26

Representation of Evidence by Hints

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Abstract. This paper introduces a mathematical model of a hint as a body of imprecise and uncertain information. Hints are used to judge hypotheses: the degree to which a hint supports a hypothesis and the degree to which a hypothesis appears as plausible in the light of a hint are defined. This leads in turn to support- and plausibility functions. Those functions are characterized as set functions which are normalized and monotone or alternating of order ∞. This relates the present work to G. Shafer's mathematical theory of evidence. However, whereas Shafer starts out with an axiomatic definition of belief functions, the notion of a hint is considered here as the basic element of the theory. It is shown that a hint contains more information than is conveyed by its support function alone. Also hints allow for a straightforward and logical derivation of Dempster's rule for combining independent and dependent bodies of information. This paper presents the mathematical theory of evidence for general, infinite frames of discernment from the point of view of a theory of hints.

Key words: Hints, Evidence, Support functions, Plausibility functions, Dempster's rule

1 Hints—An Intuitive Introduction

Intuitively, a hint is a body of information relative to some question which is in general **imprecise** in that it does not point to a precise answer but rather to a range of possible answers. It is also often **uncertain** in the sense that the information allows for several possible interpretations and it is not entirely sure which is the correct one. There may be internal conflict within a hint because different interpretations may lead to contradictory answers. Also there can be external contradictions between distinct and different hints relative to the same question. The goal of this paper is to develop a mathematical model of this intuitive notion of a hint and to study some of its basic properties. It takes as its starting point A. Dempster's (1967) multivalued mapping and develops into similar lines as G. Shafer's (1976) mathematical theory of evidence. The theory will however be developed for the most general case and not be limited to the case of finite frames as in Shafer's book.

For an introduction and as a motivation the simpler case of finite hints will first be discussed. Let Θ be an arbitrary finite set whose elements θ represent the possible answers to a given question which has to be considered. One of the elements of Θ represents the true, but unknown answer. Θ is called the **frame of discernment**. The subsets of Θ represent possible propositions about the answer to the question considered. Let Ω denote the finite **set of possible interpretations** of the information contained in the hint to be represented. One of the elements $\omega \in \Omega$ must be the **correct** interpretation, but it is unknown which one. However, not all possible interpretations are equally likely. Thus, a probability $p(\omega)$ for the interpretations $\omega \in \Omega$ is introduced.

Each possible interpretation ω restricts the possible answers within Θ somehow. If ω is the correct interpretation, then the correct answer θ is known to be within some nonempty subset $\Gamma(\omega)$ of Θ, the **focal set** of the interpretation. Alternatively, for any possible interpretation ω, the family $\mathcal{S}$ of the propositions (subsets of Θ) **implied** by the interpretation ω can be considered. $\mathcal{S}$ is simply the family of supersets of the focal set $\Gamma(\omega)$. It has thus trivially the following properties:

(1) $H \in \mathcal{S}$ and $H \subseteq H'$ imply $H' \in \mathcal{S}$
(2) $H_1 \in \mathcal{S},\ H_2 \in \mathcal{S}$ imply $H_1 \cap H_2 \in \mathcal{S}$.
(3) Θ belongs to $\mathcal{S}$, $\emptyset$ does not belong to $\mathcal{S}$.

In addition, the intersection of all implied sets of an interpretation equals $\Gamma(\omega)$. Furthermore, for any possible interpretation, one can also look at the family $\mathcal{P}$ of propositions which are **possible** under the interpretation. A subset $H \subseteq \Theta$ is possible, when H intersects the focal set $\varGamma(\omega)$ of the interpretation. Equivalently, H is possible, iff its complement is not implied, $H^c \notin \mathcal{S}$. $\mathcal{P}$ has the following properties:

(1′) $H \in \mathcal{P}$ and $H \subseteq H'$ imply $H' \in \mathcal{P}$
(2′) $H_1 \in \mathcal{P},\ H_2 \in \mathcal{P}$ imply $H_1 \cup H_2 \in \mathcal{P}$.
(3′) Θ belongs to $\mathcal{P}, \emptyset$ does not belong to $\mathcal{P}$.

Furthermore, if $H \in \mathcal{S}$, then $H^c \notin \mathcal{S}$ and thus $\mathcal{S} \subseteq \mathcal{P}$.

A **hint** is thus defined by a frame of discernment Θ to which it refers, a set of possible interpretations Ω together with a probability $p(\omega)$ and finally a multivalued mapping Γ from the set of interpretations into the frame Θ. If the interpretation ω happens to be the correct one, then the answer to the question considered is restricted to the set $\Gamma(\omega)$. So far, any hint $\mathcal{H}$ is a quadruple $(\Omega, p, \Gamma, \Theta)$.

If a proposition $H \subseteq \Theta$ is fixed as a hypothesis about the correct answer, then it will be interesting to judge this hypothesis in the light of a hint $\mathcal{H}$. Let $\mathcal{S}(\omega)$ and $\mathcal{P}(\omega)$ denote the families of implied and possible propositions of an interpretation ω. Then one can look at the subsets of interpretations under which H is **implied**, $u(H)$, or **possible**, $v(H)$

$$
\begin{aligned}
u(H) &= \{\omega \in \Omega : H \in \mathcal{S}(\omega)\} \\
v(H) &= \{\omega \in \Omega : H \in \mathcal{P}(\omega)\}
\end{aligned} \tag{1}
$$

A hypothesis H, which is implied or supported by many possible interpretations, or more important, by very probable interpretations, is very **credible** in the light of the hint. Also, if the hypothesis is possible under many interpretations, or under very probable interpretations, then the hypothesis is very **plausible** in the light of the hint. Thus, in order to measure the **degree of credibility** or **support** $sp(H)$ and the **degree of plausibility** $Pl(H)$, the probabilities of $u(H)$ and $v(H)$ can be considered:

$$
\begin{aligned}
sp(H) &= P(u(H)) \\
pl(H) &= P(v(H)).
\end{aligned} \tag{2}
$$

The values $sp(H)$ and $Pl(H)$ are defined for all subsets of Θ. sp is called a **support** (or belief) **function** and pl a **plausibility function** (or upper probability). These concepts were introduced by A. Dempster (1967) and extensively studied by Shafer (1976) for finite frames of discernment.

The goal of this contribution is to study hints with respect to arbitrary, especially infinite frames. To the best of our knowledge, only very few papers study evidence theory in this general case (Goodman, Nguyen, 1985; Nguyen, 1978; Shafer, 1979; Strat, 1984) The case of belief functions on infinite frames of discernment was in particular studied by Shafer (1979). In this paper belief functions are axiomatically defined as Choquet capacities, monotone of order ∞. Using an integral representation theorem of Choquet (1953, 1969) an **allocation of probability** for belief functions is derived. This concept provides for an interpretation of the meaning of belief. However, with this interpretation, the definition of Dempster's rule for the combination of belief functions is less straightforward. In an unpublished paper G. Shafer (1978) defines first product belief functions on a product space $\Theta \times \Theta$ and then Dempster's rule as a conditioning of the product belief function to the diagonal of $\Theta \times \Theta$. This seems somehow to be a detour. Hints on the other hand allow for a straightforward and logical derivation of Dempster's rule for combining independent and also dependent bodies of information.

Furthermore and more importantly, it will be seen that in the general case a hint contains more information than is conveyed by its support function alone. Therefore, hints cannot be combined on the base of their support functions alone as proposed in Shafer's paper (1978)! This would result in a loss of information. This will be one of the main results of this paper. Another main result is that support- and plausibility functions as defined by (2) can be characterized as Choquet capacities, monotone of order ∞. The proof of this result rejoins Shafer's (1979) development and will only be sketched here. Finally, a new inclusion relation between hints will be introduced in this paper which generalizes a similar relation between support functions introduced by Yager (1985, see also Dubois, Prade, 1986).

In Sect. 2 the general mathematical concept of a hint will be defined. In Sect. 3 support- and plausibility functions will be introduced. A process of refining hints is presented in Sect. 4. It leads to a relation of inclusion between hints. Section 5 studies inclusion relations between hints which are equivalent in the sense that they define partially the same support- and plausibility functions. Finally, in Sect. 6, the combination of hints will be discussed and Dempster's rule derived. In particular, it will be shown that inclusion of hints is maintained under Dempster's rule. The results of this section show that Dempster's rule cannot be defined in terms of support functions only.

2 The Mathematical Model of Hints

The frame of discernment Θ is now an arbitrary set and in particular it can be infinite. The set of possible interpretations Ω can then also be arbitrary. However, Ω will be a probability space $(\Omega, \mathcal{A},\ P)$ with a σ-algebra $\mathcal{A}$ and a probability measure P on it. As before (Sect. 1) any possible interpretation $\omega \in \Omega$ restricts the possible answers in Θ somehow. It will be assumed here that to any $\omega \in \Omega$ a family $\mathcal{S}(\omega)$ of implied propositions $H \subseteq \Theta$, satisfying conditions (1) to (3) of Sect. 1, is assigned. A family of subsets satisfying conditions (1) to (3) of Sect. 1 is called a **filter**. The family $\mathcal{P}(\omega) = \{H \subseteq \Theta : H^c \notin \mathcal{S}(\omega)\}$ of possible propositions satisfies conditions (1′) to (3′) of Sect. 1 above. A pair of such dual families $\mathcal{R} = (\mathcal{S},\ \mathcal{P})$ will be called a **restriction**.

A restriction $\mathcal{R}$ is called **vacuous**, if $\mathcal{S}$ contains only Θ (and $\mathcal{P}$ all subsets of Θ except the empty set). A vacuous restriction does not restrict at all the possible answers. It is used to represent the situation that, under some interpretations, a hint contains possibly no information at all concerning the question considered.

The set $R = \cap\{H : H \in \mathcal{S}\}$ is called the **base** of the restriction $\mathcal{R}$. One might wonder, whether $\mathcal{S}$ should not be closed under arbitrary intersections and thus $R \in \mathcal{S}$. This will not be assumed here — for reasons which become clear later. However, a restriction $\mathcal{R}$ with $R \in \mathcal{S}$ will be called **set-based**, because in this case $\mathcal{S} = \{H \subseteq \Theta : R \subseteq H\}$ and $\mathcal{P} = \{H \subseteq \Theta : R \cap H \neq \emptyset\}$. For a set-based restriction we write $\mathcal{R} = R$. Similarly, if (2) and (2′) of Sect. 1 hold for countable families, the restriction will be called a **σ-restriction**.

To go back to the model of a hint, it will thus be assumed, that every possible interpretation $\omega \in \Omega$ has assigned a **nonempty** restriction $\Gamma(\omega) = (\mathcal{S}(\omega), \mathcal{P}(\omega))$ describing its implied and possible propositions. Γ is a mapping from Ω into the set $\mathcal{R}(\Theta)$ of restrictions on Θ. This is a generalization of the multivalued mappings considered by A. Dempster (1967). A **hint** $\mathcal{H}$ is thus finally a quintuple $\mathcal{H} = (\Omega, \mathcal{A}, P, \Gamma, \Theta)$ of elements as described above.

A hint $\mathcal{H} = (\Omega, \mathcal{A}, P, \Gamma, \Theta)$ is called **set-focussed**, iff its restrictions $\Gamma(\omega)$ are set-based for all $\omega \in \Omega$. The bases of $\Gamma(\omega)$ are then called **focal sets**. If Θ is a finite set, then all restrictions and thus all hints are set-based. But even in the general case many important classes of hints are set-focussed. For all $\omega \in \Omega$, if $\Gamma(\omega)$ is either a fixed set-based restriction R or the vacuous restriction, then the hint is called **simple**. If $\Gamma(\omega)$ equals the vacuous restriction for all $\omega \in \Omega$, then the hint is called **vacuous**; it represents full ignorance about the question at hand. If $\mathcal{H}$ is a set-focussed hint whose focal sets $\Gamma(\omega)$ all contain only one single point $\theta(\omega)$ of Θ, then the hint is called **precise**. A precise hint corresponds essentially to a random variable (under reserve of the appropriate measurability condition).

Restrictions are fundamental to the theory. In many respects they behave like ordinary subsets of Θ. Especially the operation of **intersection** or **conjunction** can be defined: If $\mathcal{R}_1$ and $\mathcal{R}_2$ are two restrictions known to hold on Θ, then their conjunction forms a new restriction $\mathcal{R} = \mathcal{R}_1 \cap \mathcal{R}_2$ defined by $\mathcal{S} = \{H_1 \cap H_2 : H_1 \in \mathcal{S}_1, H_2 \in \mathcal{S}_2\}$. It is easily verified, that $\mathcal{S}$ is a filter if $\emptyset$ does not belong to $\mathcal{S}$. If $\emptyset \in \mathcal{S}$, then $\mathcal{R}_1$ and $\mathcal{R}_2$ are called **contradictory**. If $\mathcal{R}_1 = R_1$ and $\mathcal{R}_2 = R_2$, then $\mathcal{R}_1 \cap \mathcal{R}_2 = R_1 \cap R_2$. In the same way, the intersection is defined for arbitrary families of restrictions, not only for finite ones.

In order to judge hypotheses $H \subseteq \Theta$ in the light of a hint $\mathcal{H}$, the subset $u(H)$ of interpretations which imply H and the subset $v(H)$ of interpretations under which H is possible are defined as in (1). This defines mappings u and v from the power set $\mathcal{P}(\Theta)$ to the power set $\mathcal{P}(\Omega)$. The following theorem lists some of their elementary properties:

Theorem 1. *(1)* $u(\emptyset) = v(\emptyset) = \emptyset$.
(2) $u(\Theta) = v(\Theta) = \Omega$.
(3) $u(H) = v(H^c)^c$.
(4) $v(H) = u(H^c)^c$.
(5) $u(\cap\{H_i : i \in \mathcal{C}\}) = \cap\{u(H_i) : i \in \mathcal{C}\}$, *where* $\mathcal{C}$ *is* **finite** *in general,* **countable** *for hints with* σ*-restrictions* $\Gamma(\omega), \omega \in \Omega$, *and* **arbitrary** *for set-focussed hints.*
(6) $u(\cup\{H_i : i \in \mathcal{C}\}) \supseteq \cup\{u(H_i) : i \in \mathcal{C}\}$ *for an* **arbitrary** $\mathcal{C}$.
(7) $v(\cup\{H_i : i \in \mathcal{C}\}) = \cup\{v(H_i) : i \in \mathcal{C}\}$, *where* $\mathcal{C}$ *is* **finite** *in general,* **countable** *for hints with* σ*-restrictions* $\Gamma(\omega), \omega \in \Omega$, *and* **arbitrary** *for set-focussed hints.*
(8) $v(\cap\{H_i : i \in \mathcal{C}\}) \subseteq \cap\{v(H_i) : i \in \mathcal{C}\}$ *for an* **arbitrary** $\mathcal{C}$.
(9) $u(H') \subseteq u(H'')$ *if* $H' \subseteq H''$.
(10) $v(H') \subseteq v(H'')$ *if* $H' \subseteq H''$.

Proof. (1) and (2) are trivial. By definition, $v(H)^c = \{\omega \in \Omega : H^c \in \mathcal{S}(\omega)\} = u(H^c)$ and (4) is proved. (3) follows by applying (4) to H^c. (5): If $\omega \in u(H_i)$ for all $i \in \mathcal{C}$, then $H_i \in \mathcal{S}(\omega)$, thus $\cap\{H_i : i \in \mathcal{C}\} \in \mathcal{S}(\omega)$ and therefore $\omega \in u(\cap\{H_i : i \in \mathcal{C}\})$. Inversely, $\omega \in u(\cap\{H_i : i \in \mathcal{C}\})$ implies $\cap\{H_i : i \in \mathcal{C}\} \in \mathcal{S}(\omega)$, hence $H_i \in \mathcal{S}(\omega)$ and $\omega \in u(H_i)$ for all $i \in \mathcal{C}$. (6): If $\omega \in u(H_i)$ for some $i \in \mathcal{C}$, then $H_i \in \mathcal{S}(\omega)$, thus $\cup\{H_i : i \in \mathcal{C}\} \in \mathcal{S}(\omega)$ and $\omega \in u(\cup\{H_i : i \in \mathcal{C}\})$. (7) and (8) are proved using (3),(4),(5) and (6) together with de Morgan laws. (9) and (10) follow immediately from the definitions of u and v. Q.E.D.

In view of (5) u is called a $\cap$ - homomorphism and in view of (7) v is called a $\cup$ - homomorphism.

3 Support and Plausibility Functions

For a hint $\mathcal{H} = (\Omega, \mathcal{A}, P, \Gamma, \Theta)$ the degree of support $sp(H)$ and the degree of plausibility $Pl(H)$ are defined by (2) for any subset H of Θ for which $u(H) \in \mathcal{A}$ and $v(H) \in \mathcal{A}$ respectively. Let $\mathcal{E}_s$ be the class of all subsets H of Θ for which $u(H) \in \mathcal{A}$, i.e. for which the degree of support is defined. The sets of $\mathcal{E}_s$ are called **s-measurable** and $\mathcal{E}_s$ is the domain of the set-function *sp*. Similarly let $\mathcal{E}_p$ be the class of all subsets $H \subseteq \Theta$ for which $v(H) \in \mathcal{A}$, i.e. for which the degree of plausibility is defined. The sets of $\mathcal{E}_p$ are called **p-measurable** and $\mathcal{E}_p$ is the domain of the set-function *pl*.

Note that there is a strong link between the support- and the plausibility function. In fact, according to theorem 1 (4) and (3)

$$\begin{aligned} pl\,(H) &= P\,(v\,(H)) = P\,(u\,(H^c)^c) = 1 - sp\,(H^c) \\ sp\,(H) &= P\,(u\,(H)) = P\,(v\,(H^c)^c) = 1 - pl\,(H^c) \end{aligned} \qquad (3)$$

whenever the corresponding probabilities are defined.

Theorem 2. *(1) $\mathcal{E}_s$ is a multiplicative class (i.e. closed under finite intersections) or a σ-multiplicative class (closed under countable intersections) depending on whether $\Gamma(\omega),\ \omega \in \Omega$ are general restrictions or σ-restrictions.*

(2) $\mathcal{E}_p$ is an additive class (i.e. closed under finite unions) or a σ-additive class (closed under countable unions) depending on whether $\Gamma(\omega),\ \omega \in \Omega$ are general restrictions or σ-restrictions.

(3) $\mathcal{E}_p = \{H \subseteq \Theta : H^c \in \mathcal{E}_s\},\ \mathcal{E}_s = \{H \subseteq \Theta : H^c \in \mathcal{E}_p\}$ and $\emptyset, \Theta$ belong to both $\mathcal{E}_s$ and $\mathcal{E}_p$.

Proof. (1) and (2) are direct consequences of theorem 1 (5) and (7) and the fact that $\mathcal{A}$ is a σ-algebra. (3): $H \in \mathcal{E}_s$ is equivalent to $u(H) \in \mathcal{A}$, which is equivalent to $v(H^c) \in \mathcal{A}$ (theorem 1 (3) and (4)) which finally is equivalent to $H^c \in \mathcal{E}_p$. $\emptyset, \Theta$ belong to $\mathcal{E}_s$ and $\mathcal{E}_p$ because of theorem 1 (1) and (2).

Q.E.D.

$\mathcal{E}_s$ and $\mathcal{E}_p$ are called **dual** classes of s- and p-measurable sets. If Ω is a finite set, then **all** subsets of Θ are s- and p-measurable. However, in general $\mathcal{E}_s$ and $\mathcal{E}_p$ are strict subclasses of the power set of Θ. Let's illustrate theorem 2 by a simple, albeit somewhat pathological example: If $(\Omega, \mathcal{A}, P)$ is a probability space and $B \subseteq \Omega$ a subset which does not belong to $\mathcal{A}, \Gamma(\omega) = F \subseteq \Theta$ for all $\omega \in B, \Gamma(\omega) = \Theta$ otherwise, then $\mathcal{E}_s$ contains all subsets of Θ which do not contain F plus the set Θ. We have $u(H) = \emptyset$ for all $H \in \mathcal{E}_s,\ H \neq \Theta$ and thus $sp(H) = 0$, unless $H = \Theta$. $\mathcal{E}_p$ contains all subsets of Θ which are not contained in F^c plus $\emptyset$.

Theorem 3. *The* **support***- and* **plausibility functionns** *of a hint* $\mathcal{H} = (\Omega, \mathcal{A}, P, \Gamma, \Theta)$*, sp:* $\mathcal{E}_s \to [0,1]$ *and pl:* $\mathcal{E}_p \to [0,1]$ *respectively, satisfy the following conditions:*

(1) $sp(\emptyset) = pl(\emptyset) = 0$ *and* $sp(\Theta) = pl(\Theta) = 1$.
(2) sp is **monotone of order** ∞*, i.e.*

$$sp\,(E) \geq \sum \left\{ (-1)^{|I|+1}\, sp\,(\cap_{i\in I} E_i) : \emptyset \neq I \subseteq \{1,\ldots,n\} \right\} \tag{4}$$

for all $n \geq 1$ *and sets* $E, E_i \in \mathcal{E}_s$*, such that* $E \supseteq E_i$*; and pl is* **alternating of order** ∞*, i.e.*

$$pl\,(E) \leq \sum \left\{ (-1)^{|I|+1}\, pl\,(\cup_{i\in I} E_i) : \emptyset \neq I \subseteq \{1,\ldots,n\} \right\} \tag{5}$$

for all $n \geq 1$ *and sets* $E, E_i \in \mathcal{E}_p$*, such that* $E \subseteq E_i$*.*

Furthermore, if all $\Gamma(\omega)$*,* $\omega \in \Omega$ *are* σ*-restrictions, then the following conditions hold:*

(3) sp and pl are **continuous***, i.e. if* $E_1 \supseteq E_2 \supseteq \ldots$ *is a monotone decreasing sequence of sets of* $\mathcal{E}_s$*, then*

$$sp\,(\cap_{i=1}^{\infty} E_i) = \lim_{i\to\infty} sp\,(E_i) \tag{6}$$

and if $E_1 \subseteq E_2 \subseteq \ldots$ *is a monotone increasing sequence of sets of* $\mathcal{E}_p$*, then*

$$pl\,(\cap_{i=1}^{\infty} E_i) = \lim_{i\to\infty} pl\,(E_i)\,. \tag{7}$$

Proof. (1) follows from theorem 1 (1) and (2). In order to prove (2) for the support function, the well-known inclusion-exclusion formula of probability theory, together with theorem 1 (5), (6) and (9) is used:

$$\begin{aligned}
sp\,(E) &= P\,(u\,(E)) \geq P\,(u\,(\cup \{E_i : i = 1,2,\ldots,n\})) \\
&\geq P\,(\cup\,[u\,(E_i) : i = 1,2,\ldots,n]) \\
&= \sum \left\{ (-1)^{|I|+1}\, P\,(\cap_{i\in I} u\,(E_i)) : \emptyset \neq I \subseteq \{1,\ldots,n\} \right\} \\
&= \sum \left\{ (-1)^{|I|+1}\, P\,(u\,(\cap_{i\in I} E_i)) : \emptyset \neq I \subseteq \{1,\ldots,n\} \right\} \\
&= \sum \left\{ (-1)^{|I|+1}\, sp\,(\cap_{i\in I} E_i) : \emptyset \neq I \subseteq \{1,\ldots,n\} \right\}.
\end{aligned}$$

Condition (2) for the plausibility function is proved in the same way or by using (4) together with (3).

$E_1 \supseteq E_2 \supseteq \ldots$ implies $u(E_1) \supseteq u(E_2) \supseteq \ldots$ (theorem 1 (9)) and $\cap_{i=1}^{\infty} E_i \in \mathcal{E}_s$ (theorem 2 (1)). By the continuity of probabilities and theorem 1 (5)

$$\begin{aligned} sp\left(\cap_{i=1}^{\infty} E_i\right) &= P\left(u(\cap_{i=1}^{\infty} E_i)\right) \\ &= P\left(\cap_{i=1}^{\infty} u(E_i)\right) \\ &= \lim_{i \to \infty} P\left(u\left(E_i\right)\right) = \lim_{i \to \infty} sp\left(E_i\right) \end{aligned}$$

and condition (3) is proved.

Q.E.D.

Note that in particular set-focussed hints have continuous support-and plausibility functions.

Does it make sense to define the degree of support for a hypothesis $H \subseteq \Theta$ outside the class $\mathcal{E}_s$ of s-measurable subsets? If $u(H) \subseteq \Omega$ is not measurable, the model of the hint $\mathcal{H}$ does not contain the necessary information to determine the probability of the set of interpretations supporting H. But any measurable set of interpetations $A \subseteq \Omega$ which is contained in $u(H)$ is a support for H. Hence one may say that the unknown support for H must be at least $P(A)$, for any $A \subseteq u(H)$ and $A \in \mathcal{A}$. Thus, in the absence of further information the support of H could be defined as

$$sp_e\left(H\right) = sup\left\{P\left(A\right) : A \subseteq u\left(H\right), A \in \mathcal{A}\right\} = P_*\left(u\left(H\right)\right) \tag{8}$$

where P_* is the inner probability to P. This is an extension of the support function sp onto the whole power set $\mathcal{P}(\Theta)$ because the restriction of sp_e to $\mathcal{E}_s$ equals sp. We call sp_e the **vacuous extension** of sp to underline that no information not contained in the hint $(\Omega, \mathcal{A}, P, \Gamma, \Theta)$ has been added.

By duality, we may also extend the plausibility functions pl from $\mathcal{E}_p$ to $\mathcal{P}(\Theta)$:

$$pl_e\left(H\right) = 1 - sp_e\left(H^c\right). \tag{9}$$

This is similarly called the vacuous extension of pl. This name is justified by the following proposition:

Theorem 4. *The equality*

$$pl_e\left(H\right) = inf\ \left\{P\left(A\right) : A \supseteq v\left(H\right), A \in \mathcal{A}\right\} = P^*\left(v\left(H\right)\right) \tag{10}$$

holds. P^* is the outer probability to P.

Proof. From the definitions (8) and (9) and theorem 1 (4) it follows that

$$\begin{aligned} pl_e(H) &= 1 - sp_e\left(H^c\right) \\ &= 1 - sup\left\{P\left(A\right) : A \in \mathcal{A}, A \subseteq u\left(H^c\right)\right\} \\ &= 1 - sup\left\{P\left(A\right) : A \in \mathcal{A}, u\left(H^c\right)^c \subseteq A^c\right\} \\ &= inf\left\{P\left(A^c\right) : A \in \mathcal{A}, v\left(H\right) \subseteq A^c\right\} \\ &= inf\left\{P\left(A\right) : A \in \mathcal{A}, v\left(H\right) \subseteq A\right\}. \end{aligned}$$

Q.E.D.

Furthermore, it turns out that sp_e and Pl_e satisfy also the conditions of theorem 3.

Theorem 5. *Let sp_e and Pl_e be the extended support- and plausibility functions of a hint $\mathcal{H} = (\Omega, \mathcal{A}, P, \Gamma, \Theta)$. Then*

(1) sp_e and Pl_e are **monotone** *and* **alternating of order** ∞ *respectively on $\mathcal{P}(\Theta)$.*
(2) If $\Gamma(\omega)$ is a σ-restriction for all ω, then sp_e and Pl_e are also **continuous***.*

The proof of this theorem will not be given here. It seems to be surprisingly difficult and relies on the notion of an allocation of probability (Shafer, 1979). See Kohlas (1990) for a proof of this theorem. The connection between inner probability measures and support or belief functions have also been noted by Ruspini (1987) and Fagin and Halpern (1989), see also Shafer (1990).

4 Refining Hints

A hint $\mathcal{H} = (\Omega, \mathcal{A}, P, \Gamma, \Theta)$ can be refined in several respects by adding supplementary information to it:

(1) The restrictions $\Gamma(\omega)$ associated with the interpretations ω may become more precise: A restrictions $(\mathcal{S}', \mathcal{P}')$ is said to be more precise than (or included in) a restriction $(\mathcal{S}, \mathcal{P})$ iff $\mathcal{S}' \supseteq \mathcal{S}$ (or equivalently $\mathcal{P}' \subseteq \mathcal{P}$), i.e. if it implies more propositions and if less propositions are possible. We write then $(\mathcal{S}', \mathcal{P}') \subseteq (\mathcal{S}, \mathcal{P})$.
(2) Some interpretations which originally are considered as possible may become known as impossible: The new set of possible interpretations Ω' becomes a subset of Ω. This implies also that the original probability P must be conditionned on Ω'. This leads to a new probability space $(\Omega', \mathcal{A}', P')$ of possible interpretations, where $\mathcal{A}' = \mathcal{A} \cap \Omega'$ and $P'(A) = P^*(A \cap \Omega')/P^*(\Omega')$, provided that $P^*(\Omega') > 0$. Note that Ω' is not necessarily measurable; P' is still a probability measure on $\mathcal{A}'$ (Neveu, 1964).
(3) The probability measure P' on the set of possible interpretations Ω' may be extended from the σ-algebra $\mathcal{A}'$ to a probability measure P'' on a larger σ-algebra $\mathcal{A}''$ containing $\mathcal{A}'$. Let's note that in this case

$$P'_*(A) \leq P''(A) \leq P'^*(A) \tag{11}$$

for all $A \in \mathcal{A}''$.

Thus, combining all three refining steps in the above sequence, a new, refined hint $\mathcal{H}'' = (\Omega'', \mathcal{A}'', P'', \Gamma'', \Theta)$ may be obtained, such that $\Omega'' \subseteq \Omega, \mathcal{A}'' \supseteq \mathcal{A} \cap \Omega''$, P'' is an extension to $\mathcal{A}''$ of the probability measure $P'(A) = P^*(A \cap \Omega'')/P^*(\Omega'')$ on $\mathcal{A} \cap \Omega''$ and $\Gamma''(\omega) \subseteq \Gamma(\omega)$ for all $\omega \in \Omega''$. In this case we write $\mathcal{H}'' \subseteq \mathcal{H}$ and say that $\mathcal{H}''$ is **included** in or is **finer** than

$\mathcal{H}$ (and $\mathcal{H}$ is **coarser** then $\mathcal{H}''$). Of course, many times not all three refining steps are present; in particular often only step (1) or steps (1) and (3) are considered. These particular cases correspond to Yager's (1985) definition of inclusion.

This notion of inclusion of hints leads to the following comparison of the corresponding support- and plausibility functions:

Theorem 6. *Let $\mathcal{H}'' = (\Omega'', \mathcal{A}'', P'', \Gamma'', \Theta)$ and $\mathcal{H} = (\Omega, \mathcal{A}, P, \Gamma, \Theta)$ be two hints such that $\mathcal{H}'' \subseteq \mathcal{H}$ and with sp''_e, pl''_e and sp_e, pl_e as their respective extended support- and plausibility functions. If $k = P^*(\Omega'')$, then*

(1) $sp_e(H) \leq k \cdot sp''_e(H) + (1-k)$ for all $H \subseteq \Theta$
(2) $Pl_e(H) \geq k \cdot pl''_e(H)$ for all $H \subseteq \Theta$.

Proof. Let $v''(H)$ and $v(H)$ be the subsets of interpretations of Ω'' and Ω respectively under which H is possible. Then clearly $v''(H) \subseteq v(H) \cap \Omega''$ by the refining step (1).

Now, for any $H \subseteq \Theta$,

$$pl_e(H) = P^*(v(H)) \geq P^*(v(H) \cap \Omega'') \geq P^*(v''(H)) = P^*(v''(H) \cap \Omega'') .$$

Let $P'(A) = P^*(A \cap \Omega'')/P^*(\Omega'')$ for $A \in \mathcal{A} \cap \Omega''$ and $P'^*(A)$ denote the outer probability measure with respect to P'. Then it follows easily that $P'^*(v''(H) \cap \Omega'') = P^*(v''(H) \cap \Omega'')/P^*(\Omega'')$ and hence

$$pl_e(H) \geq P'^*(v''(H) \cap \Omega'') P^*(\Omega'') .$$

If $P''^*(A)$ is the outer measure with respect to the probability measure P'' on $\mathcal{A}''$, then clearly $P'^*(A) \geq P''^*(A)$ for any $A \subseteq \Omega''$. Thus

$$\begin{aligned} pl_e(H) &\geq P''^*(v''(H) \cap \Omega'') P^*(\Omega'') = P''^*(v''(H)) P^*(\Omega'') \\ &= pl''_e(H) P^*(\Omega'') = k \cdot pl''_e(H) . \end{aligned}$$

This proves (2).

By (9) we have

$$\begin{aligned} sp_e(H) &= 1 - pl_e(H^c) \leq 1 - k \cdot pl''_e(H^c) \\ &= 1 - k \cdot (1 - sp''_e(H)) = k \cdot sp''_e(H) + (1-k) . \end{aligned}$$

This proves (1).

Q.E.D.

If only refining steps (1) and possibly (3) are present, then $k = 1$ and $[sp''_e(H), pl''_e(H)] \subseteq [sp_e(H), pl_e(H)]$.

To any hint $\mathcal{H} = (\Omega, \mathcal{A}, P, \Gamma, \Theta)$ a vacuous hint $\mathcal{V} = (\Omega, \mathcal{A}, P, \Gamma_{vac}, \Theta)$ can be associated, where $\Gamma_{vac}(\omega)$ is the vacuous restriction for all ω. Clearly $\Gamma(\omega) \subseteq \Gamma_{vac}(\omega)$ for all ω and therefore we have always $\mathcal{H} \subseteq \mathcal{V}$.

5 Families of Hints Related to a Support Function

A hint generates a support function sp on some multiplicative class $\mathcal{E}_s$. This function has the properties (1) and (2), possibly (3) as stated in theorem 3. If now sp is a function on a multiplicative class $\mathcal{E}_s$, satisfying conditions (1) and (2) of theorem 3, is there always a hint which generates this support function? The answer is affirmative. This is a consequence of an integral theorem of Choquet (1953) as was noted by Shafer (1979). But it can easily be seen that different hints may generate the **same** support function sp on $\mathcal{E}_s$, but with **different** extensions sp_e to $\mathcal{P}(\Theta)$. In fact, let $(\Omega, \mathcal{A}, P)$ be a probability space and let B_1, B_2 be two different non-measurable subsets of Ω which have different inner probabilities. Furthermore, let Θ be a frame of discernment and F a strict subset of Θ. This allows to define two distinct hints $\mathcal{H}_i = (\Omega, \mathcal{A}, P, \Gamma_i, \Theta), i = 1, 2$, where

$$\Gamma_i(\omega) = \begin{cases} F & \text{if } \omega \in B_i \\ \Theta & \text{otherwise.} \end{cases}$$

For both hints, the class $\mathcal{E}_s$ equals all subsets of Θ which do not contain F plus the set Θ and the support functions of $\mathcal{H}_1$ and $\mathcal{H}_2$ coincide. But if sp_{1e} and sp_{2e} denote their respective extended support functions, then

$$sp_{1e}(F) = P_*(B_1) \neq P_*(B_2) = sp_{2e}(F).$$

Thus there exists a whole family of hints related to a support function sp on $\mathcal{E}_s$. The goal of this section is to study this family of hints. In a similar vain, Shafer (1979) studied various extensions of support (or belief) functions. This section puts some of his results into the perspective of hints.

In the context of the theory of hints Choquet's theorem can be stated as follows:

Theorem 7. *Let $\mathcal{E}_s$ be a multiplicative class and sp: $\mathcal{E}_s \to [0, 1]$ a function satisfying conditions (1) and (2) of theorem 3. Then there exists a hint whose support function is sp. If furthermore $\mathcal{E}_s$ is a σ-multiplicative class and sp satisfies condition (3) of theorem 3 (continuity), then there exists a hint whose restrictions are all σ-restrictions and whose support function is sp.*

For a formal proof we refer to Choquet (1953) (see also Shafer, 1978 and Kohlas, 1990). Let's only describe the hint constructed in this proof: As set of possible interpretations the set $\mathcal{R}(\mathcal{E}_s)$ of all filters on the multiplicative class $\mathcal{E}_s$ is selected. Note that to any restriction $\mathcal{R} = (\mathcal{S}, \mathcal{P})$ in $\mathcal{R}(\Theta)$ can be associated a filter $\varphi(\mathcal{R}) = \mathcal{S} \cap \mathcal{E}_s$ on $\mathcal{E}_s$. The maping φ from $\mathcal{R}(\Theta)$ to $\mathcal{R}(\mathcal{E}_s)$ is onto because for any filter $\mathcal{F} \in \mathcal{R}(\mathcal{E}_s)$ the restriction $\mathcal{R}_c(\mathcal{F}) \in \mathcal{R}(\Theta)$ defined by its class of implied propositions $\mathcal{S} = \{H \subseteq \Theta$: *there is an* $E \in \mathcal{F}$ such that $E \subseteq H\}$ is in $\varphi^{-1}(\mathcal{F})$. This shows that $\{\varphi^{-1}(\mathcal{F}) : \mathcal{F} \in \mathcal{R}(\mathcal{E}_s)\}$ is a partition of $\mathcal{R}(\Theta)$. Moreover, $\mathcal{R}_c(\mathcal{F})$ is the coarsest restriction in $\varphi^{-1}(\mathcal{F})$: if $\mathcal{R}' \in \varphi^{-1}(\mathcal{F})$, then $\mathcal{R}' \subseteq \mathcal{R}_c(\mathcal{F})$. Define $\Gamma''(\mathcal{F}) = \mathcal{R}_c(\mathcal{F})$ for any $\mathcal{F} \in \mathcal{R}(\mathcal{E}_s)$. Then there is according to Choquet (1953) a σ-algebra $\mathcal{A}''$ in $\mathcal{R}(\mathcal{E}_s)$ and a probability measure P'' defined on it such that the hint $(\mathcal{R}(\mathcal{E}_s), \mathcal{A}'', P'', \Gamma'', \Theta)$ has *sp* as support function.

Note that using φ the probability space $(\mathcal{R}(\mathcal{E}_s), \mathcal{A}'', P'')$ induces a probability space $(\mathcal{R}(\Theta), \mathcal{A}', P')$. If we define $\Gamma_c(\mathcal{R}) = \mathcal{R}_c(\varphi(\mathcal{R}))$, then the hint $(\mathcal{R}(\Theta), \mathcal{A}', P', \Gamma_c, \Theta)$ generates clearly also the support function *sp* on $\mathcal{E}_s$. Let $u_c(H), v_c(H)$ be the functions (1) defined with respect to Γ_c and let $u_c(\mathcal{E}_s), v_c(\mathcal{E}_p)$ (where $\mathcal{E}_p$ is the dual class to $\mathcal{E}_s$) be the images of $\mathcal{E}_s$ and $\mathcal{E}_p$ with respect to u_c and v_c respectively. By theorem 1 (5) and (7), $u_c(\mathcal{E}_s)$ is a multiplicative class and $v_c(\mathcal{E}_p)$ an additive class. Both $u_c(\mathcal{E}_s)$ and $v_c(\mathcal{E}_p)$ are contained in $\mathcal{A}'$. Now, let $\mathcal{A}_c$ be the smallest σ-algebra containing $u_c(\mathcal{E}_s)$ and $v_c(\mathcal{E}_p)$; $\mathcal{A}_c$ is a subalgebra of $\mathcal{A}'$. Let finally P_c be the restriction of P' to $\mathcal{A}_c$. Then the hint $\mathcal{H}_c = (\mathcal{R}(\Theta), \mathcal{A}_c, P_c, \Gamma_c, \Theta)$ still has *sp* on $\mathcal{E}_s$ as support function. This hint is called the **canonical hint** of the support function *sp* on $\mathcal{E}_s$. We shall see that $\mathcal{H}_c$ is in some sense the coarsest hint which generates *sp* on $\mathcal{E}_s$: among all hints generating *sp*, it contains the least information. This will be formulated more precisely using the inclusion relation between hints introduced in the previous section.

Thus, let $\mathcal{H} = (\Omega, \mathcal{A}, P, \Gamma, \Theta)$ be any hint, which defines the support function *sp* on $\mathcal{E}_s$. More precisely, suppose that the class of s-measurable sets of $\mathcal{H}$ contains $\mathcal{E}_s$ and that on $\mathcal{E}_s$ its support function equals *sp*. Hints which define in this sense identical support functions on $\mathcal{E}_s$ are called **equivalent**. In order to compare equivalent hints among themselves and in particular with the canonical hint, they must be represented with respect to an identical set of possible interpretations. By the mapping Γ, the σ-algebra $\mathcal{A}$ and the probability measure P can be transported to the set $\mathcal{R}(\Theta)$ in the usual way: Consider the σ-algebra $\mathcal{A}'$ of all subsets $B \subseteq \mathcal{R}(\Theta)$ for which $\Gamma^{-1}(B) \in \mathcal{A}$ and define a probability P' on $\mathcal{A}'$ by $P'(B) = P(\Gamma^{-1}(B))$. This leads to an equivalent hint $(\mathcal{R}(\Theta), \mathcal{A}', P', id, \Theta)$ where *id* stands for the identical mapping $id(\mathcal{R}) = \mathcal{R}$. This is called the **canonical representation** of $\mathcal{H}$. In particular, note that this new hint defines the same extended support function sp'_e as $\mathcal{H}$. In this sense $\mathcal{H}$ and its canonical representation $\mathcal{H}_{cr}$ contain exactly the same information.

The following theorem states now that the canonical hint is the coarsest hint among all equivalent hints with respect to a support function *sp* on $\mathcal{E}_s$.

Theorem 8. *Let $\mathcal{H}_c$ be the canonical hint with respect to a support function sp on a multiplicative class $\mathcal{E}_s$. If $\mathcal{H}$ is any equivalent hint with respect to this support function and $\mathcal{H}_{cr}$ its canonical representation, then $\mathcal{H}_{cr} \subseteq \mathcal{H}_c$.*

Proof. Both $\mathcal{H}_{cr}$ and $\mathcal{H}_c$ have the same set of possible interpretations $\mathcal{R}(\Theta)$. Moreover, clearly $id(\mathcal{R}) \subseteq \mathcal{R}_c(\varphi(\mathcal{R})), \mathcal{A}' \supseteq \mathcal{A}_c$ and the restriction of P' to $\mathcal{A}_c$ equals P_c.

Q.E.D.

As a consequence of this theorem, it follows that $[sp_e(H), pl_e(H)] \subseteq [sp_{ce}(H), pl_{ce}(H)]$ for all $H \subseteq \Theta$, if sp_{ce}, pl_{ce} denote the extended support and plausibility functions of the canonical hint and sp_e, pl_e the extended support and plausibility functions of the hint $\mathcal{H}$. Shafer (1979) studied extensions of support functions and identified among others the minimal extension of a support function sp on $\mathcal{E}_s$. It turns out that this minimal extension is in fact as one expects the extension of the canonical hint with respect to sp on $\mathcal{E}_s$.

Theorem 9. *If sp_{ce}, pl_{ce} are the extended support and plausibility functions of the canonical hint $\mathcal{H}_c$ with respect to a support and plausibility function sp and pl on a multiplicative class $\mathcal{E}_s$ and its dual additive class $\mathcal{E}_p$, then*

$$sp_{ce}(H) = sup\left\{\sum\left\{(-1)^{|I|+1} sp\left(\cap_{i\in I} E_i\right) : \emptyset \neq I \subseteq \{1,\ldots,n\}\right\} : E_i \subseteq H, E_i \in \mathcal{E}_s, i = 1,\ldots,n; n = 1,2,\ldots\right\}, \quad (12)$$

$$pl_{ce}(H) = inf\left\{\sum\left\{(-1)^{|I|+1} pl(\cup_{i\in I} E_i) : \emptyset \neq I \subseteq \{1,\ldots,n\}\right\} : E_i \supseteq H, E_i \in \mathcal{E}_p, i = 1,\ldots,n; n = 1,2,\ldots\right\}. \quad (13)$$

Proof. Note that by theorem 1 (6) $\cup_{i=1}^n u_c(E_i) \subseteq u_c(\cup_{i=1}^n E_i)$. Furthermore

$$\begin{aligned}
&sp_{ce}(H) = P_{c*}(u_c(H)) \\
&\geq sup\left\{P_c\left(\cup_{i=1}^n u_c(E_i)\right) : E_i \subseteq H, E_i \in \mathcal{E}_s, i = 1,\ldots,n; n = 1,2,\ldots\right\} \\
&= sup\left\{\sum\left\{(-1)^{|I|+1} P_c\left(\cap_{i\in I} u_c(E_i)\right) : \emptyset \neq I \subseteq \{1,\ldots,n\}\right\} : \right. \\
&\qquad \left. E_i \subseteq H, E_i \in \mathcal{E}_s, i = 1,\ldots,n; n = 1,2\ldots\right\} \\
&= sup\left\{\sum\left\{(-1)^{|I|+1} sp\left(\cap_{i\in I} u_c E_i\right) : \emptyset \neq I \subseteq \{1,\ldots,n\}\right\} : \right. \\
&\qquad \left. E_i \subseteq H, E_i \in \mathcal{E}_s, i = 1,\ldots,n; n = 1,2\ldots\right\}
\end{aligned}$$

On the other hand, Shafer (1979) proves that the right hand side of (12) defines indeed a support function sp_m on the power set $\mathcal{P}(\Theta)$ satisfying the conditions of theorem 7. There exists therefore a hint $\mathcal{H}'$ which generates this support function and let $\mathcal{H}'_{cr}$ its canonical representation. But theorem 8 implies that $\mathcal{H}'_{cr} \subseteq \mathcal{H}_c$ and by theorem 6 $sp_{ce}(H) \leq sp_{cre}(H) = sp_m(H)$ since $k = 1$. Thus we obtain finally $sp_{ce}(H) = sp_m(H)$ which proves (12).

(13) is deduced from (12) using (3) and theorem 1 (3) and (4) together with the de Morgan laws.

Q.E.D.

Theorem 9 together with theorems 6 and 8 show that sp_m is the smallest support function which extends sp from $\mathcal{E}_s$ to all of $\mathcal{P}(\Theta)$.

If the support function sp on a σ-multiplicative class $\mathcal{E}_s$ is **continuous** (satisfies condition (3) of theorem 3), then a canonical hint associated to this support function can be constructed in a similar way with respect to the set of σ-restrictions $\mathcal{R}_\sigma(\Theta)$ on Θ. For any hint for which all restrictions are σ-restrictions, a canonical representation with respect to $\mathcal{R}_\sigma(\Theta)$ can be defined along similar lines as above. Then two further results corresponding to theorems 8 and 9 can be proved:

Theorem 10. *Let $\mathcal{H}_c$ be the canonical hint with respect to a* **continuous** *support function sp on a σ-multiplicative class $\mathcal{E}_s$. If $\mathcal{H}$ is any equivalent hint with respect to this support function and $\mathcal{H}_{cr}$ its canonical representation, then $\mathcal{H}_{cr} \subseteq \mathcal{H}_c$.*

Theorem 11. *If sp_{ce}, pl_{ce} are the extended support and plausibility functions of the canonical hint $\mathcal{H}_c$ with respect to continuous support and plausibility functions sp and pl on a σ-algebra $\mathcal{E}_s = \mathcal{E}_p$, then*

$$sp_{ce}(H) = \sup\left\{\lim_{i\to\infty} sp(E_i) : E_1 \supseteq E_2 \supseteq \ldots, E_i \in \mathcal{E}_s, \cap E_i \subseteq H\right\} \qquad (14)$$

$$pl_{ce}(H) = \inf\left\{\lim_{i\to\infty} pl(E_i) : E_1 \subseteq E_2 \subseteq \ldots, E_i \in \mathcal{E}_p, \cup E_i \supseteq H\right\}. \qquad (15)$$

These theorems will not be proved here. The proofs develop along similar lines as those of theorems 8 and 9. Note that for theorem 11 Shafer (1979) showed that the right hand side of (14) is indeed a continuous support function. This theorem shows that it is the smallest continuous support function which extends the continuous support function sp from $\mathcal{E}_s$ to $\mathcal{P}(\Theta)$.

6 Combining Hints

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two hints relative to the same frame Θ and defined by $(\Omega_1, \mathcal{A}_1, P_1, \Gamma_1, \Theta)$ and $(\Omega_2, \mathcal{A}_2, P_2, \Gamma_2, \Theta)$. The basic idea for the combination of these hints into a combined body of information is that in each hint there must be exactly one correct interpretation $\omega_i, i = 1, 2$ such that — looking at both hints together — ω_1 and ω_2 must be simultaneously correct interpretations. Hence (ω_1, ω_2) must be the correct combined correct interpretation. Therefore, in order to combine the two hints $\mathcal{H}_1$ and $\mathcal{H}_2$ into one new combined hint, we form first the product space of the combined interpretations from the two hints $(\Omega_1 \times \Omega_2, \mathcal{A}_1 \otimes \mathcal{A}_2, P')$ where P' is any probability measure on $\mathcal{A}_1 \otimes \mathcal{A}_2$ reflecting the common likelihood of combined interpretations. The two hints are called **independent**, if the interpretations of the two hints are stochastically independent. Then P' is the product measure of P_1 and P_2. This is the case which will be pursued here although other cases would be equally possible.

If the combined interpretation (ω_1, ω_2) is the correct one, then the restriction

$$\Gamma(\omega_1, \omega_2) = \Gamma_1(\omega_1) \cap \Gamma_2(\omega_2) \tag{16}$$

must necessarily hold. Note that it is possible that $\Gamma_1(\omega_1)$ and $\Gamma_2(\omega_2)$ are contradictory. Then ω_1 and ω_2 are called contradictory interpretations.

Define now

$$\begin{aligned} u'(H) &= \{(\omega_1, \omega_2) \in \Omega_1 \times \Omega_2 : \ H \textit{ is implied by } \Gamma(\omega_1, \omega_2)\} \\ v'(H) &= \{(\omega_1, \omega_2) \in \Omega_1 \times \Omega_2 : \ H \textit{ is possible under } \Gamma(\omega_1, \omega_2)\}. \end{aligned} \tag{17}$$

Theorem 1 — except (1) and (2) — clearly applies to u' and v'; (1) is replaced by $v'(\emptyset) = \emptyset$ and (2) by $u'(\Theta) = \Omega_1 \times \Omega_2$. $u'(\emptyset)$ represents the set of contradictory interpretation pairs. Such a pair can never be the correct one because contradictions are not possible. Therefore contradictory interpretations must be eliminated and the probability must be conditioned on the event that there is no contradiction. Provided that $u'(\emptyset)$ is measurable, i.e. $u'(\emptyset) \in \mathcal{A}_1 \otimes \mathcal{A}_2$ and $P'(u'(\emptyset)) < 1$, the new combined hint $\mathcal{H}_1 \oplus \mathcal{H}_2 = (\Omega, \mathcal{A}, P, \Gamma, \Theta)$ can be formed, where

$$\begin{aligned} \Omega &= u'(\emptyset)^c = v'(\Theta), \\ \mathcal{A} &= u'(\emptyset)^c \cap \mathcal{A}_1 \otimes \mathcal{A}_2, \\ P(A) &= P'(A) / P'(u'(\emptyset)^c) \end{aligned}$$

and Γ is defined by (16) (and restricted to Ω). This way to combine hints is called **Dempster's rule** (A. Dempster (1967)).

Let u and v be defined by (1) relative to the hint $\mathcal{H}_1 \oplus \mathcal{H}_2$. Then $u(H) = u'(H) \cap \Omega = u'(H) - u'(\emptyset)$ and $v(H) = v'(H)$.

Dempster's rule may be extended even to the case where $u'(\emptyset)$ is not measurable. In this case the conditional probability space $(\Omega, \mathcal{A}, P)$ can be considered, where $(\Omega, \mathcal{A})$ is defined as above and $P(A) = P'^*(A \cap u'(\emptyset)^c / P'^*(u'(\emptyset)^c)$, provided that $P'^*(u'(\emptyset)^c) > 0$. This leads to the combined hint $\mathcal{H}_1 \oplus \mathcal{H}_2 = (\Omega, \mathcal{A}, P, \Gamma, \Theta)$.

As before, we have $u(H) = u'(H) \cap \Omega = u'(H) - u'(\emptyset)$ and $v(H) = v'(H)$. Let $\mathcal{E}_s$ and $\mathcal{E}_p$ be the classes of s- and p-measurable sets relative to the hint $\mathcal{H}_1 \oplus \mathcal{H}_2$. Denote by $\mathcal{E}'_s$ and $\mathcal{E}'_p$ the classes of sets H such that $u'(H)$ and $v'(H)$ are measurable with respect to $\mathcal{A}_1 \otimes \mathcal{A}_2$. From $u'(H) \in \mathcal{A}_1 \otimes \mathcal{A}_2$ it follows that $u(H) \in \Omega \cap \mathcal{A}_1 \otimes \mathcal{A}_2$ and thus $\mathcal{E}'_s \subseteq \mathcal{E}_s$. Similarly, because $v'(H) \subseteq \Omega, v'(H) \in \mathcal{A}_1 \otimes \mathcal{A}_2$ implies $v(H) \in \Omega \cap \mathcal{A}_1 \otimes \mathcal{A}_2$ or $\mathcal{E}'_p \subseteq \mathcal{E}_p$. If $u'(\emptyset)$ is measurable, then $\mathcal{E}'_s = \mathcal{E}_s$ and $\mathcal{E}'_p = \mathcal{E}_p$.

The next theorem states that inclusion of hints is maintained under Dempster's rule:

Theorem 12. *Let $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}''_1, \mathcal{H}''_2$ be four hints such that $\mathcal{H}''_1 \subseteq \mathcal{H}_1$ and $\mathcal{H}''_2 \subseteq \mathcal{H}_2$. Then $\mathcal{H}''_1 \oplus \mathcal{H}''_2 \subseteq \mathcal{H}_1 \oplus \mathcal{H}_2$.*

Proof. $\Gamma''_1(\omega_1) \subseteq \Gamma_1(\omega_1)$ and $\Gamma''_2(\omega_2) \subseteq \Gamma_2(\omega_2)$ imply $\Gamma''_1(\omega_1) \cap \Gamma''_2(\omega_2) \subseteq \Gamma_1(\omega_1) \cap \Gamma_2(\omega_2)$. This, together with $\Omega''_1 \subseteq \Omega_1$ and $\Omega''_2 \subseteq \Omega_2$ implies $\Omega'' \subseteq \Omega$. Also $\mathcal{A}''_1 \supseteq \mathcal{A}_1 \cap \Omega''_1$ and $\mathcal{A}''_2 \supseteq \mathcal{A}_2 \cap \Omega''_2$ imply that

$$\begin{aligned}\mathcal{A}'' &= \mathcal{A}_1'' \otimes \mathcal{A}_2'' \cap \Omega'' \supseteq (\mathcal{A}_1 \cap \Omega_1'') \otimes (\mathcal{A}_2 \cap \Omega_2'') \cap \Omega'' \\ &= (\mathcal{A}_1 \otimes \mathcal{A}_2) \cap (\Omega_1'' \times \Omega_2'') \cap \Omega'' \\ &= \mathcal{A}_1 \otimes \mathcal{A}_2 \cap \Omega'' = (\mathcal{A}_1 \otimes \mathcal{A}_2 \cap \Omega) \cap \Omega'' = \mathcal{A} \cap \Omega''.\end{aligned}$$

It remains to show that

$$P''(A) = P^*(A)/P^*(\Omega'')$$

for any $A \in \mathcal{A}\cap\Omega''$. Let Q'', Q denote the product measures of P_1'' and P_2'' and P_1 and P_2 on the product spaces $(\Omega_1'' \times \Omega_2'',\ \mathcal{A}_1'' \otimes \mathcal{A}_2'')$ and $(\Omega_1 \times \Omega_2, \mathcal{A}_1 \otimes \mathcal{A}_2)$ respectively. Then by definition $P''(A) = Q''^*(A)/Q''^*(\Omega'')$ for any $A \in \mathcal{A} \cap \Omega''$. It is thus sufficient to show that $Q''^*(A) = k \cdot P^*(A)$ for some constant k independent of A.

To begin with, let's suppose that the sets $\Omega_1'', \Omega_2'', \Omega''$ and Ω are measurable with respect to $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}$ and $\mathcal{A}_1 \otimes \mathcal{A}_2$ respectively. Then $P''(A) = Q''(A)/Q''(\Omega'')$ and $P^*(A)/P^*(\Omega'') = P(A)/P(\Omega'')$ for $A \in \mathcal{A} \cap \Omega''$ and we must prove that $Q^*(A) = k \cdot P(A)$. Let X_A denote the indicator function of A. Then

$$Q''(A) = \int P_1''(d\omega_1)\, P_2''(d\omega_2)\, X_A.$$

Because X_A is a measurable function with respect to $\mathcal{A}$, it is sufficient to take the restrictions of the probability measures P_1'' and P_2'' to $\mathcal{A}_1$ and $\mathcal{A}_2$. But there these probabilities are conditional probabilities such that

$$\begin{aligned}Q''(A) &= \int P_1(d\omega_1)\, P_2(d\omega_2)\, X_A / P_1(\Omega_1'')\, P_2(\Omega_2'') \\ &= Q(A)/P_1(\Omega_1'')\, P_2(\Omega_2'') = P(A)\left(Q(\Omega)/P_1(\Omega_1'')\, P_2(\Omega_2'')\right).\end{aligned}$$

This proves the theorem in the case of measurable sets $\Omega_1'', \Omega_2'', \Omega''$ and Ω. If Ω is not measurable, then there exists a measurable set $\bar{\Omega}$, containing Ω, such that $Q^*(\Omega) = Q(\bar{\Omega})$. If $A \in \mathcal{A} \cap \Omega$, then $\bar{A} = A \cap \bar{\Omega}$ is measurable, contains A, and $Q^*(A) = Q(\bar{A})$.

Thus $P(\bar{A}) = P(A)$ for all $A \in \mathcal{A} \cap \Omega$ and Ω may be replaced by $\bar{\Omega}$ and $\mathcal{A} \cap \Omega$ by $\mathcal{A} \cap \bar{\Omega}$ without changing the relevant probability values. In this way the case where some or all sets $\Omega_1'', \Omega_2'', \Omega''$ and Ω are not measurable can be reduced to the former case. This proves the theorem.

Q.E.D.

In the case of theorem 12, the constant k appearing in theorem 6 equals $P^*(\Omega'')$, where Ω'' contains all combined interpretations (ω_1, ω_2) which are not contradictory under $\mathcal{H}_1'' \oplus \mathcal{H}_2''$. Some combined interpretations, which are not contradictory under $\mathcal{H}_1 \oplus \mathcal{H}_2$ may however be contradictory under $\mathcal{H}_1'' \oplus \mathcal{H}_2''$. This accounts for the possible difference between Ω and Ω''. If the situation is such that $\Omega'' = \Omega$, then $k = 1$ and $[sp_e''(H),\ pl_e''(H)] \subseteq [sp_e(H),\ pl_e(H)]$.

Let $\mathcal{V}$ be the vacuous hint associated with $\mathcal{H}_2$. Then theorem 12 implies that $\mathcal{H}_1 \oplus \mathcal{H}_2 \subseteq \mathcal{H}_1 \oplus \mathcal{V}$. Similariy $\mathcal{H}_1 \oplus \mathcal{H}_2 \subseteq \mathcal{V} \oplus \mathcal{H}_2$. As the combination of a hint with a vacuous hint does not add new information to the hint, this result shows that a combined hint $\mathcal{H}_1 \oplus \mathcal{H}_2$ is always finer than each of the two hints $\mathcal{H}_1$ and $\mathcal{H}_2$ alone. And in particular, if sp is the support function of $\mathcal{H}_1 \oplus \mathcal{H}_2$, then we have $[sp_e(H), pl_e(H)] \subseteq [sp_{1e}(H), pl_{1e}(H)]$ and $[sp_e(H), pl_e(H)] \subseteq [sp_{2e}(H), pl_{2e}(H)]$, if $\mathcal{H}_1$ and $\mathcal{H}_2$ have no contradictory interpretations.

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