New stability analysis of singular systems with mixed interval time-varying delays

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Abstract. This paper deals with the problem of exponential stability of linear singular systems with mixed interval discrete and distributed time-varying delays. By using a set of improved Lyapunov-Krasovskii functionals, new delay-range-dependent conditions are established in terms of linear matrix inequalities (LMIs) which guarantee that the system is regular, impulse-free and \(\alpha\)-exponentially stable. This allows us to compute simultaneously the two bounds that characterize the exponential stability rate of the solution by various efficient convex optimization algorithms. A numerical example is given to show the effectiveness of the obtained results.

Key Words: Singular system, exponential stability, \(\alpha\)-stability, interval delay, linear matrix inequality.

1. Introduction

In many practical systems, such as electrical circuit network, power systems, aerospace engineering, chemical engineering systems, biological systems, network analysis, and lossless transition lines, etc, the state variables may be related both differentially and algebraically, which form a general class of systems, called singular systems \([4, 9, 10, 15, 29, 31]\). Depending on the area of application, these models are called singular \([9, 29]/\)implicit \([1]/\)differential-algebraic \([6, 23]/\)descriptor systems \([8, 13, 30]\). Singular systems are governed by the so-called singular differential equations, which endow the systems with many special features that are not found in classical systems such as impulse terms and input derivatives in the state response, nonproperness of transfer matrix, noncausality between input and state (or output), consistent initial conditions, etc., that make the study of singular systems are
much more complicated than that for classical systems [9]. Thus, these systems have been widely studied in the past two decades and a great number of fundamental results including solvability, regularity, canonical forms, system equivalence, stability analysis and control, etc, have been developed (for more detail, see, e.g., [29]).

On the other hand, time-delay often occurs in various practical systems such as population models, physical and chemical processes, biological and artificial neural networks, communication and long transmission lines, and microwave oscillators, etc, and frequently is a source bad performance, oscillation or instability [21, 25]. Thus, the investigation of stability analysis for systems with time delays play a important role in applied models and great efforts from researchers have been dedicated to the study of time delay systems. For more detail, see [3, 16-20, 24, 25] and references therein. The main approach in recently works is use of Lyapunov-Krasovskii functional method to derive sufficient conditions guarantee the stability of the systems. These conditions can be classified in to two type, namely, delay-dependent stability conditions and delay-independent ones. Generally speaking, delay-dependent stability criteria, which utilize information on the size of delays, are less conservative and reasonable than delay-independent ones. Thus, more attentions have been paid to derive less conservative delay-dependent stability criteria for time delay systems during the past decade with a key role to get maximum delay bounds is how to choose Lyapunov-Krasovskii functional candidates to derive stability conditions guaranteeing the asymptotic or exponential stability of time delay systems [13, 17, 18, 27, 31].

Singular systems with time delay, which have both delay and algebraic constrains, often appear in various engineering systems. The study of such systems is much more complicated than that for standard state-space time delay systems due to their structure, in general, consist of delay differential equations coupled with difference equations [13, 15]. During the last decades, we witnessed an increasing interest to the class of singular systems with time delay. Much more efforts of researchers from mathematics and control communities have been paid to develop the study of stability analysis and control of such systems. For more detail, we refer the readers to [6, 7, 12-14, 26, 30] for stability analysis and [3, 4, 11, 23, 29] for control of singular systems. Both delay-independent and delay-dependent stability conditions for singular time delay systems have been derived using the time domain method. However, there are few results concern with the problem of exponential stability of singular systems with time-varying delays and most of the delay-dependent results in the literature tackled only the case of constant or slowly time-varying delays [8, 15, 26, 28]. Particularly, in [15], the authors consider the problem of $\alpha$-exponential stability for singular system with multiple slowly time-varying delays. By decomposing the system into fast and slow subsystems and prove the stability of the the slow subsystem using Lyapunov-Krasovskii functional method. To prove the stability of the fast subsystem, some terminologies have been borrowed from graph theory to model the dependency of the fast variables on pass instants and express them in terms of the slow variables. However, this approach is rather complicated for application and further improvement. In [8], the authors considered a class of singular systems also with slowly time-varying delay of descriptor form. Based on some mild conditions so that the system can be converted to delay differential-algebraic equations. The exponential stability problem in such a way that by using Lyapunov functional method to get the decay rate of the
slow variables and then the stability of the fast variables is proved by perturbation approach. It is worth mentioning another recently work, in [10], a class of singular systems with interval time-varying delay and nonlinear perturbations is considered. Based on Lyapunov functional and free weighting matrices approach, a uniform asymptotic stability criterion in terms of linear matrix inequalities (LMIs) was addressed. Unfortunately, this result was derived from a not correct estimation of Lyapunov functional. Thus, the problem of exponential stability analysis for singular systems with interval time-varying delays without restricted on variation of state delay is still remain open, which motivated the present paper.

In this paper, we consider a class of singular systems with mixed interval discrete and distributed time-varying delays. Delay-range-dependent exponential stability conditions are established in terms of LMIs which guarantee the regularity, impulse-free and exponential stability of the system.

The rest of this presentation is organized as follows. In section 2, the problem and some definitions, technical propositions are stated. Main results are given in section 3. Numerical examples to illustrate the effectiveness of our conditions are given in Section 4. The paper ends with conclusions and cited references.

Notations: The following notations will be used throughout this paper. $R^+$ denotes the set of all nonnegative real numbers; $R^n$ denotes the $n$-dimensional Euclidean space with the norm $\| \cdot \|$ and scalar product $\langle x, y \rangle = x^T y$ of two vectors $x, y$; $\lambda_{\text{max}}(A)$ ($\lambda_{\text{min}}(A)$, resp.) denotes the maximal (the minimal, resp.) number of the real part of eigenvalues of $A$; $A^T$ denotes the transpose of the matrix $A$ and $I_d$ denotes the identity matrix in $R^d$; $Q \geq 0$ ($Q > 0$, resp.) means that $Q$ is semi-positive definite (positive definite, resp.) i.e. $\langle Qx, x \rangle \geq 0$ for all $x \in R^n$ (resp. $\langle Qx, x \rangle > 0$ for all $x \neq 0$); $A \geq B$ means $A - B \geq 0$; $C^1([-\tau, 0], R^n)$ denotes the Banach space of continuous functions from interval $[-\tau, 0]$ to $R^n$ with the norm $\| \varphi \|_{\tau} = \sup_{-\tau \leq t \leq 0} \{ \| \varphi(t) \|, \| \dot{\varphi}(t) \| \}$.

2. Problem statement and preliminaries

Consider the following singular system with mixed discrete and distributed time-varying delays
\[
\begin{cases}
E \dot{x}(t) = Ax(t) + Dx(t - h(t)) + \int_{t-d(t)}^{t} Gx(s)ds, & t \geq 0, \\
x(t) = \varphi(t), & t \in [-\tau, 0],
\end{cases}
\tag{2.1}
\]
where $x(t) \in R^n$ is the system state; $E, A, D, G \in R^{n \times n}$ are real known matrices with appropriate dimensions; matrix $E$ may be singular with rank($E$) = $r \leq n$. The time varying delays $h(t), d(t)$ are continuous functions satisfying $0 \leq h_1 \leq h(t) \leq h_2$, $0 \leq d_1 \leq d(t) \leq d_2$ and $\dot{h}(t) \leq \mu$, where $h_1, d_1$ and $h_2, d_2$ are lower and upper bounds of the time varying delays $h(t)$ and $d(t)$, respectively; $\tau = \max\{h_2, d_2\}$, and $0 \leq \mu < 1$ is the variation rate of the discrete delay function $h(t)$. $\varphi(t) \in C_{a}^\tau \subset C^1([-\tau, 0], R^n)$ is the compatible initial function.

The following definitions for singular time delay system are adopted (e.g. see [15, 27])
Definition 2.1.

(i) System (2.1) is said to be regular if the characteristic polynomial \( \det (sE - A) \) is not identically zero.

(ii) System (2.1) is said to be impulse-free if \( \deg \det (sE - A) = \text{rank}(E) \). It means that, the pair \((E, A)\) has index one.

(iii) For \( \alpha > 0 \), system (2.1) is said to be \( \alpha \)-exponentially stable if there exist \( \nu > 0 \) such that, for any compatible initial conditions \( \varphi(t) \) the solution \( x(t, \varphi) \) satisfies

\[
\|x(t, \varphi)\| \leq \nu \|\varphi\| e^{-\alpha t}, \quad \forall t \geq 0.
\]

(iv) System (2.1) is said to be \( \alpha \)-exponentially admissible if it is regular, impulse-free and \( \alpha \)-exponentially stable.

The goal of this paper is to establish linear matrix inequalities conditions guaranteeing that the system is regular, impulse-free and \( \alpha \)-exponentially stable.

The following technical propositions are given for using in the proof of our results.

Proposition 2.1. [17] For any matrices \( N, M = M^T > 0 \), with appropriate dimensions and \( x, y \in \mathbb{R}^n \), the following inequality holds

\[
2x^T Ny \leq x^T Mx + y^T N^T M^{-1} Ny.
\]

Proposition 2.2. [14] For any symmetric positive definite matrix \( M \), scalar \( \nu > 0 \) and vector function \( \omega : [0, \nu] \rightarrow \mathbb{R}^n \) such that the integrals concerned are well defined, then

\[
\left( \int_0^\nu \omega(s)ds \right)^T M \left( \int_0^\nu \omega(s)ds \right) \leq \nu \int_0^\nu \omega^T(s) M \omega(s)ds.
\]

Proposition 2.3. (Schur complement Lemma) [5] For given matrices \( X, Y, Z \) with appropriate dimensions satisfying \( X = X^T, Y = Y^T > 0 \). Then

\[
\begin{bmatrix} X & Z^T \\ Z & -Y \end{bmatrix} < 0
\]

if and only if \( X < 0 \) and \( X + Z^T Y^{-1} Z < 0 \).

Proposition 2.4. Let \( \tau > 0, \delta > 0, \gamma \in (0; 1) \) be given and \( v(t) \) be a continuous function satisfy

\[
0 \leq v(t) \leq \gamma \sup_{-\tau \leq s \leq 0} v(t + s) + \delta, \quad \forall t \geq 0.
\]  

(2.2)

Then the following inequality holds

\[
v(t) \leq \gamma \sup_{-\tau \leq s \leq 0} v(s) + \frac{\delta}{1 - \gamma}, \quad \forall t \geq 0.
\]  

(2.3)
Proof. Denote \( v_\ast = \sup_{-\tau \leq s \leq 0} v(s) \). It is easy to verify the following inequality holds

\[
v(t) \leq \gamma \max \left\{ v_\ast, \sup_{0 \leq s \leq t} v(s) \right\} + \delta, \quad \forall t \geq 0.
\]

Thus, from (2.2) we have, \( v(0) \leq \gamma v_\ast + \frac{\delta}{1 - \gamma} =: \eta \). If (2.3) does not hold for all \( t \geq 0 \), then there exists a \( t_\ast > 0 \) such that \( v(t_\ast) = \eta \) and \( v(t) < \eta \) for all \( t \in [0, t_\ast) \). Therefore, \( \sup_{0 \leq s \leq t} v(s) \leq \eta \). Note that, \( \gamma \eta + \delta < \eta \), and we have

\[
v(t_\ast) \leq \gamma \max \left\{ v_\ast, \sup_{0 \leq s \leq t_\ast} v(s) \right\} + \delta \leq \gamma \max\{v_\ast, \eta\} + \delta < \eta.
\]

This contradiction shows that \( v(t) \leq \eta \) for all \( t \geq 0 \), which completes the proof. \( \square \)

Proposition 2.5. Let \( \tau, \lambda, \delta_1, \delta_2 \) are given positive numbers, \( \delta_1 e^{\lambda \tau} < 1 \), and \( f(t) \) be a continuous function satisfy

\[
0 \leq f(t) \leq \delta_1 \sup_{-\tau \leq s \leq 0} f(t + s) + \delta_2 e^{-\lambda t}, \quad \forall t \geq 0.
\]

Then, the following inequality holds

\[
f(t) \leq \left[ \delta_1 e^{\lambda \tau} \sup_{-\tau \leq s \leq 0} f(s) + \frac{\delta_2}{1 - \delta_1 e^{\lambda \tau}} \right] e^{-\lambda t}, \quad \forall t \geq 0.
\]

Proof. Denote \( v(t) = e^{\lambda t} f(t), \gamma = \delta_1 e^{\lambda \tau} < 1 \). Noting that, \( e^{\lambda(t-\tau)} \leq e^{\lambda(t+s)} \) for all \( s \in [-\tau, 0] \), we have

\[
v(t) \leq \delta_1 e^{\lambda t} \sup_{-\tau \leq s \leq 0} f(t + s) + \delta_2 \leq \gamma \sup_{0 - \tau \leq s \leq 0} v(t + s) + \delta_2.
\]

By proposition 2.4, we obtain

\[
e^{\lambda t} f(t) = v(t) \leq \gamma \sup_{-\tau \leq s \leq 0} v(s) + \frac{\delta_2}{1 - \gamma} \leq \delta_1 e^{\lambda \tau} \sup_{-\tau \leq s \leq 0} f(s) + \frac{\delta_2}{1 - \delta_1 e^{\lambda \tau}},
\]

which implies

\[
f(t) \leq \left[ \delta_1 e^{\lambda \tau} \sup_{-\tau \leq s \leq 0} f(s) + \frac{\delta_2}{1 - \delta_1 e^{\lambda \tau}} \right] e^{-\lambda t}, \quad \forall t \geq 0.
\]

\( \square \)

3. Main results

In this section, we propose new conditions guaranteeing that system (2.1) is regular, impulse-free and exponentially stable. Before stating main result, some notations of matrices are defined for given \( \alpha > 0 \) as follows

\[
\Phi_{11} = A^T P^T + PA + Q + Q_1 + Q_2 + d^2_1 R_1 + (d_2 - d_1)^2 R_2 + 2\alpha P E + X_1 E + E^T X_1^T,
\]
Theorem 3.1. For given symmetric positive definite matrices $Q, Q_2, R_i, W_i$, $i = 1, 2$, and matrices $P, X_i, Y_i, Z_i$, $i = 1, 2$, with appropriate dimensions satisfying the following LMIs:

$$PE = E^TP^T \succeq 0,$$  \hspace{1cm} (3.1)

\[
\begin{bmatrix}
\Phi_{11} & \Phi_{12} & Z_1 E & -Y_1 E & PG & 0 & \gamma_{21} X_1 & \gamma_{22} Y_1 & \gamma_{22} Z_1 & \Phi_{110} \\
* & \Phi_{22} & Z_2 E & -Y_2 E & 0 & 0 & \gamma_{21} X_2 & \gamma_{22} Y_2 & \gamma_{22} Z_2 & \Phi_{210} \\
* & * & \Phi_{33} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & \Phi_{44} & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & \Phi_{55} & \Phi_{56} & 0 & 0 & 0 & \Phi_{510} \\
* & * & * & * & * & \Phi_{66} & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & -\gamma_{21} W_1 & 0 & 0 & 0 \\
* & * & * & * & * & * & * & \Phi_{88} & 0 & 0 \\
* & * & * & * & * & * & * & * & -\gamma_{22} W_2 & 0 \\
* & * & * & * & * & * & * & * & * & \Phi_{1010}
\end{bmatrix} < 0, \hspace{1cm} (3.2)
\]

\[
\tau_{\alpha,d} R_2 - Q < 0, \hspace{1cm} \tau_{\alpha,h} Q - Q_2 < 0. \hspace{1cm} (3.3)
\]

Proof. Firstly, we prove the regularity and impulse-free of system (2.1). Since $\text{rank}(E) = r \leq n$, these exist two nonsingular matrices $M, N$ such that

$$\bar{E} = MEN = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}.$$

We denote

$$\bar{A} = MAN = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \hspace{1cm} \bar{D} = MDN = \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix},$$

Now, main result is addressed in the following theorem.
\[
\tilde{G} = MGN = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix}, \quad \tilde{P} = N^TPM^{-1} = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix},
\]

\[
\tilde{Q} = N^TQN = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^T & Q_{22} \end{bmatrix}, \quad \tilde{Q}_i = N^TQ_iN = \begin{bmatrix} Q_{i11} & Q_{i12} \\ Q_{i12}^T & Q_{i22} \end{bmatrix},
\]

\[
\tilde{R}_i = N^T\tilde{R}_iN = \begin{bmatrix} R_{i11} & R_{i12} \\ R_{i12} & R_{i22} \end{bmatrix},
\]

and

\[
\tilde{X}_i = N^TX_iM^{-1}, \quad \tilde{Y}_i = N^TY_iM^{-1}, \quad \tilde{Z}_i = N^TZ_iM^{-1}, \quad \tilde{W}_i = M^{-T}W_iM^{-1} \quad i = 1, 2.
\]

It follows from (3.1) that

\[
\tilde{P}\tilde{E} = \tilde{E}^T\tilde{P}^T \geq 0.
\]

By using the expression of \( \tilde{E} \) and \( \tilde{P} \), we have \( P_{21} = 0, P_{11} \geq 0 \). On the other hand, by Schur complement lemma (Proposition 2.3), it follows from (3.2) that

\[
\begin{bmatrix}
\Phi_{11} & \Phi_{12} & Z_1E & -Y_1E \\
* & \Phi_{22} & Z_2E & -Y_2E \\
* & * & \Phi_{33} & 0 \\
* & * & * & \Phi_{44}
\end{bmatrix} < 0,
\]

(3.4)

Pre- and post- multiply by \( [I \quad I \quad I \quad I] \) and \( [I \quad I \quad I \quad I]^T \), from (3.4), we obtain

\[
(A + D)^TP^T + P(A + D) + 2\alpha PE + (1 - e^{-2\alpha h_1})Q_1 + (1 - e^{-2\alpha h_2})[(1 - \mu)Q + Q_2] + d_1^2R_1 + (d_2 - d_1)^2R_2 < 0.
\]

Thus,

\[
(A + D)^TP^T + P(A + D) < 0,
\]

which proves that \( P, \tilde{P} \) are nonsingular matrices, and hence \( P_{11} > 0 \). On the other hand, by pre- and post- multiplying by \( \text{diag}\{N^T, N^T, N^T, N^T\} \) and \( \text{diag}\{N, N, N, N\} \), it follows from (3.4) that

\[
\begin{bmatrix}
\Phi_{11} & \Phi_{12} & \tilde{Z}_1E & -\tilde{Y}_1E \\
* & \Phi_{22} & \tilde{Z}_2E & -\tilde{Y}_2E \\
* & * & \Phi_{33} & 0 \\
* & * & * & \Phi_{44}
\end{bmatrix} < 0,
\]

(3.5)

where

\[
\begin{align*}
\Phi_{11} &= A^TP^T + \tilde{P}A + \tilde{Q} + \tilde{Q}_1 + \tilde{Q}_2 + d_1^2\tilde{R}_1 + (d_2 - d_1)^2\tilde{R}_2 + 2\alpha \tilde{P}\tilde{E} + \tilde{X}_1E + \tilde{E}^T\tilde{X}_1^T, \\
\Phi_{12} &= \tilde{P}\tilde{D} - X_1E + \tilde{E}^T\tilde{X}_2^T + \tilde{Y}_1E - \tilde{Z}_1E, \\
\Phi_{22} &= -(1 - \mu)e^{-2\alpha h_2}\tilde{Q} + \tilde{Y}_2E + \tilde{E}^T\tilde{Y}_2^T - \tilde{X}_2E - \tilde{E}^T\tilde{X}_2^T - \tilde{Z}_2E - \tilde{E}^T\tilde{Z}_2^T, \\
\Phi_{33} &= -e^{-2\alpha h_1}Q_1, \quad \Phi_{44} = -e^{-2\alpha h_2}Q_2.
\end{align*}
\]
By using Proposition 2.3 again, from (3.5) we have \( \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ * & \Phi_{22} \end{bmatrix} < 0 \) and therefore, 
\[
\begin{bmatrix} \Gamma & P_{22}D_{22} \\ * & -(1 - \mu)e^{-2ah_2}Q_{22} \end{bmatrix} < 0,
\]
where, \( \Gamma = P_{22}A_{22} + A_{22}^TP_{22} + Q_{22} + Q_{122} + Q_{222} + d_2^2R_{122} + (d_2 - d_1)^2R_{222} \), which gives \( P_{22}A_{22} + A_{22}^TP_{22} < 0 \), and hence \( P_{22} \) and \( A_{22} \) are nonsingular matrices.

Let us denote 
\[
\tilde{M} = \begin{bmatrix} I & -A_{12}A_{22}^{-1} \\ 0 & I \end{bmatrix} M, \quad \tilde{N} = N \begin{bmatrix} I & 0 \\ -A_{22}^{-1}A_{21} & A_{22}^{-1} \end{bmatrix}.
\]

It is easy to verify that 
\[
\tilde{E} := \tilde{M}E\tilde{N} = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}, \quad \tilde{A} := \tilde{M}A\tilde{N} = \begin{bmatrix} \hat{A}_{11} & 0 \\ 0 & I_{n-r} \end{bmatrix},
\]
where, \( \hat{A}_{11} = A_{11} - A_{12}A_{22}^{-1}A_{21} \).

Therefore, 
\[
\det(sE - A) = \det(\tilde{M}^{-1}) \det(s\tilde{E} - \tilde{A}) \det(\tilde{N}^{-1}) = \det(\tilde{M}^{-1})(-1)^{(n-r)} \det(sI_r - \hat{A}_{11}) \det(\tilde{N}^{-1}),
\]
which implies that the polynomial \( \det(sE - A) \) is not identically zero and \( \deg \det(sE - A) = r = \text{rank}(E) \). Hence, the system (2.1) is regular and impulse-free.

Next, we prove that the system (2.1) is exponentially stable.

Let \( y(t) = N^{-1}x(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} \), where \( y_1(t) \in \mathbb{R}^r, y_2(t) \in \mathbb{R}^{n-r} \), system (2.1) is equivalent to the following system

\[
\begin{align*}
\dot{E}y(t) &= \tilde{A}y(t) + Dy(t - h(t)) + \int_{t - d(t)}^{t} \tilde{G}y(s)ds, \quad t \geq 0, \\
y(t) &= N^{-1}\varphi(t) := \psi(t), \quad t \in [-\tau, 0],
\end{align*}
\]

that is,

\[
\begin{align*}
\dot{y}_1(t) &= A_{11}y_1(t) + D_{11}y_1(t - h(t)) + D_{12}y_2(t - h(t)) \\
&\quad + \int_{t - d(t)}^{t} G_{11}y_1(s)ds + \int_{t - d(t)}^{t} G_{12}y_2(s)ds, \quad t \geq 0, \\
0 &= y_2(t) + D_{21}y_1(t - h(t)) + D_{22}y_2(t - h(t)) \\
&\quad + \int_{t - d(t)}^{t} G_{21}y_1(s)ds + \int_{t - d(t)}^{t} G_{22}y_2(s)ds, \\
y(t) &= \psi(t) := \begin{bmatrix} \psi_1(t) \\ \psi_2(t) \end{bmatrix}, \quad t \in [-\tau, 0],
\end{align*}
\]

(3.6)
where $\psi_1(t) \in \mathbb{R}^r, \psi_2(t) \in \mathbb{R}^{n-r}$.

Consider the following Lyapunov-Krasovskii functional

\[ V(t, y_t) = V_1 + V_2 + V_3 + V_4 + V_5 + V_6 + V_7 + V_8, \]

where

\[

\begin{align*}
V_1 &= y^T(t) \hat{P} \hat{E} y(t), \\
V_2 &= \int_{t-h_1}^t e^{2\alpha(s-t)} y^T(s) \hat{Q}_1 y(s) ds, \\
V_3 &= \int_{t-h_2}^t e^{2\alpha(s-t)} y^T(s) \hat{Q}_2 y(s) ds, \\
V_4 &= \int_{t-h(t)}^t e^{2\alpha(s-t)} y^T(s) \bar{Q} y(s) ds, \\
V_5 &= d_1 \int_{t-s}^t \int_{t-s}^t e^{2\alpha(u-t)} y^T(u) \bar{R}_1 y(u) duds, \\
V_6 &= (d_2 - d_1) \int_{t-s}^t \int_{t-s}^t e^{2\alpha(u-t)} y^T(u) \bar{R}_2 y(u) duds, \\
V_7 &= \int_{t-h_2}^t \int_{t+h(t)}^t e^{2\alpha(u-t)} y^T(u) \bar{E}^T \bar{W}_1 \bar{E} y(u) duds, \\
V_8 &= \int_{t-h_2}^t \int_{t+h(t)}^t e^{2\alpha(u-t)} y^T(u) \bar{E}^T \bar{W}_2 \bar{E} y(u) duds.
\end{align*}
\]

It is easy to verify that

\[ \lambda_1 \| y_1(t) \|^2 \leq V(t, y_t) \leq \lambda_2 \| y_t \|^2, \tag{3.8} \]

where $\lambda_1 = \lambda_{\text{min}}(P_{11})$ and

\[
\lambda_2 = \lambda_{\text{max}}(P_{11}) + h_1 \lambda_{\text{max}}(\bar{Q}_1) + h_2 \lambda_{\text{max}}(\bar{Q}_2) + h_2 \lambda_{\text{max}}(\tilde{Q}) + \frac{1}{2} d_1^2 \lambda_{\text{max}}(\bar{R}_1) + \frac{1}{2} (d_2 - d_1)^2 (d_2 + d_1) \lambda_{\text{max}}(\bar{R}_2) + \frac{1}{2} \left[ h_2^2 \lambda_{\text{max}}(\bar{W}_1) + (h_2^2 - h_1^2) \lambda_{\text{max}}(\bar{W}_2) \right].
\]

Taking derivative of $V_1$ in $t$ along the trajectory of the system, we have

\[
\dot{V}_1 = 2y^T(t) \hat{P} \hat{E} \dot{y}(t) = 2y^T(t) \hat{P} \left[ \dot{A} y(t) + \dot{D} y(t - h(t)) + G \int_{t-h(t)}^t y(s) ds \right] \\
= y^T(t) \left[ \hat{P} \bar{A} + \bar{A}^T \hat{P} + 2\alpha \hat{P} \hat{E} \right] y(t) + 2y^T(t) \hat{P} \bar{D} y(t - h(t)) + 2y^T(t) \hat{P} \bar{G} \int_{t-h(t)}^t y(s) ds - 2\alpha V_1. \tag{3.9}
\]
By using the following differentiation formula (see, \[17\]):

\[
\frac{d}{dt} \left[ \int_{-h}^{0} \int_{-h}^{t} f(z) dz ds \right] = hf(t) - \int_{-h}^{0} \dot{u}(t + s) f(u(t + s)) ds,
\]

the derivative of $V_k$, $k = 2, 3, \ldots, 8$, respectively, is given as follows

\[
\begin{align*}
\dot{V}_2 &= y^T(t) \vec{Q}_1 y(t) - e^{-2\alpha_h} y^T(t - h_1) \vec{Q}_1 y(t - h_1) - 2\alpha V_2; \\
\dot{V}_3 &= y^T(t) \vec{Q}_2 y(t) - e^{-2\alpha_h} y^T(t - h_2) \vec{Q}_2 y(t - h_2) - 2\alpha V_3; \\
\dot{V}_4 &= y^T(t) \vec{Q}_3 y(t) - (1 - \mu) e^{-2\alpha_h} y^T(t - h(t)) \vec{Q}_3 y(t - h(t)) - 2\alpha V_4 \\
&\leq y^T(t) \vec{Q}_4 y(t) - (1 - \mu) e^{-2\alpha_h} y^T(t - h(t)) \vec{Q}_4 y(t - h(t)) - 2\alpha V_4; \\
\dot{V}_5 &= d_1^2 y^T(t) \vec{R}_1 y(t) - d_1 \int_{-d_1}^{t} e^{2\alpha s} y^T(t + s) \vec{R}_1 y(t + s) ds - 2\alpha V_5 \\
&\leq d_1^2 y^T(t) \vec{R}_1 y(t) - d_1 e^{-2\alpha d_1} \int_{t-d_1}^{t} y^T(s) \vec{R}_1 y(s) ds - 2\alpha V_5 \\
&\leq d_1^2 y^T(t) \vec{R}_1 y(t) - e^{-2\alpha d_1} \left( \int_{t-d_1}^{t} y(s) ds \right)^T \vec{R}_1 \left( \int_{t-d_1}^{t} y(s) ds \right) - 2\alpha V_5; \\
\dot{V}_6 &= (d_2 - d_1)^2 y^T(t) \vec{R}_2 y(t) \\
&\quad - (d_2 - d_1) \int_{-d_2}^{-d_1} e^{2\alpha s} y^T(t + s) \vec{R}_2 y(t + s) ds - 2\alpha V_6 \\
&\leq (d_2 - d_1)^2 y^T(t) \vec{R}_2 y(t) \\
&\quad - (d_2 - d_1) e^{-2\alpha d_2} \int_{t-d_2}^{t-d_1} y^T(s) \vec{R}_2 y(s) ds - 2\alpha V_6 \\
&\leq (d_2 - d_1)^2 y^T(t) \vec{R}_2 y(t) \\
&\quad - e^{-2\alpha d_2} \left( \int_{t-d_2}^{t-d_1} y(s) ds \right)^T \vec{R}_2 \left( \int_{t-d_2}^{t-d_1} y(s) ds \right) - 2\alpha V_6 \\
&\leq (d_2 - d_1)^2 y^T(t) \vec{R}_2 y(t) - 2\alpha V_6 \\
&\quad - e^{-2\alpha d_2} \left[ \int_{t-d_2}^{t-d_1} y(s) ds - \int_{t-d_1}^{t} y(s) ds \right]^T \vec{R}_2 \left[ \int_{t-d_2}^{t-d_1} y(s) ds - \int_{t-d_1}^{t} y(s) ds \right]; \\
\dot{V}_7 &= h_2 y^T(t) \vec{E}^T \vec{W}_1 \vec{E} \dot{y}(t) - \int_{t-h_2}^{t} e^{2\alpha(s-t)} \dot{y}^T(s) \vec{E}^T \vec{W}_1 \vec{E} \dot{y}(s) ds - 2\alpha V_7, \\
\dot{V}_8 &= (h_3 - h_1) y^T(t) \vec{E}^T \vec{W}_2 \vec{E} \dot{y}(t) - \int_{t-h_3}^{t} e^{2\alpha(s-t)} \dot{y}^T(s) \vec{E}^T \vec{W}_2 \vec{E} \dot{y}(s) ds - 2\alpha V_8.
\end{align*}
\]

On the other hand, by partitioning the integral intervals and using Newton-Leibniz formula,
we have the following identities

\[ - \int_{t-h_2}^{t} e^{2\alpha(s-t)} y^T(s) \bar{E}^T \bar{W}_1 \bar{E} \dot{y}(s) ds = - \int_{t-h_2}^{t-h(t)} e^{2\alpha(s-t)} y^T(s) \bar{E}^T \bar{W}_1 \bar{E} \dot{y}(s) ds \]

\[ - \int_{t-h(t)}^{t} e^{2\alpha(s-t)} y^T(s) \bar{E}^T \bar{W}_1 \bar{E} \dot{y}(s) ds, \]

\[ - \int_{t-h_2}^{t-h_1} e^{2\alpha(s-t)} y^T(s) \bar{E}^T \bar{W}_2 \bar{E} \dot{y}(s) ds = - \int_{t-h_2}^{t-h(t)} e^{2\alpha(s-t)} y^T(s) \bar{E}^T \bar{W}_2 \bar{E} \dot{y}(s) ds \]

\[ - \int_{t-h(t)}^{t-h_1} e^{2\alpha(s-t)} y^T(s) \bar{E}^T \bar{W}_2 \bar{E} \dot{y}(s) ds, \]

and

\[ 2 \left[ y^T(t) \bar{X}_1 + y^T(t-h(t)) \bar{X}_2 \right] \left[ \bar{E} \dot{y}(t) - \bar{E} \dot{y}(t-h(t)) - \int_{t-h(t)}^{t} \bar{E} \dot{y}(s) ds \right] = 0, \]

\[ 2 \left[ y^T(t) \bar{Y}_1 + y^T(t-h(t)) \bar{Y}_2 \right] \left[ \bar{E} \dot{y}(t-h(t)) - \bar{E} \dot{y}(t-h_2) - \int_{t-h_2}^{t-h(t)} \bar{E} \dot{y}(s) ds \right] = 0, \]

\[ 2 \left[ y^T(t) \bar{Z}_1 + y^T(t-h(t)) \bar{Z}_2 \right] \left[ \bar{E} \dot{y}(t-h_1) - \bar{E} \dot{y}(t-h(t)) - \int_{t-h(t)}^{t-h_1} \bar{E} \dot{y}(s) ds \right] = 0. \]

Combining equations (3.9) to (3.15), we obtain

\[ \dot{V}(t, y_t) + 2\alpha V(t, y_t) \leq \eta^T(t) \bar{F} \eta(t) + h_2 \bar{y}^T(t) \bar{E}^T \bar{W}_1 \bar{E} \dot{y}(t) + (h_2 - h_1) \bar{y}^T(t) \bar{E}^T \bar{W}_2 \bar{E} \dot{y}(t) \\
- \int_{t-h(t)}^{t} e^{2\alpha(s-t)} y^T(s) \bar{E}^T \bar{W}_1 \bar{E} \dot{y}(s) ds \\
- \int_{t-h_2}^{t-h(t)} e^{2\alpha(s-t)} y^T(s) \bar{E}^T \bar{W}_2 \bar{E} \dot{y}(s) ds \\
- 2 \left[ y^T(t) \bar{X}_1 + y^T(t-h(t)) \bar{X}_2 \right] \int_{t-h(t)}^{t} \bar{E} \dot{y}(s) ds \\
- \int_{t-h_2}^{t-h(t)} e^{2\alpha(s-t)} y^T(s) \bar{E}^T (\bar{W}_1 + \bar{W}_2) \bar{E} \dot{y}(s) ds \\
- 2 \left[ y^T(t) \bar{Y}_1 + y^T(t-h(t)) \bar{Y}_2 \right] \int_{t-h_2}^{t-h(t)} \bar{E} \dot{y}(s) ds \\
- \int_{t-h(t)}^{t-h_1} e^{2\alpha(s-t)} y^T(s) \bar{E}^T \bar{W}_2 \bar{E} \dot{y}(s) ds \\
- 2 \left[ y^T(t) \bar{Z}_1 + y^T(t-h(t)) \bar{Z}_2 \right] \int_{t-h(t)}^{t-h_1} \bar{E} \dot{y}(s) ds \]
where, \( \bar{\Phi} \) are defined in (3.5) and
\[
\bar{\Phi}_{55} = -e^{-2\alpha d_2} \bar{R}_2, \quad \bar{\Phi}_{56} = e^{2\alpha d_2} \bar{R}_2, \quad \bar{\Phi}_{66} = -e^{-2\alpha d_1} \bar{R}_2 - e^{-2\alpha d_1} \bar{R}_1.
\]
This proves that the slow variables, i.e. the first \( r \) components of the system state \( \eta(t) \), is exponentially stable.

Next, we will prove the fast variables, i.e. the remain components of the system state \( y(t) \) is exponentially stable with the same exponential decay rate \( \alpha \). Let us denote

\[
p(t) = D_{21} y_1(t - h(t)) + G_{21} \int_{t - d(t)}^{t} y_1(s) ds.
\]
Observe that, for \( t \geq 0 \), if \( t - h(t) \geq 0 \) then
\[
\|y_1(t - h(t))\|^2 \leq \frac{\lambda_2}{\lambda_1} \|\psi\|^2 e^{-2\alpha(t-h(t))} \leq \frac{\lambda_2}{\lambda_1} \|\psi\|^2 e^{2\alpha h_2 e^{-2\alpha t}},
\]
only otherwise, \( \|y_1(t - h(t))\|^2 \leq \|\psi\|^2 \leq \|\psi\|^2 e^{-2\alpha(t-h(t))} \leq \|\psi\|^2 e^{2\alpha h_2 e^{-2\alpha t}}, \) and therefore,
\[
\|y_1(t - h(t))\| \leq \nu_1 e^{\alpha h_2 \|\psi\|e^{-\alpha t}}, \quad \forall t \geq 0.
\]
By the same arguments, it is easy to check that
\[
\|y_1(s)\| \leq \nu_1 \|\psi\| e^{-\alpha s}, \quad \forall s \in [t - d(t), t], \ t \geq 0.
\]
Substituting the above estimations into vector \( p(t) \), we obtain
\[
\|p(t)\| \leq \|D_{21}\| \|y_1(t - h(t))\| + \|G_{21}\| \int_{t-d(t)}^t \|y_1(s)\| ds
\]
\[
\leq \left( \sigma_1 e^{\alpha h_2} e^{-\alpha t} + \sigma_2 \int_{t-d(t)}^t e^{-\alpha s} ds \right) \nu_1 \|\psi\|_\tau,
\]
\[
\leq \left[ \sigma_1 e^{\alpha h_2} + \frac{\sigma_2}{\alpha} (e^{\alpha d_2} - 1) \right] \nu_1 \|\psi\|_\tau e^{-\alpha t} \leq \gamma_p \|\psi\|_\tau e^{-\alpha t}, \quad t \geq 0,
\]
where \( \sigma_1 = \|D_{21}\| \), \( \sigma_2 = \|G_{21}\| \) and \( \gamma_p = \nu_1 \left[ \sigma_1 e^{\alpha h_2} + \frac{\sigma_2}{\alpha} (e^{\alpha d_2} - 1) \right] \).

From (3.7), we have
\[
0 = y_2(t) + D_{22} y_2(t - h(t)) + G_{22} \int_{t-d(t)}^t y_2(s) ds + p(t),
\]
thus,
\[
2y_2^T(t) P_{22} \left[ y_2(t) + D_{22} y_2(t - h(t)) + G_{22} \int_{t-d(t)}^t y_2(s) ds + p(t) \right] = 0. \tag{3.19}
\]
Consider the following functional
\[
J(t) := (1 - \mu) y_2^T(t) Q_{22} y_2(t) - (1 - \mu) e^{-2\alpha h_2} y_2^T(t - h(t)) Q_{22} y_2(t - h(t))
\]
\[
- e^{-2\alpha d_2} d(t) \int_{t-d(t)}^t y_2^T(s) R_{222} y_2(s) ds. \tag{3.20}
\]
Adding (3.19) into (3.20) and re-arranging obtained terms, we have
\[
J(t) = y_2^T(t) \left[ P_{22} + P_{22}^T + (1 - \mu) Q_{22} \right] y_2(t)
\]
\[
+ \int_{t-d(t)}^t \left[ 2y_2^T(t) P_{22} G_{22} y_2(s) - d(t) e^{-2\alpha d_2} y_2^T(s) R_{222} y_2(s) \right] ds
\]
\[
- (1 - \mu) e^{-2\alpha h_2} y_2^T(t - h(t)) Q_{22} y_2(t - h(t))
\]
\[
+ 2y_2^T(t) P_{22} D_{22} y_2(t - h(t)) + 2y_2^T(t) P_{22} p(t).
\]
By applying proposition 1, with note that \( Q_{122} > 0 \), we have

\[
\int_{t-d(t)}^{t} 2y_{2}^T(t)P_{22}G_{22}y_{2}(s)ds \leq e^{2\alpha d_{2}}y_{2}^T(t)P_{22}G_{22}R_{222}^{-1}G_{22}^T P_{22}^{T}y_{2}(t) + e^{-2\alpha d_{2}}d(t) \int_{t-d(t)}^{t} y_{2}^T(s)R_{222}y_{2}(s)ds,
\]

\[
2y_{2}^T(t)P_{22}p(t) \leq y_{2}^T(t)Q_{122}y_{2}(t) + p^T(t)P_{22}^{T}Q_{122}^{-1}P_{22}p(t).
\]

Therefore,

\[
J(t) \leq \chi^T(t) \begin{bmatrix} J_{1} & P_{22}D_{22} \\ * & -(1-\mu)e^{-2\alpha h_{2}Q_{22}} \end{bmatrix} \chi(t) + y_{2}^T(t)Q_{122}y_{2}(t) + p^T(t)P_{22}^{T}Q_{122}^{-1}P_{22}p(t), \quad (3.21)
\]

where, \( \chi^T(t) = [y_{2}^T(t) \ y_{2}^T(t - h(t))] \) and

\[
J_{1} = P_{22} + P_{22}^{T} + (1-\mu)Q_{22} + e^{2\alpha d_{2}}P_{22}G_{22}R_{222}^{-1}G_{22}^T P_{22}^{T}.
\]

On the other hand, it follows from the fact that matrix

\[
\Phi = \begin{bmatrix} \Phi_{11} & \Phi_{12} & \bar{Z}_{1}\bar{E} & -\bar{Y}_{1}\bar{E} & \bar{P}\bar{G} & 0 \\ * & \Phi_{22} & \bar{Z}_{2}\bar{E} & -\bar{Y}_{2}\bar{E} & 0 & 0 \\ * & * & \Phi_{33} & 0 & 0 & 0 \\ * & * & * & \Phi_{44} & 0 & 0 \\ * & * & * & * & \Phi_{55} & \Phi_{56} \\ * & * & * & * & * & \Phi_{66} \end{bmatrix}
\]

then we have

\[
\begin{bmatrix} \Phi_{11} & \Phi_{12} & \bar{P}\bar{G} \\ * & \Phi_{22} & 0 \\ * & * & \Phi_{55} \end{bmatrix} < 0,
\]

and hence,

\[
\begin{bmatrix} J_{2} & P_{22}D_{22} \\ * & -(1-\mu)e^{-2\alpha h_{2}Q_{22}} \end{bmatrix} P_{22}G_{22}^{-1} \begin{bmatrix} P_{22}G_{22} \\ 0 \\ -e^{-2\alpha d_{2}}R_{222} \end{bmatrix} < 0,
\]

where, \( J_{2} = P_{22} + P_{22}^{T} + (1-\mu)Q_{22} + Q_{122} + Q_{222} + d_{1}^{2}R_{122} + (d_{2} - d_{1})^{2}R_{222} \).

By Schur complement lemma (Proposition 2.3), we have

\[
\begin{bmatrix} J_{2} + e^{2\alpha d_{2}}P_{22}G_{22}R_{222}^{-1}G_{22}^T P_{22}^{T} & P_{22}D_{22} \\ * & -(1-\mu)e^{-2\alpha h_{2}Q_{22}} \end{bmatrix} < 0.
\]

Let us denote \( \Delta(d_{1}, d_{2}) = d_{1}^{2}R_{122} + (d_{2} - d_{1})^{2}R_{222} \), then from (3.21), we have

\[
J(t) \leq -y_{2}^T(t) [Q_{222} + \Delta(d_{1}, d_{2})] y_{2}(t) + p^T(t)P_{22}^{T}Q_{122}^{-1}P_{22}p(t).
\]

(3.22)
Combining equations (3.20) and (3.22) we obtain
\[
y_2^T(t) \left[ (1 - \mu)Q_{22} + Q_{222} + \Delta(d_1, d_2) \right] y_2(t)
\leq (1 - \mu)e^{-2\alpha h_2} y_2^T(t - h(t)) Q_{222} y_2(t - h(t))
+ d_2e^{-2\alpha d_2} \int_{t-d(t)}^{t} y_2^T(s) R_{222} y_2(s) ds
+ p^T(t)P_{22}^T Q_{1222}^{-1} P_{22} p(t).
\] (3.23)

From (3.3), by pre- and post-multiplying with \(N^T, N\), respectively, and due to Schur complement lemma, we have \(d_2^2 e^{-2\alpha d_2} R_{222} < (1 - \mu)e^{-2\alpha \tau} Q_{222}\), and \(\tau_{o,h} Q_{22} < Q_{222}\). Combining with (3.23) we have
\[
(1 - \mu + \tau_{o,h}) y_2^T(t) Q_{222} y_2(t) \leq (1 - \mu) \left( e^{-2\alpha h_2} + e^{-2\alpha \tau} \right) \sup_{-\tau \leq s \leq 0} y_2^T(t + s) Q_{222} y_2(t + s)
+ p^T(t)P_{22}^T Q_{1222}^{-1} P_{22} p(t).
\]

Let \(f(t) = y_2^T(t) Q_{222} y_2(t), t \geq -\tau\), then we have
\[
f(t) \leq \delta_1 \sup_{-\tau \leq s \leq 0} f(t + s) + \delta_2 e^{-2\alpha t}, \quad t \geq 0,
\] (3.24)

where, \(\delta_1 = \frac{(1 - \mu) \left( e^{-2\alpha h_2} + e^{-2\alpha \tau} \right)}{1 - \mu + \tau_{o,h}}\), and \(\delta_2 = \frac{\lambda_{\text{max}}(P_{22}^T P_{22}) \gamma_p^2}{\lambda_{\text{min}}(Q_{122}) (1 - \mu + \tau_{o,h})} \|\psi\|_{2}^2\).

Note that, \(\delta_1 e^{2\alpha \tau} = \frac{1 + e^{2\alpha (\tau - h_2)}}{2 + e^{2\alpha (\tau - h_2)}} < 1\), thus, by Proposition 2.5, from (3.24) we have
\[
f(t) \leq \left[ \delta_1 e^{2\alpha \tau} \sup_{-\tau \leq s \leq 0} f(s) + \frac{\delta_2}{1 - \delta_1 e^{2\alpha \tau}} \right] e^{-2\alpha t}, \quad t \geq 0.
\]

Moreover, by some simple computations we have
\[
\sup_{-\tau \leq s \leq 0} f(s) \leq \lambda_{\text{max}}(Q_{22}) \|\psi\|_{2}^2, \quad \frac{\delta_2}{1 - \delta_1 e^{2\alpha \tau}} = \frac{\lambda_{\text{max}}(P_{22}^T P_{22}) \gamma_p^2}{(1 - \mu) \lambda_{\text{min}}(Q_{122})} \|\psi\|_{2}^2,
\]
and
\[
\lambda_{\text{min}}(Q_{22}) \|y_2(t)\|^2 \leq f(t)
\leq \left[ \frac{(1 + e^{2\alpha (\tau - h_2)}) \lambda_{\text{max}}(Q_{22})}{2 + e^{2\alpha (\tau - h_2)}} + \frac{\lambda_{\text{max}}(P_{22}^T P_{22}) \gamma_p^2}{(1 - \mu) \lambda_{\text{min}}(Q_{122})} \right] \|\psi\|_{2}^2 e^{-2\alpha t}, \quad t \geq 0.
\]

Therefore,
\[
\|y_2(t)\| \leq \nu_2 \|\psi\|_{2} e^{-\alpha t}, \quad t \geq 0,
\]
where,
\[

nu_2 = \left[ \frac{(1 + e^{2\alpha (\tau - h_2)}) \lambda_{\text{max}}(Q_{22})}{2 + e^{2\alpha (\tau - h_2)}} + \frac{\lambda_{\text{max}}(P_{22}^T P_{22}) \gamma_p^2}{(1 - \mu) \lambda_{\text{min}}(Q_{122}) \lambda_{\text{min}}(Q_{22})} \right].
\]
Finally, turning back to the state $x(t) = Ny(t)$, we get
\[
\|x(t, \phi)\| \leq \|N\| \sqrt{\|y_1(t)\|^2 + \|y_2(t)\|^2} \leq \nu \|\phi\| e^{-\alpha t}, \quad t \geq 0,
\]
where, $\nu = \|N\|\|N^{-1}\|\sqrt{\nu_1^2 + \nu_2^2}$. This completes the proof. \hfill \square

**Remark 3.1.** Generally speaking, strict LMI conditions are more desirable than non-strict ones. For tackling this, conditions (3.1) and (3.2) can be combined into a single strict LMI. Let $P$ be a symmetric positive definite matrix, $S \in \mathbb{R}^{n \times (n-\tau)}$ be any full column rank matrix such that $E^T S = 0$ and $\Theta$ be any matrix with appropriate dimension, denote $P = (PE + S \Theta)^T$ then $PE = E^T P^T = E^T PE \geq 0$. Thus, by changing $P$ to $P = (PE + S \Theta)^T$ in (3.2) yields a strict LMI.

**Remark 3.2.** In Wu et al. [27], the problem of exponential stability and stabilization for a class of descriptor systems with constant discrete and distributed delays is considered. The authors use Lyapunov functional method to get the exponential stability of the slow subsystem and homotopy-based approach to investigate the exponential stability of the fast subsystem. However, the exponential convergence rate $\alpha$ is defined by matrix functional inequality, which the convergence rate can not be estimated explicitly for such stable system.

### 4. Numerical examples

In this section, we use a numerical example to demonstrate the obtained results.

**Example 4.1.** Consider system (2.1) with system matrices
\[
E = \begin{bmatrix} -1 & 2 \\ -2 & 4 \end{bmatrix}, \quad A = \begin{bmatrix} -4 & -1 \\ 1 & -2 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix}, \quad G = \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}.
\]

Delay functions $h(t) = 0.3 + 0.2 \sin(2.5t)$ and $d(t) = 0.425 + 0.325 C(t)$, where $C(t)$ is a Cellérier function, $C(t) = \sum_{k=1}^{\infty} \frac{1}{2^k} \sin (2^k t)$.

It is well known that, $C(t)$ be a continuous but nowhere differentiable function. Therefore, the stability criteria in [7, 8, 15, 26, 28, 31] are not applicable to this system. We have $h_1 = 0.1, h_2 = 0.5, d_1 = 0.1, d_2 = 0.75, \mu = 0.5$ and $\tau = \max \{h_2, d_2\} = 0.75$. Let $\alpha = 1$, by using LMI toolbox of Matlab, inequalities (3.1)-(3.3) are feasible with

\[
P = \begin{bmatrix} 1.5197 & -0.8329 \\ 0.8583 & -0.2831 \end{bmatrix}, \quad Q = \begin{bmatrix} 2.1100 & 0.3373 \\ 0.3373 & 1.5839 \end{bmatrix},
\]

\[
Q_1 = \begin{bmatrix} 1.5322 & -0.1665 \\ -0.1665 & 1.6227 \end{bmatrix}, \quad Q_2 = \begin{bmatrix} 3.6576 & 0.0152 \\ 0.0152 & 3.4802 \end{bmatrix},
\]

\[
R_1 = \begin{bmatrix} 6.5777 & 2.1182 \\ 2.1182 & 4.7878 \end{bmatrix}, \quad R_2 = \begin{bmatrix} 1.1922 & 0.4093 \\ 0.4093 & 0.8011 \end{bmatrix},
\]

\[
W_1 = \begin{bmatrix} 0.6159 & 0.4791 \\ 0.4791 & 1.3009 \end{bmatrix}, \quad W_2 = \begin{bmatrix} 0.5965 & 0.2427 \\ 0.2427 & 0.9386 \end{bmatrix}.
\]
\[
X_1 = \begin{bmatrix}
0.4222 & 0.8278 \\
-0.6725 & -1.3183
\end{bmatrix}, \quad X_2 = \begin{bmatrix}
-0.1087 & -0.2136 \\
0.1135 & 0.2233
\end{bmatrix}, \\
Y_1 = \begin{bmatrix}
0.1281 & 0.2487 \\
-0.3070 & -0.5996
\end{bmatrix}, \quad Y_2 = \begin{bmatrix}
0.1780 & 0.3484 \\
-0.3285 & -0.6425
\end{bmatrix}, \\
Z_1 = \begin{bmatrix}
-0.0310 & -0.0606 \\
0.0729 & 0.1426
\end{bmatrix}, \quad Z_2 = \begin{bmatrix}
-0.0427 & -0.0837 \\
0.0792 & 0.1552
\end{bmatrix}.
\]

We choose two nonsingular matrices
\[
M = \begin{bmatrix}
-1 & 1 \\
-2 & 1
\end{bmatrix}, \quad N = \begin{bmatrix}
3 & 2 \\
2 & 1
\end{bmatrix}
\]
such that \( \bar{E} = MEN = \begin{bmatrix}
1 & 0 \\
0 & 0
\end{bmatrix} \). Taking some computations as in the proof of theorem 3.1 we obtain
\[
\lambda_1 = 0.146, \quad \lambda_2 = 62.4414, \quad \nu_1 = 20.6804, \quad \gamma_p = 587.3900, \quad \nu_2 = 180.3492.
\]
By theorem 3.1, system (2.1) is exponentially admissible with \( \alpha = 1 \). Moreover, every solution \( x(t, \varphi) \) of the system satisfies
\[
\|x(t, \varphi)\| \leq 3.2574e + 0.003\|\varphi\|e^{-t}, \quad t \geq 0.
\]

5. Conclusions

In this paper, the problem of exponential stability for a class of linear singular systems with mixed interval time-varying delays is considered. By using an improved Lyapunov-Krasovskii functional, new delay-range-dependent conditions are established in terms of linear matrix inequalities (LMIs) which guarantee that the system is regular, impulse-free and exponentially stable. This allows us to compute simultaneously the two bounds that characterize the exponential stability rate of the solution by various efficient convex optimization algorithms.

A numerical example is given to show the effectiveness of the obtained results.

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