (they become much smaller for the escape probability (39), modeling FO in a finite layer) and 1D character of the model [26].

VIII. CONCLUSIONS

The FO process was discussed in the 4-dimensional space-time in a layer of finite thickness. Arising from the physical process this layer is directed, it has an inside and outside boundary, which are not identical. The processes in the layer are not isotropic, they must be sensitive to the direction of the layer. It is shown that, as a consequence, the basic assumptions of the Boltzmann Transport Equation are not satisfied in this layer, and the equation should be modified. It is also shown that earlier, ad hoc kinetic models of the FO process, can be obtained from this approach in a fully covariant way, and freeze out in space-like and time-like directions can be handled on the same covariant footing.

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[27] In refs. [1, 12] the author discusses two physically different situations, which should be clearly separated, otherwise one can run into some confusions. The first one is the "transition" from hydrodynamical description of the system created in heavy ion collisions to cascade description. This is not a physical phase transition, just a switch from one theoretical model to another, which can be justified, strictly speaking, only in the region where both models give adequate description of the system, i.e. the same result for the all possible observables. In this overlapping ST region such a transition can be realized at any infinitely narrow dividing hypersurface. On the other hand, for the case of a real physical phase transition, like FO or/and hadronization, the infinitely narrow dividing hypersurface is an idealization of the layer of some finite thickness, as it was discussed above. We basically assume that if we use the correct particle distributions on the "post" side, we do not make a big mistake changing distributions to the new ones sharply instead of changing them gradually in some layer.